KALMAN-YAKUBOVICH-POPOV LEMMA FOR TWO-DIMENSIONAL SYSTEMS [1]

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Abstract: This paper is concerned with the development of a version of Kalman-Yakubovich-Popov (KYP) lemma for two-dimensional (2-D) systems characterized by the Roesser model. We shall establish the 2-D KYP lemma over any given finite frequency range which contains the KYP lemma over the infinite frequency range as a special case. Note that the latter has not been known for 2-D systems even though its one-dimensional (1-D) counterpart has been available for a long time. Our result is given in terms of a linear matrix inequality (LMI) which enables efficient computations for both analysis and design. As important applications of the lemma, 2-D bounded realness and positive realness will be investigated. A numerical example on the design of 2-D digital filter will be demonstrated.

Keywords: Two-dimensional systems; Roesser model; Kalman-Yakubovich-Popov lemma; Bounded and positive realness; 2-D digital filters.

1. INTRODUCTION

It is well known that the celebrated Kalman-Yakubovich-Popov (KYP) Lemma for 1-D systems is one of the most basic tools of system theory (Kalman 1963, Rantzer 1996), which connects two areas of system theory, frequency domain methods and state space methods. Together with Lyapunov stability theory, the KYP Lemma broadens the applications of state space methods and makes them dominant in modern control theory especially in robust control theory. Some useful computation tools such as linear matrix inequality (LMI) technique, can accordingly be applied in analysis and synthesis of systems. Recently, a so-called generalized KYP Lemma for 1-D systems was developed which establishes the equivalence between a frequency domain inequality for a transfer function over a finite frequency range and an LMI for its state space realization. The result generalizes the classical KYP Lemma from infinite frequency range to finite frequency range (Iwasaki *et al.* 2000). The generalized KYP Lemma contains the positive real lemma and the bounded real lemma as special cases and hence has a wide range of applications. For example, it can be used to design systems with different levels of H_∞ constraint over different ranges of frequency which is traditionally approximated by using a frequency weighting function.

On the other hand, the bounded realness of 2-D systems and the corresponding H_∞ control were recently approached by (Du and Xie 2002) and the positive real control was studied by (Xu *et al.*

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2003). However, there has been no KYP Lemma for 2-D systems over infinite frequency range, not to mention a KYP Lemma over a finite frequency one, even though its 1-D result has been known for four decades. Motivated by this, we investigate 2-D systems and develop a KYP Lemma for 2-D systems in Roesser model over a finite frequency range which naturally contains the KYP Lemma over infinite frequency range as a special case. Some useful results on the conditions of the finite frequency bounded realness and positive realness of 2-D systems, can be obtained immediately from this 2-D KYP Lemma.

As an important application of the 2-D KYP Lemma, we develop a systematic method for the design of 2-D digital filters using Roesser model (Roesser 1975). Note that many 2-D system theories including the stability of 2-D systems (Anderson *et al.* 1986, Lu 1993), 2-D system control (Du and Xie 2002, Li and Fadali 1991, Xu *et al.* 2003), 2-D state estimation (Du and Xie 2002) and 2-D H_∞ deconvolution filtering (Du and Xie 2002), have been associated with the Roesser model.

Presently, the design of 2-D digital filters is commonly conducted based on a transfer function model in frequency domain. One popular approach is to approximate the desired frequency response behavior by minimizing the error function under some norm measure, which includes the linear programming method (Hu and Rabiner 1972), the minimax method (Harris and Mersereau 1977), and the weighted least-squares method (Lu and Yeh 2000). However, in order to meet frequency response specifications, the frequency range is sampled and there is a frequency response specification at each sample point, which results in a complicated computation.

In this paper, we show that this difficulty can be overcome by using the proposed 2-D KYP Lemma. And the finite frequency response condition can be specified in terms of an LMI based on the Roesser model realization, which is computationally simpler and explicit. As compared with the frequency weighting approach, the proposed approach is exact in imposing constraints on magnitude response of filter over various ranges of frequency.

We use the following notation. We denote $\mathbb{C}$ the complex number set, $\mathbb{R}$ the real number set, $\mathbb{Z}$ the integer set, $\mathbb{R}_+$ the non-negative real number set and $\mathbb{Z}_+$ the non-negative integer set. Furthermore, $\mathbb{R}_{[0,\pi]}$ represents the set $\{x|x \in [0,\pi] \subset \mathbb{R}\}$. $\mathbb{H}_n$ stands for the set of $n\times n$ Hermitian matrices and $\mathbb{U}_n$ the set of $n \times n$ unitary matrices. For a matrix M, its transpose and complex conjugate transpose are denoted by M^T and M^*, respectively. $M > 0$ ($M \geq 0$) means M is positive definite (positive semi-definite). I_n represents the identity matrix of dimension $n \times n$.

2. 2-D KALMAN-YAKUBOVICH-POPOV LEMMA

2.1 Frequency property characterisation

Since the aim of KYP Lemma is to represent frequency-domain conditions of linear systems by linear matrix inequality conditions suitable for numerical computation, we firstly develop the following lemma to show the equivalence between a finite frequency condition and an LMI.

Lemma 1. Given two scalars $\bar{\omega}_h, \bar{\omega}_v \in \mathbb{R}_{[0,\pi]}$ and two complex vectors $f := \begin{bmatrix} f_h \\ f_v \end{bmatrix} \in \mathbb{C}^{n_h+n_v}$, $g := \begin{bmatrix} g_h \\ g_v \end{bmatrix} \in \mathbb{C}^{n_h+n_v}$, the following statements are equivalent

(i) There exist scalars $\omega_h, \omega_v \in \mathbb{R}$ such that

$$f = \Omega g, \quad \Omega := \begin{bmatrix} e^{j\omega_h} I_{n_h} & 0 \\ 0 & e^{j\omega_v} I_{n_v} \end{bmatrix}$$

and $cos(\omega_h) \geq cos(\bar{\omega}_h)$, $cos(\omega_v) \geq cos(\bar{\omega}_v)$;

(ii) For all $Q := \begin{bmatrix} Q_h & 0 \\ 0 & Q_v \end{bmatrix} \in \mathbb{H}_{(n_h+n_v)} > 0$, $P := \begin{bmatrix} P_h & 0 \\ 0 & P_v \end{bmatrix} \in \mathbb{H}_{(n_h+n_v)}$, the following matrix inequality

$$\begin{bmatrix} f \\ g \end{bmatrix}^* \begin{bmatrix} P & -Q \\ -Q & 2WQ - P \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} \leq 0 \qquad (1)$$

holds where $W = \begin{bmatrix} cos(\bar{\omega}_h) I_{n_h} & 0 \\ 0 & cos(\bar{\omega}_v) I_{n_v} \end{bmatrix}$.

Proof. Suppose (i) holds, then it follows that

$$\begin{aligned} &\begin{bmatrix} f \\ g \end{bmatrix}^* \begin{bmatrix} P & -Q \\ -Q & 2WQ - P \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} \\ &= f^*Pf - f^*Qg - g^*Qf + g^*(2WQ-P)g \\ &= g^*\Omega^*P\Omega g - g^*\Omega^*Qg - g^*Q\Omega g + g^*(2WQ-P)g \\ &= [2cos(\bar{\omega}_h) - 2cos(\omega_h)] g_h^* Q_h g_h \\ &\quad + [2cos(\bar{\omega}_v) - 2cos(\omega_v)] g_v^* Q_v g_v \leq 0. \end{aligned}$$

Conversely, (ii) yields

$$trace(ff^* - gg^*)P + trace(-fg^* - gf^* + gg^*2W)Q \leq 0$$

for all Hermitian block diagonal matrices P and positive definite matrices Q, which implies

$$f_h f_h^* - g_h g_h^* = 0, \quad f_v f_v^* - g_v g_v^* = 0,$$

and $-fg^* - gf^* + gg^*2W \leq 0$.

It can be verified that there exist two scalars $\omega_h, \omega_v \in \mathbb{R}_{[0,\pi]}$ such that $f_h = e^{j\omega_h} g_h$, $f_v = e^{j\omega_v} g_v$, and

$$-fg^* - gf^* + gg^* 2W = \begin{bmatrix} (-2cos(\omega_h) + 2cos(\bar{\omega}_h))g_h g_h^* & \star \\ \star & (-2cos(\omega_v) + 2cos(\bar{\omega}_v))g_v g_v^* \end{bmatrix} \leq 0$$

Obviously, it can be concluded that

$$cos(\omega_h) \geq cos(\bar{\omega}_h), \quad cos(\omega_v) \geq cos(\bar{\omega}_v).$$

This completes the proof. □

2.2 Kalman-Yakubovich-Popov Lemma for Roesser Model

In this subsection, the KYP Lemma will be established for the following 2-D Roesser model

$$\begin{bmatrix} x^h(i+1, j) \\ x^v(i, j+1) \end{bmatrix} = A \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix} + Bu(i, j)$$
$$y(i, j) = C \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix} + Du(i, j) \tag{2}$$

where $x^h \in \mathbb{C}^{n_h}, x^v \in \mathbb{C}^{n_v}, u \in \mathbb{C}^m$ and $y \in \mathbb{C}^l$ are, respectively, the horizontal state, vertical state, input and the output of the system, $A \in \mathbb{C}^{(n_h+n_v)\times(n_h+n_v)}$, $B \in \mathbb{C}^{(n_h+n_v)\times m}$, $C \in \mathbb{C}^{r\times(n_h+n_v)}$ and $D \in \mathbb{C}^{r\times m}$ are the system matrices.

Let $x(i, j) := \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix}$ and $n = n_h + n_v$, then (2) provides the relationship between the state variable x and the input variable u, that is,

$$\begin{bmatrix} X \\ U \end{bmatrix} \in \mathbb{C}^{n+m} \text{ and } \begin{bmatrix} e^{j\omega_h} I_{n_h} & 0 \\ 0 & e^{j\omega_v} I_{n_v} \end{bmatrix} X = AX + BU,$$

for $\omega_h, \omega_v \in \mathbb{R}_{[0,\pi]}$ where X, U denote the $\mathcal{Z}$-transforms of the state variable x and the control variable u, respectively. By applying Lemma 1, the 2-D KYP Lemma can be obtained.

Theorem 1. Given scalars $\bar{\omega}_h, \bar{\omega}_v \in \mathbb{R}_{[0,\pi]}$ and matrices $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$ and a Hermitian matrix $\Theta \in \mathbb{C}^{(n+m)\times(n+m)}$ with $n = n_h + n_v$, suppose A has no eigenvalue on the unit circle. The finite frequency condition

$$\begin{bmatrix} (e^{j\omega} I - A)^{-1} B \\ I \end{bmatrix}^* \Theta \begin{bmatrix} (e^{j\omega} I - A)^{-1} B \\ I \end{bmatrix} < 0, \tag{3}$$

for all $cos(\omega_h) \geq cos(\bar{\omega}_h), cos(\omega_v) \geq cos(\bar{\omega}_v)$ is satisfied, where $\omega = \begin{bmatrix} \omega_h I_{n_h} & 0 \\ 0 & \omega_v I_{n_v} \end{bmatrix}$, if there exist matrices $Q := \begin{bmatrix} Q_h & 0 \\ 0 & Q_v \end{bmatrix} \in \mathbb{H}_n > 0$ and $P := \begin{bmatrix} P_h & 0 \\ 0 & P_v \end{bmatrix} \in \mathbb{H}_n$ such that

$$\Theta < \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} P & -Q \\ -Q & 2WQ - P \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}, \tag{4}$$

where $W = \begin{bmatrix} cos(\bar{\omega}_h) I_{n_h} & 0 \\ 0 & cos(\bar{\omega}_v) I_{n_v} \end{bmatrix}$.

Proof. Note that the matrix $e^{j\omega} I - A$ is invertible for any $\omega \in \mathbb{R}$ since A has no eigenvalue on the unit circle.

Let $\begin{bmatrix} f \\ g \end{bmatrix} := \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} \begin{bmatrix} X \\ U \end{bmatrix}$, it can be known that

$$f = e^{j\omega} g := \begin{bmatrix} e^{j\omega_h} I_{n_h} & 0 \\ 0 & e^{j\omega_v} I_{n_v} \end{bmatrix} g.$$

Since $cos(\omega_h) \geq cos(\bar{\omega}_h), cos(\omega_v) \geq cos(\bar{\omega}_v)$, by Lemma 1, it provides that for any

$$\xi \in \mathcal{K} =: \left\{ \begin{bmatrix} f \\ g \end{bmatrix} \in \mathbb{C}^{n+m} : f = e^{j\omega} g \right.$$
$$\left. \text{where } cos(\omega_h) \geq cos(\bar{\omega}_h), cos(\omega_v) \geq cos(\bar{\omega}_v) \right\},$$

the inequality $\xi^* T \xi \leq 0$ holds for any

$$T \in \mathcal{T} := \left\{ \begin{bmatrix} P & -Q \\ -Q & 2WQ - P \end{bmatrix} : \forall\, Q = Q^* := \begin{bmatrix} Q_h & 0 \\ 0 & Q_v \end{bmatrix} > 0, \right.$$
$$\left. P = P^* := \begin{bmatrix} P_h & 0 \\ 0 & P_v \end{bmatrix} \right\}.$$

Furthermore, it can be said that for all

$$\zeta \in \mathcal{G} := \left\{ \begin{bmatrix} X \\ U \end{bmatrix} \in \mathbb{C}^{n+m} : e^{j\omega} X = AX + BU \right.$$
$$\left. \text{for } cos(\omega_h) \geq cos(\bar{\omega}_h), cos(\omega_v) \geq cos(\bar{\omega}_v) \right\},$$

and all

$$S \in \mathcal{S} := \left\{ \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} P & -Q \\ -Q & 2WQ - P \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} : \right.$$
$$\left. \forall\, Q = Q^* > 0, P = P^* \right\},$$

there holds $\zeta^* S \zeta \leq 0$. Therefore, if there exists a matrix $S \in \mathcal{S}$ such that (4) exists, it is readily known that (3) holds. This completes the proof. □

Note that for $\bar{\omega}_h, \bar{\omega}_v \in [0,\ \pi]$ and $\omega_h, \omega_v \in [-\pi,\ \pi]$, $cos(\omega_h) \geq cos(\bar{\omega}_h), cos(\omega_v) \geq cos(\bar{\omega}_v)$ if and only if $0 \leq |\omega_h| \leq \bar{\omega}_h$ and $0 \leq |\omega_v| \leq \bar{\omega}_v$. Theorem 1 thus gives a sufficient condition for the existence of a performance characterization specified on low frequency domain. The next corollary extends the result presented in Theorem 1, and provides a sufficient condition for a performance characterization specified on arbitrarily specified frequency interval.

Corollary 1. Given scalars $\bar{\omega}_{h_1}, \bar{\omega}_{h_2}, \bar{\omega}_{v_1}, \bar{\omega}_{v_2}, \in \mathbb{R}_{[0,\pi]}$ with $\bar{\omega}_{h_1} \leq \bar{\omega}_{h_2}$, $\bar{\omega}_{v_1} \leq \bar{\omega}_{v_2}$, and matrices $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$ and a Hermitian matrix $\Theta \in \mathbb{C}^{(n+m)\times(n+m)}$ with $n = n_h + n_v$, suppose A has no eigenvalue on the unit circle. For all $\omega_h, \omega_v \in \mathbb{R}_{[0,\pi]}$ satisfying $0 \leq \bar{\omega}_{h_1} \leq \omega_h \leq \bar{\omega}_{h_2} \leq \pi$ and $0 \leq \bar{\omega}_{v_1} \leq \omega_v \leq \bar{\omega}_{v_2} \leq \pi$, the following finite frequency condition

$$\begin{bmatrix} (e^{j\omega} I - A)^{-1} B \\ I \end{bmatrix}^* \Theta \begin{bmatrix} (e^{j\omega} I - A)^{-1} B \\ I \end{bmatrix} < 0, \tag{5}$$

holds where $\omega = \begin{bmatrix} \omega_h I_{n_h} & 0 \\ 0 & \omega_v I_{n_v} \end{bmatrix}$, if there exist matrices $Q := \begin{bmatrix} Q_h & 0 \\ 0 & Q_v \end{bmatrix} \in \mathbb{H}_n > 0$ and $P := \begin{bmatrix} P_h & 0 \\ 0 & P_v \end{bmatrix} \in \mathbb{H}_n$ such that

$$\Theta < \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} P & -\Lambda^* Q \\ -Q\Lambda & 2WQ - P \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}, \quad (6)$$

where

$$\Lambda = \begin{bmatrix} e^{j\bar{\omega}_h^p} I_{n_h} & 0 \\ 0 & e^{j\bar{\omega}_v^p} I_{n_v} \end{bmatrix}, W = \begin{bmatrix} cos(\bar{\omega}_h^m) I_{n_h} & 0 \\ 0 & cos(\bar{\omega}_v^m) I_{n_v} \end{bmatrix},$$

and $\bar{\omega}_h^p = (\bar{\omega}_{h_1} + \bar{\omega}_{h_2})/2$, $\bar{\omega}_v^p = (\bar{\omega}_{v_1} + \bar{\omega}_{v_2})/2$, $\bar{\omega}_h^m = (\bar{\omega}_{h_2} - \bar{\omega}_{h_1})/2$, $\bar{\omega}_v^m = (\bar{\omega}_{v_2} - \bar{\omega}_{v_1})/2$.

Proof. Note that $0 \leq \bar{\omega}_{h_1} \leq \omega_h \leq \bar{\omega}_{h_2} \leq \pi$ or $-1 \leq cos(\bar{\omega}_{h_2}) \leq cos(\omega_h) \leq cos(\bar{\omega}_{h_1}) \leq 1$, is equivalent to

$$|\omega_h - \bar{\omega}_h^p| \leq \bar{\omega}_h^m, \quad or \quad cos(\omega_h - \bar{\omega}_h^p) \geq cos(\bar{\omega}_h^m)$$

and $0 \leq \bar{\omega}_{v_1} \leq \omega_v \leq \bar{\omega}_{v_2} \leq \pi$ or $-1 \leq cos(\bar{\omega}_{v_2}) \leq cos(\omega_v) \leq cos(\bar{\omega}_{v_1}) \leq 1$ is equivalent to

$$|\omega_v - \bar{\omega}_v^p| \leq \bar{\omega}_v^m, \quad or \quad cos(\omega_v - \bar{\omega}_v^p) \geq cos(\bar{\omega}_v^m).$$

Hence, introducing the following transformation

$$\hat{A} = \Lambda A, \quad \hat{B} = \Lambda B,$$

it can be found that

$$\left(\begin{bmatrix} e^{j(\omega_h - \bar{\omega}_h^p)} I_{n_h} & \\ & e^{j(\omega_v - \bar{\omega}_v^p)} I_{n_v} \end{bmatrix} - A \right)^{-1} B = \left(\begin{bmatrix} e^{j\omega_h} I_{n_h} & \\ & e^{j\omega_v} I_{n_v} \end{bmatrix} - \hat{A} \right)^{-1} \hat{B}.$$

According to Theorem 1, (5) holds if and only if there exist a Hermitian matrix $P := \begin{bmatrix} P_h & 0 \\ 0 & P_v \end{bmatrix} \in \mathbb{H}_n$ and a positive definite matrix $Q := \begin{bmatrix} Q_h & 0 \\ 0 & Q_v \end{bmatrix} \in \mathbb{H}_n$ such that

$$\Theta < \begin{bmatrix} \hat{A} & \hat{B} \\ I & 0 \end{bmatrix}^* \begin{bmatrix} P & -Q \\ -Q & 2WQ - P \end{bmatrix} \begin{bmatrix} \hat{A} & \hat{B} \\ I & 0 \end{bmatrix} \quad (7)$$

exists. Obviously, (7) is equivalent to

$$\Theta < \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} \Lambda & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} P & -Q \\ -Q & 2WQ - P \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}$$

Due to the fact that $\Lambda^* P \Lambda = P$, this corollary follows. □

2.3 Some applications of 2-D Kalman-Yakubovich -Popov Lemma

Motivated by the important roles of 1-D KYP Lemma in systems theory, the corresponding applications of 2-D KYP Lemma can be found. Here we just give two immediate examples.

The first example is the finite frequency positive realness for Roesser model. Let

$$\Theta = \begin{bmatrix} 0 & C^* \\ C & D + D^* \end{bmatrix},$$

then the condition of finite frequency positive realness of 2-D systems in Roesser model follows from the 2-D KYP Lemma.

Corollary 2. Given scalars $\bar{\omega}_h, \bar{\omega}_v \in \mathbb{R}_{[0,\pi]}$, consider the system (2) that has no poles on the unit circle. Then, the finite frequency condition

$$(C[e^{j\omega}I - A]^{-1}B + D)^* + (C[e^{j\omega}I - A]^{-1}B + D) > 0,$$

is satisfied for all ω_h and ω_v satisfying

$$cos(\omega_h) \geq cos(\bar{\omega}_h), cos(\omega_v) \geq cos(\bar{\omega}_v)$$

if there exist matrices $Q := \begin{bmatrix} Q_h & 0 \\ 0 & Q_v \end{bmatrix} \in \mathbb{H}_n > 0$ and $P := \begin{bmatrix} P_h & 0 \\ 0 & P_v \end{bmatrix} \in \mathbb{H}_n$ such that

$$\begin{bmatrix} 0 & C^* \\ C & D + D^* \end{bmatrix} < \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} P & -Q \\ -Q & 2WQ - P \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} \quad (8)$$

where $W = \begin{bmatrix} cos(\bar{\omega}_h) I_{n_h} & 0 \\ 0 & cos(\bar{\omega}_v) I_{n_v} \end{bmatrix}$.

The second example is the finite frequency bounded realness of 2-D systems.

Corollary 3. Given scalars $\bar{\omega}_h, \bar{\omega}_v \in \mathbb{R}_{[0,\pi]}$, consider the system of (2) that has no poles on the unit circle. The H_∞ performance

$$\left\| C[e^{j\omega}I - A]^{-1}B + D \right\|_\infty < \gamma \quad (9)$$

holds for all ω_h and ω_v satisfying $cos(\omega_h) \geq cos(\bar{\omega}_h)$ and $cos(\omega_v) \geq cos(\bar{\omega}_v)$ if there exist matrices $Q := \begin{bmatrix} Q_h & 0 \\ 0 & Q_v \end{bmatrix} \in \mathbb{H}_n > 0$ and $P := \begin{bmatrix} P_h & 0 \\ 0 & P_v \end{bmatrix} \in \mathbb{H}_n$ such that

$$\begin{bmatrix} C^*C & C^*D \\ D^*C & -\gamma^2 I + D^*D \end{bmatrix} < \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} P & -Q \\ -Q & 2WQ - P \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}$$

where $W = \begin{bmatrix} cos(\bar{\omega}_h) I_{n_h} & 0 \\ 0 & cos(\bar{\omega}_v) I_{n_v} \end{bmatrix}$.

Proof. This corollary follows from Theorem 1 and the fact that (17) can be rewritten as (3) with

$$\Theta = \begin{bmatrix} C^*C & C^*D \\ D^*C & -\gamma^2 I + D^*D \end{bmatrix}.$$

□

3. 2-D DIGITAL FILTER DESIGN

In this section, the problem of 2-D digital filter design will be addressed based on the 2-D KYP Lemma. First of all, we specify some requirements for the frequency response of 2-D digital filters. Without loss of generality, the frequency range $[-\pi, \pi] \times [-\pi, \pi]$ is considered.

3.1 Performance specification for 2-D filter design

In many applications of 2-D signal processing, a *linear phase* frequency response is often desired. In image processing, for example, nonlinear phase responses tend to destroy lines and edges. The frequency response of a linear phase filter can be represented by

$$G(\omega_h, \omega_v) = r(\omega_h, \omega_v) e^{-j\omega_h d_h} e^{-j\omega_v d_v}$$

where $d_h, d_v \in \mathbb{R}$ are two scalars and $r : \mathbb{R}_{[-\pi,\pi]} \times \mathbb{R}_{[-\pi,\pi]} \mapsto \mathbb{R}$.

Moreover, a frequency response with different magnitudes over different frequency ranges is desired. For example, the frequency response of a low-pass filter has high magnitude over the low frequency range and low magnitude over the high frequency range to pass low frequency signals and block high frequency signals.

Accordingly, the filter design problem can be reformulated as an optimization problem, that is, given $(k_h, k_v) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, $d_h, d_v \in \mathbb{R}$ and four real scalars $0 \le \bar\omega_{h_1} < \bar\omega_{h_2} \le \pi$, $0 \le \bar\omega_{v_1} < \bar\omega_{v_2} \le \pi$, which partition the frequency domain $[-\pi, \pi] \times [-\pi, \pi]$ into three bands:

$$\begin{aligned}
\Pi_1 &:= \{(\omega_h, \omega_v) | 0 \le |\omega_h| \le \bar\omega_{h_1}, 0 \le |\omega_v| \le \bar\omega_{v_1}\} \\
\Pi_2 &:= \{(\omega_h, \omega_v) | \bar\omega_{h_1} < |\omega_h| < \bar\omega_{h_2}, 0 < |\omega_v| < \bar\omega_{v_2}\} \\
&\quad \cup \{(\omega_h, \omega_v) | 0 < |\omega_h| < \bar\omega_{h_2}, \bar\omega_{v_1} < |\omega_v| < \bar\omega_{v_2}\} \\
\Pi_3 &:= \{(\omega_h, \omega_v) | \bar\omega_{h_2} \le |\omega_h| \le \pi, 0 \le |\omega_v| \le \pi\} \\
&\quad \cup \{(\omega_h, \omega_v) | 0 \le |\omega_h| \le \pi, \bar\omega_{v_2} \le |\omega_v| \le \pi\}
\end{aligned}$$

find a filter G of order $k_h \times k_v$ in the form of (2) such that

$$\left\| G(\omega_h, \omega_v) - a e^{-j\omega_h d_h} e^{-j\omega_v d_v} \right\|_\infty < \gamma_1 \quad (10)$$

for $(\omega_h, \omega_v) \in \Pi_1$ and

$$\left\| G(\omega_h, \omega_v) - b e^{-j\omega_h d_h} e^{-j\omega_v d_v} \right\|_\infty < \gamma_2 \quad (11)$$

for $(\omega_h, \omega_v) \in \Pi_3$, where $a = 1, b = 0$ for a low-pass filter, and $a = 0, b = 1$ for a high-pass filter.

Note that we choose $\bar\omega_{h_1} \ne \bar\omega_{h_2}$ and $\bar\omega_{v_1} \ne \bar\omega_{v_2}$, otherwise, the performances (10) and (11) can not hold for satisfactorily small parameters γ_1 and γ_2. This is because of the fact that $G(\omega_h, \omega_v)$ is a continuous function and cannot jump at the points $\omega_h = \bar\omega_{h_1}$ and $\omega_v = \bar\omega_{v_1}$.

Besides low-pass filters and high pass filters, band-pass filters are also widely applied in practice. For the design of 2-D digital band-pass filters, the frequency domain is divided into five bands. For each pass band and stop band, similar performances as those of (10) and (11) can be established. So the design of band-pass filters can be regarded as a natural extension of the design of low-pass or high pass filters, and it will not be specially discussed in this paper.

3.2 2-D FIR filer design

In this subsection, we consider the problem of 2-D FIR filter design. Similar to the state space realization of 1-D FIR filters, 2-D FIR filters in polynomial form

$$G(z_h, z_v) = \sum_{i=0}^{k_h} \sum_{j=0}^{k_v} h_{i,j} z_h^{-i} z_v^{-j}, \quad h_{i,j} \in \mathbb{R} \quad (12)$$

can be realized in the canonical form of the Roesser model (Kaczorek 1985). Given the order of a 2-D FIR filter, the structure of the system has been fixed and both the matrices A and B are in the canonical form. Hence, only matrices C and D are to be designed.

Denote the Roesser model realization of the filter to be designed as $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ and the realization of $e^{-j\omega_h d_h} e^{-j\omega_v d_v}$ as $\left[\begin{array}{c|c} A_d & B_d \\ \hline C_d & D_d \end{array}\right]$, then $G(j\omega_h, j\omega_v) - e^{-j\omega_h d_h} e^{-j\omega_v d_v}$ can be represented by

$$\left[\begin{array}{c|c} \bar A & \bar B \\ \hline \bar C & \bar D \end{array}\right] := \left[\begin{array}{cc|c} A & 0 & B \\ 0 & A_d & B_d \\ \hline C & -C_d & D - D_d \end{array}\right].$$

Since both constraints (10) and (11) can be represented by matrix inequalities according to the finite frequency bounded real lemma presented in Corollary 3, the filter design problem described in the previous subsection can be solved via an LMI optimization.

Theorem 2. Given scalars $\bar\omega_{h_1}, \bar\omega_{h_2}, \bar\omega_{v_1}, \bar\omega_{v_2} \in \mathbb{R}_{[0,\pi]}$ with $\bar\omega_{h_1} < \bar\omega_{h_2}$ and $\bar\omega_{v_1} < \bar\omega_{v_2}$ and $\gamma_1, \gamma_2 \in \mathbb{R}_+$, there exists a $(k_h \times k_v)$-th order 2-D FIR low-pass filter in the canonical form of (2) satisfying (10) and (11), if there exist $C \in \mathbb{R}^{1\times n}$, $D \in \mathbb{R}$, and block diagonal matrices $P_1, Q_1 \in \mathbb{H}_{\bar n}$, $P_2, P_3, P_4, P_5, Q_2, Q_3, Q_4, Q_5 \in \mathbb{H}_n$ and $Q_1, Q_2, Q_3, Q_4, Q_5 > 0$ such that

$$\begin{bmatrix} H_1 + \begin{bmatrix} 0 & 0 \\ 0 & \gamma_1^2 I \end{bmatrix} & \begin{bmatrix} \bar C^T \\ \bar D^T \end{bmatrix} \\ \begin{bmatrix} \bar C & \bar D \end{bmatrix} & I \end{bmatrix} > 0 \quad (13)$$

$$\begin{bmatrix} H_k + \begin{bmatrix} 0 & 0 \\ 0 & \gamma_2^2 I \end{bmatrix} & \begin{bmatrix} C^T \\ D^T \end{bmatrix} \\ \begin{bmatrix} C & D \end{bmatrix} & I \end{bmatrix} > 0, \ k = 2,3,4,5 \quad (14)$$

where $n = n_h + n_v$ is the dimension of A, $\bar n = \bar n_h + \bar n_v$ is the dimension of $\bar A$, and $H_1 = \begin{bmatrix} \bar A & \bar B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} P_1 & -Q_1 \\ -Q_1 & 2W_1 Q_1 - P_1 \end{bmatrix} \begin{bmatrix} \bar A & \bar B \\ I & 0 \end{bmatrix}$, $H_k = \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} P_k & -\Lambda_k^* Q_k \\ -Q_k \Lambda_k & 2W_k Q_k - P_k \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}$, and

$$W_1 = \begin{bmatrix} cos(\bar\omega_{h_1}) I_{\bar n_h} & 0 \\ 0 & cos(\bar\omega_{v_1}) I_{\bar n_v} \end{bmatrix}, \Lambda_2 = \begin{bmatrix} e^{j\bar\omega_h^p} I_{n_h} & 0 \\ 0 & jI_{n_v} \end{bmatrix},$$

$$W_2 = W_4 = \begin{bmatrix} cos(\bar\omega_h^m) I_{n_h} & 0 \\ 0 & 0 \end{bmatrix}, \Lambda_3 = \begin{bmatrix} jI_{n_h} & 0 \\ 0 & e^{j\bar\omega_v^p} I_{n_v} \end{bmatrix},$$

$$W_3 = W_5 = \begin{bmatrix} 0 & 0 \\ 0 & cos(\bar\omega_v^m) I_{n_v} \end{bmatrix}, \Lambda_4 = \begin{bmatrix} e^{j\bar\omega_h^p} I_{n_h} & 0 \\ 0 & -jI_{n_v} \end{bmatrix},$$

$$\Lambda_5 = \begin{bmatrix} -jI_{n_h} & 0 \\ 0 & e^{j\bar{\omega}_v^p} I_{n_v} \end{bmatrix}, \bar{\omega}_h^p = (\bar{\omega}_{h_2} + \pi)/2,$$
$$\bar{\omega}_v^p = (\bar{\omega}_{v_2} + \pi)/2, \bar{\omega}_h^m = (\pi - \bar{\omega}_{h_2})/2, \bar{\omega}_v^m = (\pi - \bar{\omega}_{v_2})/2.$$

Remark 1. Given scalars $\bar{\omega}_{h_1}, \bar{\omega}_{h_2}, \bar{\omega}_{v_1}, \bar{\omega}_{v_2} \in \mathbb{R}_{[0,\pi]}$ with $\bar{\omega}_{h_1} < \bar{\omega}_{h_2}$ and $\bar{\omega}_{v_1} < \bar{\omega}_{v_2}$ and $w_1, w_2 \in \mathbb{R}_+$, an optimal $(k_h \times k_v)$-th order 2-D FIR filter in the form of (2) with both A and B of the canonical form can be solved from

$$\min_{C,D,P_1,P_2,P_3,P_4,P_5,Q_1,Q_2,Q_3,Q_4,Q_5} w_1\gamma_1 + w_2\gamma_2 \quad (15)$$

subject to (13) and (14).

3.3 Example

To illustrate the proposed filter design algorithm, an 11×11th order low-pass FIR filter is designed. Given $\bar{\omega}_{h_1} = \bar{\omega}_{v_1} = \pi/3$, $\bar{\omega}_{h_2} = \bar{\omega}_{v_2} = 2\pi/3$, $d_0 = 3$, $d_1 = 3$, $w_1 = 1$, $w_2 = 1$, we have the solution $\gamma_1 = 0.0308$ and $\gamma_2 = 0.0295$, and the magnitude and the phase of frequency response of this designed filter is shown in Figure 1 and Figure 2, respectively. The linear phase property is well reflected in Figure 2.

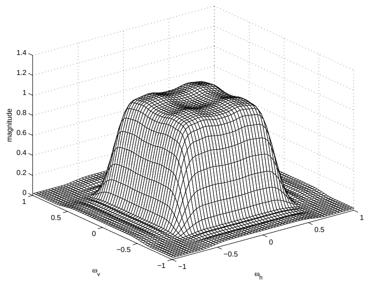

Fig. 1. The magnitude of frequency response of the obtained (11×11)th-order low-pass FIR filter

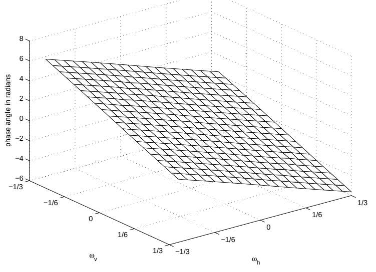

Fig. 2. The phase of frequency response of the obtained (11×11)th-order low-pass FIR filter

4. CONCLUSION

A finite frequency KYP Lemma for the 2-D Roesser model has been developed in this paper. The result is characterized in terms of an LMI, which can be easily solved via convex optimization. Following from this lemma, some useful results, such as finite frequency positive realness and finite frequency bounded realness of 2-D systems, have been obtained. As an application example, the latter has been successfully applied to the design of 2-D digital filters.

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