

ROBUST STABILITY ANALYSIS OF LINEAR SYSTEMS WITH AFFINE PARAMETER UNCERTAINTIES¹

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Abstract: This paper is concerned with the problem of robust stability of a class of uncertain linear systems, where the system state-matrices considered are affinely dependent on the uncertain parameters. Affine parameter-dependent Lyapunov functions are exploited to prove stability, and a robust stability criterion for the above class of systems to be affinely quadratically stable (AQS) is given in terms of linear matrix inequalities (LMIs). A comparison with the existing tests for AQS is given, and it is shown that the robust stability criterion provides a test that is not more conservative than the existing tests. Numerical examples are given to illustrate the results. *Copyright ©2005 IFAC*

Keywords: Linear systems, robust stability, parameter-dependent Lyapunov function, linear matrix inequalities, parameter uncertainty.

1. INTRODUCTION

Robust stability analysis for control systems with parameter uncertainties is one of fundamental issues in systems theory, many important advances have been achieved, see (Apkarian *et al.*, 2001; Feron *et al.*, 1996; Gahinet *et al.*, 1996; Scherer, 1997; Tuan and Apkarian, 2002; Zhou *et al.*, 1996) and the references therein. For linear systems with affine parameter uncertainties, a comprehensive development via the notion of quadratic stability (i.e., use of a single quadratic Lyapunov function of the form $V(x) = x^T P x$ to prove stability, (Barmish, 1982)) is presented by the Linear Matrix Inequality (LMI) approach in (Boyd *et al.*, 1994). However, the use of a single Lyapunov function for robust stability often leads to overly conservative results. To overcome this, affine parameter-dependent Lyapunov functions

which depend on the real uncertain parameters are used to establish robust stability criteria in (Gahinet *et al.*, 1996), which is less conservative than quadratic stability. In (Feron *et al.*, 1996), a sufficient condition for the existence of an affine parameter-dependent Lyapunov function is presented in terms of LMIs, and the resulting stability criterion is less conservative than Popov's stability criterion.

Recently, more general parameter-dependent Lyapunov functions, the Projection Lemma (Gahinet and Apkarian, 1994) and the multiplier technique have been exploited to develop less conservative robust stability criteria (Apkarian *et al.*, 2001; de Oliveira *et al.*, 1999; Dettori and Scherer, 1998; Dettori and Scherer, 2000; Iwasaki, 1997; Iwasaki, 1998; Iwasaki *et al.*, 2001; Peaucelle and Arzelier, 2001; Scherer, 1997). The multiplier technique consists of parameter dependent multipliers (Dettori and Scherer, 2000; Iwasaki, 1998) and parameter independent multipliers (Dettori and

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Scherer, 1998; Iwasaki *et al.*, 2001; Scherer, 1997), which reduces the problems of robust stability and performance analysis to convex optimization problems, and results in various robustness tests. However, the relation among these tests appears to be unclear. Generally, it is difficult to compare the degree of conservatism associated with the proposed robustness tests. But it becomes possible when the type of parameter-dependent Lyapunov functions used to prove stability is specified. Some such comparisons have been done in (Feron *et al.*, 1996; Iwasaki, 1997; Iwasaki *et al.*, 2001)

This paper will address the problem of robust stability analysis for a class of uncertain linear time-invariant systems, where the system state-matrices considered are affinely dependent on the uncertain parameters. Affine parameter-dependent Lyapunov functions are exploited to prove stability, and a robust stability criterion for this class of systems to be affinely quadratically stable (AQS)(Gahinet *et al.*, 1996) is given in terms of linear matrix inequalities (LMIs). Comparing with the existing robust stability criteria for AQS, it is shown that the proposed robust stability criterion is no more conservative than the AQS results in (Dettori and Scherer, 2000; Feron *et al.*, 1996; Gahinet *et al.*, 1996; Iwasaki, 1998; Peaucelle and Arzelier, 2001). The paper is organized as follows. The problem statement and some preliminaries are given in Section 2. Section 3 presents the robust stability criterion, and the comparison with the existing robust stability criteria for AQS. Section 4 concludes the paper.

Notation: For a matrix E , E^T and E^{-1} denote its transpose and inverse if it exists, respectively. For a column-rank deficient matrix H , \mathbf{N}_H denotes a matrix whose columns form a basis for the null space of H . I and I_k denote the identity matrix with an appropriate dimension and $k \times k$ identity matrix, respectively.

2. PROBLEM STATEMENT AND PRELIMINARIES

Consider an uncertain linear time-invariant system described by

$$\dot{x}(t) = A(\delta)x(t) \quad (1)$$

where $x(t) \in R^n$ is the state, and $A(\delta)$ is affinely dependent on δ , i.e.,

$$A(\delta) = A_0 + \sum_{i=1}^N \delta_i A_i \quad (2)$$

The real parameter vector $\delta = [\delta_1 \cdots \delta_N]^T$ is uncertain but constant in time and satisfies

$$|\delta_i| \leq \gamma, \quad i = 1, \dots, N \quad (3)$$

Definition 1(Feron *et al.*, 1996): The system (1) is said to be affinely quadratically stable (AQS) if there exist $N + 1$ symmetric matrices P_0, P_1, \dots, P_N such that, for all δ_i satisfying (3),

$$P(\delta) := P_0 + \sum_{i=1}^N \delta_i P_i > 0 \quad (4)$$

and

$$P(\delta)A(\delta) + A(\delta)^T P(\delta) < 0 \quad (5)$$

Then the problem considered in this paper is as follows.

Affinely quadratic stability problem: For a given $\gamma > 0$, is the system (1) AQS?

Remark 1: For the robust stability problem, many significant advances have been achieved. In (Gahinet *et al.*, 1996), the S-procedure is applied to give a test for AQS, which is less conservative than Popov's stability criterion. Based on the multiconvexity condition for scalar quadratic functions, a different test for AQS is presented in (Feron *et al.*, 1996). Recently, the parameter dependent multiplier approach or the vertex separator approach are exploited to derive robust stability criteria for systems with affine parameter uncertainties in (Dettori and Scherer, 2000) and (Iwasaki, 1998), respectively, and a test for robust stability is derived in (Peaucelle and Arzelier, 2001) by using the Projection Lemma(Gahinet and Apkarian, 1994). All these robust stability criteria also provide tests for AQS. In the next section, we will use the approach in (Iwasaki *et al.*, 2001) to present a sufficient condition for AQS, and then prove that the latter is no more conservative than the above-mentioned ones.

The following preliminaries will be used in the development. The set of vertices of the cube described by (3) is denoted by δ_v , i.e.,

$$\delta_v = \{[\delta_1 \cdots \delta_N]^T : \delta_i \in \{-\gamma, \gamma\}, i = 1, \dots, N\} \quad (6)$$

Lemma 1 ((Iwasaki *et al.*, 2001; Scherer, 1997)): Let matrices $Q = Q^T$, F , and a compact subset of real matrices \mathbf{H} be given. Then the following statements are equivalent:

(i) for each $H \in \mathbf{H}$

$$\xi^T Q \xi < 0 \text{ for all } \xi \neq 0 \text{ such that } HF\xi = 0;$$

(ii) there exists $\Theta = \Theta^T$ such that

$$Q + F^T \Theta F < 0, \quad \mathbf{N}_H^T \Theta \mathbf{N}_H \geq 0 \text{ for all } H \in \mathbf{H}.$$

3. ROBUST STABILITY ANALYSIS

Denote

$$X_1 = [x_1 \quad \cdots \quad x_N]^T \in R^{Nn} \quad (7)$$

$$E_1 = [I_n \ \cdots \ I_n]^T \in R^{Nn \times n} \quad (8)$$

$$\Delta = \text{diag}[\delta_1 I_n \cdots \delta_N I_n] \quad (9)$$

$$P^1 = [P_1 \ \cdots \ P_N]^T \in R^{Nn \times n} \quad (10)$$

$$A^1 = [A_1 \ \cdots \ A_N] \in R^{n \times Nn} \quad (11)$$

Then we have

Theorem 1: Consider the uncertain system (1), and assume that A_0 is Hurwitz. Then the following two statements are equivalent:

- (i) the system (1) is affinely quadratically stable;
- (ii) there exist symmetric matrices P_0, P_1, \dots, P_N and $\Theta \in R^{2Nn \times 2Nn}$ such that

$$\begin{aligned} & \begin{bmatrix} P_0 \\ P^1 \end{bmatrix} [A_0 \ A^1] + \left(\begin{bmatrix} P_0 \\ P^1 \end{bmatrix} [A_0 \ A^1] \right)^T \\ & + \begin{bmatrix} E_1 & 0 \\ 0 & I_{Nn} \end{bmatrix}^T \Theta \begin{bmatrix} E_1 & 0 \\ 0 & I_{Nn} \end{bmatrix} < 0 \end{aligned} \quad (12)$$

and

$$\begin{bmatrix} I \\ \Delta \end{bmatrix}^T \Theta \begin{bmatrix} I \\ \Delta \end{bmatrix} \geq 0 \text{ for all } \delta_i \text{ satisfying (3)} \quad (13)$$

hold.

Proof: By definition 1, (i) is equivalent to that there exist $N+1$ symmetric matrices P_0, P_1, \dots, P_N such that (4) and (5) holds for all δ_i satisfying (3). From (7)-(11) and (2), it follows

$$\begin{aligned} x^T P(\delta) A(\delta) x &= (x^T P_0 + X_1^T P^1) (A_0 x + A^1 X_1) \\ &= \begin{bmatrix} x \\ X_1 \end{bmatrix}^T \begin{bmatrix} P_0 \\ P^1 \end{bmatrix} [A_0 \ A^1] \begin{bmatrix} x \\ X_1 \end{bmatrix} \end{aligned} \quad (14)$$

$$X_1 = \Delta E_1 x \quad (15)$$

(15) is equivalent to

$$[\Delta \ -I] \begin{bmatrix} E_1 & 0 \\ 0 & I_{Nn} \end{bmatrix} \begin{bmatrix} x \\ X_1 \end{bmatrix} = 0 \quad (16)$$

By Lemma 1, (14) and (16), the equivalence of (5) and (ii) follows. Furthermore, by the result in (Feron *et al.*, 1996), conditions (4) and (5) are equivalent to (5) under the assumption that A_0 is Hurwitz. Thus, the proof is complete. \square

Theorem 1 gives a necessary and sufficient condition for the uncertain system (1) to be AQS via the parameter dependent Lyapunov function $V(x) = x^T P(\delta) x$ with $P(\delta)$ given by (4). However, condition (13) is nonconvex, it is difficult to be verified. The following theorem presents a convex sufficient condition.

Theorem 2: The system (1) is affinely quadratically stable if A_0 is Hurwitz and, if there exist

symmetric matrices P_0, P_i ($i = 1, \dots, N$), and a symmetric matrix Θ given by

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12}^T & \Theta_{22} \end{bmatrix} \quad (17)$$

with $\Theta_{11} \in R^{Nn \times Nn}$, and

$$\Theta_{22ii} \leq 0, \quad i = 1, \dots, N \quad (18)$$

such that (12) holds and, moreover,

$$\begin{bmatrix} I \\ \Delta \end{bmatrix}^T \Theta \begin{bmatrix} I \\ \Delta \end{bmatrix} \geq 0 \text{ for all } \delta \in \delta_{\mathbf{v}} \quad (19)$$

where $\Theta_{22ii} \in R^{s_p \times s_p}$ ($i = 1, \dots, N$) are the block matrices on the diagonal of Θ_{22} .

Proof: This follows from Theorem 1, and the method in (Iwasaki *et al.*, 2001). Details are omitted. \square

Theorem 2 presents a sufficient condition for affinely quadratic stability (AQS) of affine parameter-dependent systems, which is from the vertex separator approach or the parameter-independent multiplier approach (Iwasaki *et al.*, 2001; Dettori and Scherer, 1998). The robust stability condition is convex, and can be checked by solving a set of LMIs. For the topic of AQS, there have been some applicable analysis methods (Dettori and Scherer, 2000; Feron *et al.*, 1996; Gahinet *et al.*, 1996; Iwasaki, 1998; Peaucelle and Arzelier, 2001). However, the relation among these tests appears to be unclear. The following theorem will show that the robust stability criterion given in Theorem 2 is no more conservative than the existing ones.

Theorem 3: Consider the uncertain system (1), and $P(\delta)$ given by (4). Assume that A_0 is Hurwitz. Then the condition in Theorem 2 is feasible, i.e., there exist symmetric matrices P_0, P_i ($i = 1, \dots, N$), and a symmetric matrix Θ given by (17) such that (18), (12) and (19) hold, if one of the following robust stability conditions holds:

- (i) there exist symmetric matrices P_0, P_1, \dots, P_N and V_0 with $V_0 \leq 0$, and matrices S_1 and S_2 such that

$$\begin{aligned} M_1(\Delta) := & \begin{bmatrix} S_1^T A(\delta) + A(\delta)^T S_1 - A(\delta)^T V_0 A(\delta) \\ P(\delta) - S_1^T + A(\delta)^T S_2^T \\ P(\delta) - S_1^T + A(\delta)^T S_2^T \\ -S_2 - S_2^T + V_0 \end{bmatrix} < 0, \text{ for } \delta \in \delta_{\mathbf{v}} \end{aligned} \quad (20)$$

- (ii) there exist symmetric matrices P_0, P_1, \dots, P_N and $V = [V_{ij}]_{N \times N} \in R^{Nn \times Nn}$ with $V_{ii} \in R^{n \times n}$ and

$$V_{ii} \leq 0, \quad i = 1, \dots, N \quad (21)$$

and matrices W and Z such that

$$M_2(\Delta) := \begin{bmatrix} P(\delta)A_0 + A_0^T P(\delta) & P(\delta)A^1 \\ (A^1)^T P(\delta) & 0 \end{bmatrix} \\ + \begin{bmatrix} E_1 & 0 \\ 0 & I_{Nn} \end{bmatrix}^T \Theta(\Delta) \begin{bmatrix} E_1 & 0 \\ 0 & I_{Nn} \end{bmatrix} \\ < 0, \text{ for } \delta \in \delta_v \quad (22)$$

where

$$\Theta(\Delta) = \begin{bmatrix} \Theta_{11} & W^T + \Delta^T Z^T \\ W + Z\Delta & V - Z - Z^T \end{bmatrix} \quad (23)$$

with $\Theta_{11} = -(\Delta^T V \Delta + \Delta^T W + W^T \Delta)$.

(iii) there exist symmetric matrices P_0, P_1, \dots, P_N and $M_i, i = 1, \dots, N$ such that

$$P_i A_i + A_i^T P_i + M_i \geq 0 \text{ for } i = 1, \dots, N \quad (24)$$

$$M_i \geq 0 \text{ for } i = 1, \dots, N \quad (25)$$

and

$$M_3(\Delta) := P(\Delta)A(\Delta) + A(\Delta)^T P(\Delta) \\ + \sum_{i=1}^N \delta_i^2 M_i < 0 \text{ for all } \delta \in \delta_v \quad (26)$$

(iv) there exist symmetric matrices P_0, P_1, \dots, P_N and $S = \text{diag}[S_1 \dots S_N]$ with $S_i \in R^{n \times n}$, and a skew-symmetric matrix $T_s = \text{diag}[T_{s1} \dots T_{sN}]$ with $T_{si} \in R^{n \times n}$ such that $S > 0$ and

$$\begin{bmatrix} P_0 \\ P^1 \end{bmatrix} [A_0 \ A^1] + \left(\begin{bmatrix} P_0 \\ P^1 \end{bmatrix} [A_0 \ A^1] \right)^T \\ + \begin{bmatrix} E_1 & 0 \\ 0 & I_{Nn} \end{bmatrix}^T \begin{bmatrix} \gamma^2 S & -T_s \\ T_s & -S \end{bmatrix} \begin{bmatrix} E_1 & 0 \\ 0 & I_{Nn} \end{bmatrix} < 0 \quad (27)$$

The following lemmas will be used in the proof of Theorem 3.

Lemma 2: Let $W \in R^{(N+1)n \times (N+1)n}$ be a symmetric matrix. If

$$\begin{bmatrix} I \\ F_1 \\ \vdots \\ F_N \end{bmatrix}^T W \begin{bmatrix} I \\ F_1 \\ \vdots \\ F_N \end{bmatrix} \geq 0 \quad (28)$$

holds for any $F_i \in R^{n \times n}$ ($i = 1, \dots, N$), then $W \geq 0$.

Proof: If $W \geq 0$ does not hold, then there exists a

vector $\begin{bmatrix} x \\ y_1 \\ \vdots \\ y_N \end{bmatrix}$ with $x \neq 0$ such that

$$\begin{bmatrix} x \\ y_1 \\ \vdots \\ y_N \end{bmatrix}^T W \begin{bmatrix} x \\ y_1 \\ \vdots \\ y_N \end{bmatrix} < 0$$

Since $x \neq 0$, we have matrices F_1, \dots, F_N such that $y_i = F_i x, i = 1, \dots, N$. Thus,

$$x^T \begin{bmatrix} I \\ F_1 \\ \vdots \\ F_N \end{bmatrix}^T W \begin{bmatrix} I \\ F_1 \\ \vdots \\ F_N \end{bmatrix} x < 0$$

which contradicts with (28). \square

Denote

$$F = \text{diag}[F_1 \dots F_N] \quad (29)$$

$$P(F) = P_0 + \sum_{i=1}^N F_i P_i \quad (30)$$

$$A(F) = A_0 + \sum_{i=1}^N A_i F_i \quad (31)$$

$$M_{1R}(F) = S_1^T A(F) + A(F)^T S_1 - A(F)^T V_0 A(F) \\ + (P(\delta) - S_1^T + A(\delta)^T S_2^T) \\ \times (S_2 + S_2^T - V_0)^{-1} \\ \times (P(\delta) - S_1^T + A(\delta)^T S_2^T)^T \quad (32)$$

$$M_{1RE} = [M_{1REij}]_{(N+1) \times (N+1)} \\ := \begin{bmatrix} S_1^T [A_0 \ A^1] \\ 0 \end{bmatrix} + \begin{bmatrix} S_1^T [A_0 \ A^1] \\ 0 \end{bmatrix}^T \\ - [A_0 \ A^1]^T V_0 [A_0 \ A^1] + \left(\begin{bmatrix} P_0 \\ P^1 \end{bmatrix} - \begin{bmatrix} S_1^T \\ 0 \end{bmatrix} \right) \\ + [A_0 \ A^1]^T S_2^T (S_2 + S_2^T - V_0)^{-1} \\ \times \left(\begin{bmatrix} P_0 \\ P^1 \end{bmatrix} - \begin{bmatrix} S_1^T \\ 0 \end{bmatrix} + [A_0 \ A^1]^T S_2^T \right) \quad (33)$$

with $M_{1REii} \in R^{n \times n}$ ($i = 0, 1, \dots, N$).

$$M_{2R}(F) = P(F)A_0 + A_0^T P(F)^T \\ - (E_1^T F^T V F E_1 + E_1^T F^T W + W^T F E_1) \\ + (P(F)A^1 + W^T + E_1^T F^T Z^T)(Z + Z^T - V)^{-1} \\ \times (P(F)A^1 + W^T + E_1^T F^T Z^T)^T \quad (34)$$

$$M_{2RE} = [M_{2REij}]_{(N+1) \times (N+1)} \\ := \begin{bmatrix} \begin{bmatrix} P_0 \\ P^1 \end{bmatrix} A_0 & 0 & \dots & 0 \end{bmatrix} \\ + \begin{bmatrix} \begin{bmatrix} P_0 \\ P^1 \end{bmatrix} A_0 & 0 & \dots & 0 \end{bmatrix}^T \\ - \text{diag}[0 \ V] - \begin{bmatrix} 0 & W \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & W \\ 0 & 0 \end{bmatrix}^T \\ + \left(\begin{bmatrix} P_0 \\ P^1 \end{bmatrix} A^1 + \begin{bmatrix} W^T \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ Z^T \end{bmatrix} \right) \\ \times (Z + Z^T - V)^{-1} \\ \times \left(\begin{bmatrix} P_0 \\ P^1 \end{bmatrix} A^1 + \begin{bmatrix} W^T \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ Z^T \end{bmatrix} \right)^T \quad (35)$$

with $M_{2REii} \in R^{n \times n}$ ($i = 0, 1, \dots, N$).

$$M_{3RE} = [M_{2REij}]_{(N+1) \times (N+1)} \\ := \begin{bmatrix} P_0 \\ P^1 \end{bmatrix} [A_0 \ A^1] + \left(\begin{bmatrix} P_0 \\ P^1 \end{bmatrix} [A_0 \ A^1] \right)^T \\ + \text{diag}[0 \ M_{11} \ \dots \ M_{NN}] \quad (36)$$

Lemma 3: (i) If $S_2 + S_2^T - V_0 > 0$, then

$$\begin{bmatrix} P_0 \\ P^1 \end{bmatrix} [A_0 \ A^1] + \left(\begin{bmatrix} P_0 \\ P^1 \end{bmatrix} [A_0 \ A^1] \right)^T \leq M_{1RE} \quad (37)$$

(ii) If $Z + Z^T - V > 0$, then

$$\begin{bmatrix} P_0 \\ P^1 \end{bmatrix} [A_0 \ A^1] + \left(\begin{bmatrix} P_0 \\ P^1 \end{bmatrix} [A_0 \ A^1] \right)^T \leq M_{2RE} \quad (38)$$

(iii) If $M_{ii} \geq 0$ ($i = 1, \dots, N$), then

$$\begin{bmatrix} P_0 \\ P^1 \end{bmatrix} [A_0 \ A^1] + \left(\begin{bmatrix} P_0 \\ P^1 \end{bmatrix} [A_0 \ A^1] \right)^T \leq M_{3RE} \quad (39)$$

Proof: It is omitted here, due to the space limitation. \square

Proof of Theorem 3: (i) By Schur complement and (32), (20) is equivalent to that $S_2 + S_2^T - V_0 > 0$ and

$$M_{1R}(\Delta) < 0 \quad (40)$$

By Lemma 1 and (??), it follows that there exists a symmetric matrix $\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12}^T & \Theta_{22} \end{bmatrix}$ such that

$$M_{1RE} + \begin{bmatrix} E_1 & 0 \\ 0 & I_{Nn} \end{bmatrix}^T \Theta \begin{bmatrix} E_1 & 0 \\ 0 & I_{Nn} \end{bmatrix} < 0 \quad (41)$$

and (13) hold. From (33), $V_0 \leq 0$ and $S_2 + S_2^T - V_0 > 0$, it follows that

$$M_{1Rii} \geq 0, \quad i = 1, \dots, N$$

which further implies that $\Theta_{22ii} \leq 0$, $i = 1, \dots, N$. Thus, from Lemma 2, (19) holds and (12) follows from (37).

(ii) By Schur complement and (34), it follows that (22) is equivalent to

$$M_{2R}(\Delta) < 0 \quad (42)$$

The rest of the proof can be completed by (38), (21), (42) and the arguments similar to those for (i), and the details are omitted.

(iii) By (39) and $M_{ii} \geq 0$ ($i = 1, \dots, N$), it is immediate.

(iv) Notice that $\Theta = \begin{bmatrix} \gamma^2 S & -T_s \\ T_s & -S \end{bmatrix}$ satisfies (18) and (19), so the conclusion follows. \square

Remark 2: Theorem 3 presents a comparison between Theorem 2 and the existing results for

AQS, where $P(\delta)$ is required to be in the form of (4). Condition (i) is an extension of the results for AQS in (Dettori and Scherer, 2000) and (Peaucelle and Arzelier, 2001), where (Peaucelle and Arzelier, 2001) applies the Projection Lemma (Gahinet and Apkarian, 1994) to give the following condition for AQS: A_0 is Hurwitz, and there exist symmetric matrices P_0, P_1, \dots, P_N , and matrices S_1 and S_2 such that

$$M_1(\Delta) := \begin{bmatrix} S_1^T A(\delta) + A(\delta)^T S_1 \\ (P(\delta) - S_1^T + A(\delta)^T S_2^T)^T \\ P(\delta) - S_1^T + A(\delta)^T S_2^T \\ -S_2 - S_2^T \end{bmatrix} \\ < 0, \text{ for } \delta \in \delta_v \quad (43)$$

which is same as the result in (Dettori and Scherer, 2000), and corresponds to a special case of condition (i) when $V_0 = 0$. Condition (ii) is given in (Iwasaki, 1998). The stability criteria in (Dettori and Scherer, 2000) and (Iwasaki, 1998) are obtained by using parameter-dependent multiplier or separator approach. Condition (iii) is from (Gahinet *et al.*, 1996), which is based on the result on multiconvexity (Iwasaki *et al.*, 2001). Condition (iv) is given in (Feron *et al.*, 1996), which is derived by using S-procedure approach. Theorem 3 shows that the stability criterion given by using vertex separator approach or parameter-independent multiplier approach (Iwasaki *et al.*, 2001; Dettori and Scherer, 1998) provides a stability test that is no more conservative than the above existing results for AQS.

Example 1: The considered uncertain linear time-invariant system is described by (1)-(3) with

$$A_0 = \begin{bmatrix} -5 & 2 & 1 \\ -10 & -5 & -1 \\ 5 & 1 & -6 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

We are interested in computing the maximal value of γ such that the system is AQS. The optimal γ for each method is given in Table 1. It is as expected that Theorem 2 gives the best stability margin $\gamma = 2.957$ for the system to be AQS. However, it requires more computational time than most of the existing tests.

Regarding computational complexity, the number of variables and the number of LMIs are used as measures of computational burden. It is easy to see that the methods given in (Iwasaki, 1998) and Theorem 2 require more variables than others, and the methods in (Dettori and Scherer, 2000), (Peaucelle and Arzelier, 2001), (Iwasaki, 1998), (Gahinet *et al.*, 1996), Condition (i) in Theorem 3 and Theorem 2 involve conditions on the vertices

of the parameter set and require exponentially growing numbers of LMI constraints with respect to N . So the methods given in (Iwasaki, 1998) and Theorem 2 will be computationally very demanding, while the method in (Feron *et al.*, 1996) is of less computational burden than others when n and N are large. In Table 1, DS and PA represent the methods in (Dettori and Scherer, 2000) and (Peaucelle and Arzelier, 2001), respectively.

Table 1. Optimal γ for each method

Method	γ
Theorem 2	2.957
Condition (i) in Theorem 3	2.943
DS, PA	2.943
(Iwasaki, 1998)	2.957
(Gahinet <i>et al.</i> , 1996)	2.936
(Feron <i>et al.</i> , 1996)	1.38

Remark 3: It should be pointed out that the tests for AQS given by Condition (i) in Theorem 3, (Dettori and Scherer, 2000), (Peaucelle and Arzelier, 2001) and (Iwasaki, 1998)(see (43) and (ii) in Theorem 3) are also applicable for the case where polytopic Lyapunov functions are used to prove robust stability, which is a more general type of Lyapunov functions than that affine parameter-dependent Lyapunov functions considered here.

Remarks 4: In (Tuan and Apkarian, 2002), based on monotonicity concept, a sufficient condition for AQS is given in terms of LMIs. For the uncertain system in Example 1, the optimal γ for the method in (Tuan and Apkarian, 2002) is 2.927, which shows that the method is more conservative than Theorem 2 for this numerical example. However, it is difficult to prove the relationship between Theorem 2 and the method in (Tuan and Apkarian, 2002) for the general case.

4. CONCLUSIONS

In this paper, we have investigated the problem of affinely quadratic stability (AQS) of a class of uncertain linear systems, where the system state-matrices considered are affinely dependent on the uncertain parameters, and affine parameter-dependent Lyapunov functions are exploited to prove stability. A robust stability criterion for the class of systems to be affinely quadratically stable (AQS) is given in terms of linear matrix inequalities (LMIs), which is based on the vertex separator approach or parameter-independent multiplier approach(Iwasaki *et al.*, 2001; Dettori and Scherer, 1998). The comparison with the existing tests for AQS is given, and it is shown that the robust stability criterion is no more conservative than the existing tests. A further research topic is to extend the comparison to cases where more general types of parameter-dependent Lyapunov

functions such as polytopic Lyapunov functions, and Lyapunov functions that depend on the parameters in a linear fractional manner (Iwasaki *et al.*, 2001). Obviously, such an extension would be very significant, but nontrivial.

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