A PERIODIC SCHEME FOR PIPELINING UNSTABLE DIGITAL CONTROLLERS [1]

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Abstract: Pipelining is a means of obtaining fast sampling and processing speed in hardware implementation of digital systems and it requires a look ahead system model for the pipelined system. However, existing methods for the design of look ahead system models are only for stable systems and hence are not applicable to unstable digital controllers for closed loop system control. In this paper a periodic method is presented for the design of look ahead system models for unstable digital controllers. Analysis is carried out to show that the pipelined implementation of the unstable digital controller using the proposed periodic look ahead system model can maintain the closed loop control system stability.

Keywords: Closed loop system; Digital controller; Pipelining; Stability.

1. INTRODUCTION

Most feedback controllers in modern industrial control systems are implemented with digital integrated circuits. And it has been an ever increasing demand from manufacturers and users that the equipment and devices of digital controllers have faster and more efficient processing ability, less energy consumption and more mobility in terms of size and weight. Hence, an objective of control system engineers is to design faster and more efficient algorithms for integrated circuit fabrication and implementation of digital controllers.

Pipelining is one of the approaches to speeding up processing and operations in digital processors. Methods and algorithms for pipelining IIR (infinite impulse response) dynamical systems have been given in the past years in the signal processing and circuits and systems communities, e.g. (Lim and Liu, 1992; Lapointe *et al.*, 1993; Katsushige *et al.*, 1999; Parhi, 1999; Living *et al.*, 2001; Meyer-Baese, 2001). It has been shown that a special model called *look ahead system* (Lim and Liu, 1992; Katsushige *et al.*, 1999; Parhi, 1999; Living *et al.*, 2001; Meyer-Baese, 2001) is to be used for pipelining of digital IIR systems. Further, two essential problems in the design of look ahead filters are stability and computational complexity (Lim and Liu, 1992; Parhi, 1999).

The look ahead system model of the pipelined digital system is to be quantized for hardware implementation which may introduce instability to the system. It turned out that stabilizing look ahead systems with minimum possible computational complexity is a very hard problem and so far there has been no analytical LTI (linear time invariant) solution to such a problem. Recently, a periodic method is presented to deal with the stabilization and computational complexity problems in the design of look ahead systems (Zhang and Xie, 2001). It is noted that all the LTI and pe-

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riodic methods for the design of look ahead system models are for originally stable digital systems which are to be pipelined.

Although most digital filters and processors applied in signal processing, telecommunications and circuits and systems areas are stable systems, unstable digital controllers can be often used in many industrial applications which include popular PID controllers. When the unstable controller is pipelined for hardware implementation, it is essential that the look ahead system model for the controller can maintain the closed loop system stability. While there have been no LTI methods to deal with the design of look ahead system models for unstable digital systems, this paper extends the periodic method in (Zhang and Xie, 2001) to the design of look ahead system models for unstable controllers. It will be shown that the pipelined unstable digital controller designed using the proposed periodic method can maintain the closed loop system stability.

2. PIPELINING DIGITAL CONTROLLERS

2.1 Pipelining process

Let z^{-1} denote the back shift operator such that $z^{-1}u_k = u_{k-1}$. In a digital closed loop control system, the digital controller produces a sequence of discrete control signals for controlling the plant using the sampled measurement of the plant output. Let u_k and y_k denote the control input and sampled output of the controlled plant, respectively. A linear time invariant (LTI) nth order single-input single-output digital controller is a dynamical system with y_k as input and u_k as output and can be written in the following polynomial equation form

$$A(z^{-1})u_k = B(z^{-1})y_k, \qquad (1)$$

where $A(z^{-1})$ and $B(z^{-1})$ are polynomials in z^{-1} written as

$$\begin{aligned} A(z^{-1}) &= 1 + a_1 z^{-1} + \cdots + a_n z^{-n}, \\ B(z^{-1}) &= b_1 z^{-1} + b_2 z^{-2} + \cdots + b_n z^{-n}, \end{aligned} \qquad (2)$$

with $a_i, b_j \in \mathcal{R}$, $1 \leq i, j \leq n$, $a_n \neq 0$. If $a_i = 0, \forall 1 \leq i \leq n$, the controller is an FIR (finite impulse response) system. Otherwise, it is an IIR (infinite impulse response) system. The LTI system (1) is stable if all zeros of the polynomial equation $A(z^{-1}) = 0$ in terms of z^{-1} are strictly outside the unit circle in the complex z^{-1}-plane. In this case we call $A(z^{-1})$ a stable polynomial. For simplicity and without loss of generality, we omitted the external reference input to the above described closed loop system.

In practical control systems, the digital controller is often implemented with digital integrated circuit. For given hardware resource, the control updating rate of the digital controller can be significantly improved by pipelining, leading to feedback control of the plant with fast sampling and control rate. To show the elementary mechanism and process in pipelining multipliers, we first consider that the digital controller (1) is a very simple first order FIR digital system

$$u_k = b_1 z^{-1} y_k, \qquad (3)$$

with a single coefficient b_1. Let $M(b_1)$ denote the operation of multiplication by b_1. A conventional block diagram of the first order system is shown in Figure 1 (I). Under this scheme, each multiplication operation is performed in one sampling period to obtain the system output. Assume that the available hardware can perform each elementary multiplication operation in T_M sec.. The fastest achievable sampling rate of the system is $\frac{1}{T_M}$Hz.

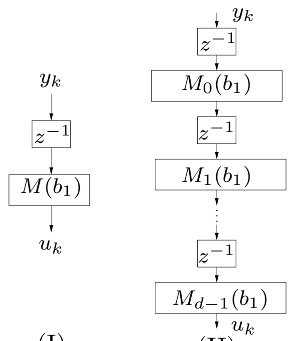

Fig. 1. (I) Conventional block diagram; (II) d-step pipelined block diagram of $u_k = b_1 z^{-1} y_k$.

According to the integrated circuit implementation scheme, the multiplication operation can be divided into a number of stages. This forms the elementary mechanism for pipelining multipliers and then the operation $M(b_1)$ can be decomposed into d-stages and expressed as

$$M(b_1) = M_{d-1}(b_1) \circ \cdots \circ M_1(b_1) \circ M_0(b_1). \qquad (4)$$

Registers are then inserted between each two connected stages. This yields the d-stage pipelined digital system, whose block diagram is shown in Figure 1 (II). Assume that the time required for carrying out each divided multiplying operation is equally allocated, i.e. each divided multiplying operation takes $\frac{T_M}{d}$ sec. to complete. With the d divided multiplying operations performing simultaneously, the fastest sampling rate of the pipelined system is $\frac{d}{T_M}$ Hz. Thus the pipelining can increase the sampling rate of the system by d times. With respect to the fast pipelined sampling rate, the input-output relationship of the pipelined system is $u_k = b_1 z^{-d} y_k$, where the d-step delay is equivalent to T_M sec.. It shows that the fast rate pipelined input-output relationship is identical to that of (3).

In fact, it is easy to verify that each of the multipliers can be pipelined following the same procedure if (1) is an nth order FIR system and the input-output relationship of the original system is preserved.

For pipelining IIR systems, we first consider that (1) is a simple first order IIR system written as

$$(1 + a_1 z^{-1})u_k = b_0 y_k. \tag{5}$$

For simplicity, let us consider the 2-stage pipelining for the first order IIR system (5), i.e. $d = 2$. If the multiplication operations $M(a_1)$ and $M(b_0)$ are replaced by their 2-stage divided multiplication operations with a delay latch inserted between the divided operations as shown in Figure 2, then the input-output relationship of this scheme is $(1 + a_1 z^{-2})u_k = b_0 z^{-1} y_k$, whose input-output relation is no longer equivalent to that of the original system (5). This shows that pipelining multiplication operations in the system feedback path cannot be implemented by simply inserting the stage divided multiplication operations into the path as that for FIR systems.

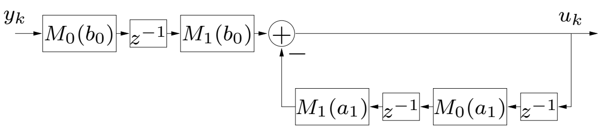

Fig. 2. Block diagram of the changed IIR system (5) after delay latches insertion.

To solve this 2-stage pipelining problem of the first order IIR system (5), we multiply both sides of the IIR system (5) by a polynomial factor $1 - a_1 z^{-1}$, yielding

$$(1 - \alpha_2 z^{-2})u_k = (\beta_0 + \beta_1 z^{-1})y_k, \tag{6}$$

where $\alpha_2 = a_1^2$, $\beta_0 = b_0$ and $\beta_1 = -a_1 b_0$. This is an equivalent expression of the IIR system (5) and it can be pipelined as shown by the block diagram in Figure 3, where the fastest achievable sampling rate of the 2-stage pipelining process is $\frac{2}{T_M}$.

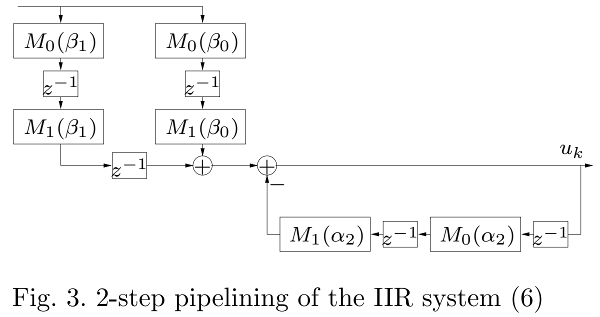

Fig. 3. 2-step pipelining of the IIR system (6)

3. LOOK AHEAD SYSTEMS FOR PIPELINING

3.1 LTI look ahead systems

As shown in the last section, the model (6) is essential for pipeline implementation of the IIR system (5), which is called *look ahead system* of (5). This is in fact also true for the general case of d-stage pipelining of the nth order IIR system in the form (1). In such a general case, the nth order IIR system is to be modelled into the look ahead system form whose output u_{k+d} is independent of its last $d-1$ outputs $\{u_{k+d-1}, u_{k+d-2}, \cdots, u_{k+1}\}$. To be specific, an LTI d-step look ahead system is written as

$$\alpha(z^{-1})\hat{u}_k = \beta(z^{-1})y_k, \tag{7}$$

where $y_k, \hat{u}_k \in \mathcal{R}$ are, respectively, the system input and output,

$$\alpha(z^{-1}) = 1 - \alpha_d z^{-d} - \alpha_{d+1} z^{-d-1} - \cdots - \alpha_{\hat{n}} z^{-\hat{n}},$$
$$\beta(z^{-1}) = \beta_1 z^{-1} + \beta_2 z^{-2} + \cdots + \beta_{\hat{n}} z^{-\hat{n}},$$

with $\alpha_i, \beta_j \in \mathcal{R}$, $1 \le i, j \le \hat{n}$ and $\hat{n}$ is the order of the look ahead system. It is noted that the coefficients of $\alpha(z^{-1})$ for the terms z^{-l}, $1 \le l \le d-1$, are all zero, so $\hat{u}_{k+d}$ is independent of its last $d-1$ outputs.

Given a stable LTI system, an analytical solution for a stable d-step LTI look ahead system whose order is d times as high as that of the original system which introduces considerably more computational load, can be easily found (Zhang and Xie, 2001). In the past years, several LTI methods were studied for the design of stable look head systems with minimum possible order for pipelining of stable IIR systems in circuits and signal processing areas (Lim and Liu, 1992; Parhi, 1999; Living *et al.*, 2001).

Moreover, if the IIR digital system is unstable there are no analytical and numerical approaches to the design of its look ahead systems for the pipelining implementation. While stable digital systems exist in many applications such as signal processing and telecommunications, many digital feedback controllers are unstable systems, which include popular PID controllers with the integral control action. In the following, we present a periodic approach to the design of stable look ahead systems for unstable controllers.

3.2 Periodic look ahead systems

An nth order linear periodic system with period N can be written in the following form

$$P(k, z^{-1})u_k = Q(k, z^{-1})y_k, \tag{8}$$

where $P(k,z^{-1})$ and $Q(k,z^{-1})$ are periodically time-varying polynomials in z^{-1} of the form

$$P(k,z^{-1}) = 1 + p_{1,k}z^{-1} + p_{2,k}z^{-2} + \cdots + p_{n,k}z^{-n}, \quad (9)$$

$$Q(k,z^{-1}) = q_{1,k}z^{-1} + q_{2,k}z^{-2} + \cdots + q_{n,k}z^{-n},$$

$p_{i,k}, q_{j,k} \in \mathcal{R}$, $1 \leq i,j \leq n$, are N-periodic coefficients satisfying $p_{i,k} = p_{i,k+N}$, $q_{j,k} = q_{j,k+N}$, and there exists some k such that $p_{n,k} \neq 0$.

The N-periodic system (8) has a state space realization which can be written as

$$x_{k+1} = A_k x_k + B_k y_k, \quad u_k = C_k x_k,$$

where $x_k \in \mathcal{R}^n$ is the state vector, $A_k \in \mathcal{R}^{n\times n}$, $B_k \in \mathcal{R}^{n\times 1}$ and $C_k \in \mathcal{R}^{1\times n}$ are N-periodic matrices satisfying $A_k = A_{k+N}$, $B_k = B_{k+N}$ and $C_k = C_{k+N}$. Let

$$\bar{A} = A_{N-1}A_{N-2}\cdots A_1 A_0 \in \mathcal{R}^{n\times n}, \quad (10)$$

and define

Definition 1. The N-periodic system (8) is stable if and only if all the eigenvalues of the matrix $\bar{A}$, denoted by $\lambda_i(\bar{A})$, $i = 1,2,\cdots,n$, satisfy $|\lambda_i(\bar{A})| < 1$. □

It can be shown that the stability of the periodic system (8) is characterized only by the polynomial $P(k,z^{-1})$. Thus we call $P(k,z^{-1})$ a *stable polynomial* if (8) is a stable system.

Let $\hat{F}(k,z^{-1})$ be an N-periodic polynomial of the form

$$\hat{F}(k,z^{-1}) = F(z^{-1}) + H(k,z^{-1})z^{-d}, \quad (11)$$

where $F(z^{-1})$ of degree *deg* $F = \hat{d} - 1 \leq d - 1$ is the unique solution of the Diophantine equation

$$F(z^{-1})A(z^{-1}) + z^{-d}G(z^{-1}) = 1 \quad (12)$$

and $G(z^{-1})$ is of degree *deg* $G = n + \hat{d} - d \leq n$, and $H(k,z^{-1})$ is a N-periodic polynomial.

Proposition 1. The periodic system

$$\hat{F}(k,z^{-1})A(z^{-1})\hat{u}_k = \hat{F}(k,z^{-1})B(z^{-1})y_k, \quad (13)$$

where $\hat{F}(k,z^{-1})$ of the form (11), is a d-step look ahead system of the LTI system (1). It is a stable d-step look ahead system of the LTI system (1) if and only if $\hat{F}(k,z^{-1})$ is stable. □

Now, we propose the periodic polynomial $H(k,z^{-1})$ of $\hat{F}(k,z^{-1})$ in (11) in the following form

$$H(k,z^{-1}) = h_{i_k}z^{-i_k}, \quad i_k = k \text{ mod } d, \quad (14)$$

where i_k satisfies $i_k = k - Md$, $Md \leq k < (M+1)d$, $\forall M \in \mathcal{Z}^+$. It is simple to verify that the index number i_k is d-periodic and takes integer values between zero and $d-1$. Thus $H(k,z^{-1})$ is a d-periodic polynomial of degree $d-1$ satisfying $H(k,z^{-1}) = H(k+d,z^{-1})$. It follows that $\hat{F}(k,z^{-1})$ is a d-periodic polynomial of degree $2d-1$ and (13) is an $(n+2d-1)$th order d-periodic d-step look ahead system of the nth order IIR system (1).

By Proposition 1, it is sufficient to design the d-periodic polynomial $H(k,z^{-1})$ of (14) to obtain a stable $\hat{F}(k,z^{-1})$ and, hence, a stable periodic d-step look ahead system (13). For this purpose, introduce the following matrices

$$\Phi = \begin{bmatrix} 0 & & & & \\ \vdots & & I & & \\ 0 & & & & \\ 0 \cdots 0 & -f_{\hat{d}-1} & \cdots & -f_2 & -f_1 \end{bmatrix} \in \mathcal{R}^{d\times d}, \quad (15)$$

$$\begin{aligned} \bar{\Phi} &= \Phi^d \in \mathcal{R}^{d\times d}, \\ \Gamma &= [0 \ \cdots \ 0 \ 1]^T \in \mathcal{R}^{d\times 1}, \\ C &= [1 \ 0 \ \cdots \ 0] \in \mathcal{R}^{1\times d}, \\ \bar{G} &= \left[\, \Phi^{d-1}\Gamma \ \cdots \ \Phi\Gamma \ \Gamma \,\right] \in \mathcal{R}^{d\times d}, \\ \bar{H} &= \left[\, h_0 \ h_1 \ \cdots \ h_{d-1} \,\right]^T \in \mathcal{R}^{d\times 1}. \end{aligned} \quad (16)$$

It is noted that $\bar{G}$ is a full rank matrix. The problem of designing the stable periodic look ahead system now is to find $\bar{H}$, which contains coefficients of $H(k,z^{-1})$ such that the polynomial $F(k,z^{-1})$ is stable.

Associated with the periodic polynomial $\hat{F}(k,z^{-1})$, there exist a periodic homogeneous system

$$\begin{aligned} \hat{F}(k,z^{-1})w_k &= w_k + f_1 w_{k-1} + \cdots + f_{\hat{d}-1}w_{k-\hat{d}+1} \\ &\quad + h_{i_k}w_{k-d-i_k}, \end{aligned} \quad (17)$$

and an LTI homogeneous state equation

$$\bar{w}_{k+d} = (\bar{\Phi} - \bar{G}\bar{H}C)\bar{w}_k, \quad (18)$$

where $w_k \in \mathcal{R}$ is the output of the homogenous system (17) and $\bar{w}_k = \left[\, w_{k-d} \ \cdots \ w_{k-2} \ w_{k-1} \,\right]^T \in \mathcal{R}^d$ is the state of the homogenous state equation (18).

Proposition 2.

(i) (13) is a stable d-step look ahead system if and only if the LTI homogeneous state equation (18) is stable, i.e. all the eigenvalues of the matrix $(\bar{\Phi} - \bar{G}\bar{H}C)$ are within the unit circle of the complex plane;

(ii) The pair $(C,\bar{\Phi})$ is observable for almost all $[f_1 \ \ f_2 \ \ \cdots \ \ f_{\hat{d}-1}]^T \in \mathcal{R}^{\hat{d}-1}$ which determines the matrix $\bar{\Phi} = \Phi^d$; i.e. the subset of $\mathcal{R}^{\hat{d}-1}$ containing $[f_1 \ \ f_2 \ \ \cdots \ \ f_{\hat{d}-1}]^T$ such that the pair $(C,\bar{\Phi})$ is unobservable is a measure zero subset. □

We now consider to assign the eigenvalues of the matrix $(\bar{\Phi} - \bar{G}\bar{H}C)$ by finding $\bar{H}$ such that the periodic look ahead system (13) is stable. Let λ_i, $i = 1, 2, \cdots, d$ be the d desirable eigenvalues to be assigned and

$$\rho(\lambda) = \lambda^d + \rho_{d-1}\lambda^{d-1} + \cdots + \rho_1\lambda + \rho_0$$

be the characteristic polynomial of degree d in λ, with coefficients $\rho_i \in \mathcal{R}$, $i = 0, 1, \cdots, d-1$ and roots λ_i, $i = 1, 2, \cdots, d$. Using Proposition 2 and Ackermann's formula (Ackermann, 1972) for eigenvalue assignment, we present the design of the stable periodic look ahead system (13) in the following theorem.

Theorem 1. If the pair $(C, \bar{\Phi})$ is observable and the eigenvalues of the matrix $(\bar{\Phi} - \bar{G}\bar{H}C)$ are to be assigned to the roots of the polynomial $\rho(\lambda)$ such that the state equation (18) is stable, then the coefficients of the polynomial $\hat{H}(k, z^{-1})$ in terms of $\bar{H}$ for achieving the specified eigenvalue assignment are given by

$$\bar{H} = \bar{G}^{-1}\rho(\bar{\Phi}) \begin{bmatrix} C \\ C\bar{\Phi} \\ \vdots \\ C\bar{\Phi}^{d-1} \end{bmatrix} \Gamma. \tag{19}$$

4. CLOSED LOOP SYSTEM STABILITY WITH PIPELINED PERIODIC CONTROLLER

Consider that

$$A_p(z^{-1})y_k = B_p(z^{-1})u_k \tag{20}$$

is an n_pth order discrete time plant with input u_k, output y_k and polynomials $A_p(z^{-1})$ and $B_p(z^{-1})$ of degree n_p. Suppose that (1) is a digital feedback controller applied to the plant (20) such that the closed loop system is stable. It is assumed that the closed system has some degree of stability robustness such that the controller coefficients can be perturbed within some range without destabilizing the closed loop system.

Consider that fast sampling control of the system is to be applied and it is required that the controller (1) is pipelined and a stable periodic d-step ahead system in the form (13) is designed in terms of periodic polynomials

$$\begin{aligned} \hat{\alpha}(k, z^{-1}) &= \hat{F}(k, z^{-1})A(z^{-1}), \\ \hat{\beta}(k, z^{-1}) &= \hat{F}(k, z^{-1})B(z^{-1}). \end{aligned} \tag{21}$$

In hardware implementation, the coefficients of $\hat{\alpha}(k, z^{-1})$ and $\hat{\beta}(k, z^{-1})$ are quantized to meet the finite word length constraint of the integrated circuit. Let the quantized periodic polynomials of $\hat{\alpha}(k, z^{-1})$ and $\hat{\beta}(k, z^{-1})$ be denoted by $\tilde{\alpha}(k, z^{-1})$ and $\tilde{\beta}(k, z^{-1})$, respectively. The actual hardware implementation of the controller is

$$\tilde{\alpha}(k, z^{-1})u_k = \tilde{\beta}(k, z^{-1})y_k, \tag{22}$$

which, together with the plant model (20), forms the closed loop control system. It is noted that the closed loop system (20) and (22) is periodic.

The quantized polynomials $\tilde{\alpha}(k, z^{-1})$ and $\tilde{\beta}(k, z^{-1})$ can be factorized as

$$\begin{aligned} \tilde{\alpha}(k, z^{-1}) &= \tilde{F}_a(k, z^{-1})\tilde{A}(k, z^{-1}), \\ \tilde{\beta}(k, z^{-1}) &= \tilde{F}_b(k, z^{-1})\tilde{B}(k, z^{-1}), \end{aligned} \tag{23}$$

where $\tilde{F}_a(k, z^{-1})$ and $\tilde{F}_b(k, z^{-1})$ are monic d-periodic polynomials of degree $2d-1$, $\tilde{A}(k, z^{-1})$ and $\tilde{B}(k, z^{-1})$ are d-periodic polynomials of degree n and $\tilde{A}(k, z^{-1})$ is monic. It is noted that the multiplication operation between periodic polynomials is, in general, not commutative, i.e.

$$\tilde{F}_a(k, z^{-1})\tilde{A}(k, z^{-1}) \neq \tilde{A}(k, z^{-1})\tilde{F}_a(k, z^{-1}).$$

In practice, it is reasonable to assume that the quantization possesses certain degree of accuracy so the coefficient quantization error is small. By continuity, the quantized polynomials $\tilde{\alpha}(k, z^{-1})$ and $\tilde{\beta}(k, z^{-1})$ approach $\hat{\alpha}(k, z^{-1})$ and $\hat{\beta}(k, z^{-1})$, respectively, if the coefficient quantization error is sufficiently small. Moreover, the factorization in (23) exists such that $\tilde{F}_a(k, z^{-1})$ and $\tilde{F}_b(k, z^{-1})$ approach $\hat{F}(k, z^{-1})$, $\tilde{A}(k, z^{-1})$ approaches $A(z^{-1})$ and $\tilde{B}(k, z^{-1})$ approaches $B(z^{-1})$, respectively, if the coefficient quantization error is sufficiently small.

Introduce the following error polynomials

$$\begin{aligned} \Delta\hat{F}_a(k, z^{-1}) &= \tilde{F}_a(k, z^{-1}) - \hat{F}_a(k, z^{-1}), \\ \Delta\hat{F}_b(k, z^{-1}) &= \tilde{F}_b(k, z^{-1}) - \hat{F}_b(k, z^{-1}), \\ \Delta A(k, z^{-1}) &= \tilde{A}(k, z^{-1}) - A(z^{-1}), \\ \Delta B(k, z^{-1}) &= \tilde{B}(k, z^{-1}) - B(z^{-1}). \end{aligned} \tag{24}$$

Using these notations, the quantized controller (22) can be written as

$$\begin{aligned} &(\hat{F}(k, z^{-1}) + \Delta\hat{F}_a(k, z^{-1}))(A(z^{-1}) + \Delta A(k, z^{-1}))u_k \\ &= (\hat{F}(k, z^{-1}) + \Delta\hat{F}_b(k, z^{-1}))(B(z^{-1}) + \Delta B(k, z^{-1}))y_k \\ &= (\hat{F}(k, z^{-1}) + \Delta\hat{F}_a(k, z^{-1}) + \Delta\hat{F}_b(k, z^{-1}) \\ &\quad -\Delta\hat{F}_a(k, z^{-1}))(B(z^{-1}) + \Delta B(k, z^{-1}))y_k. \end{aligned}$$

The above controller can be alternatively written as

$$\begin{aligned} &(A(z^{-1}) + \Delta A(k, z^{-1}))u_k \\ &= (\hat{F}(k, z^{-1}) + \Delta\hat{F}_a(k, z^{-1}))^{-1}(\hat{F}(k, z^{-1}) + \Delta\hat{F}_a(k, z^{-1}) \\ &\quad +\Delta\hat{F}_b(k, z^{-1}) - \Delta\hat{F}_a(k, z^{-1}))(B(z^{-1}) + \Delta B(k, z^{-1}))y_k \\ &= (B(z^{-1}) + \Delta B(k, z^{-1}))y_k + (\hat{F}(k, z^{-1}) + \Delta\hat{F}_a(k, z^{-1}))^{-1} \\ &\quad \cdot(\Delta\hat{F}_b(k, z^{-1}) - \Delta\hat{F}_a(k, z^{-1}))(B(z^{-1}) + \Delta B(k, z^{-1}))y_k, \end{aligned} \tag{25}$$

which, together with the plant (20), forms the closed loop system.

Theorem 2. Suppose that the closed loop system (20) and (25) possesses some degree of stability robustness against quantization errors in the controller coefficients and the stable polynomial $\hat{F}(k, z^{-1})$ in the form (11) of the d-step look ahead controller is also designed to possess some degree of stability robustness against quantization errors in its coefficients. Then the closed loop control system (20) and (25) is stable if the controller quantization error is small.

Proof: Consider the following model

$$(A(z^{-1}) + \Delta A(k, z^{-1}))u_k = (B(z^{-1}) + \Delta B(k, z^{-1}))y_k, \quad (26)$$

which is the stabilizing controller model (1) perturbed by quantization error polynomials $\Delta A(k, z^{-1})$ and $\Delta B(k, z^{-1})$. Under the condition that the closed loop system stability robustness can account for some controller quantization errors, the closed loop system resulting from applying the perturbed controller (26) to the plant (20) is stable if the quantization error in terms of the error polynomials $\Delta A(k, z^{-1})$ and $\Delta B(k, z^{-1})$ is small.

Let

$$\begin{aligned} \tilde{u}_k &= (A(z^{-1}) + \Delta A(k, z^{-1}))u_k, \\ \tilde{y}_k &= (B(z^{-1}) + \Delta B(k, z^{-1}))y_k. \end{aligned} \quad (27)$$

The stable closed loop system model (20) and (26) can be alternatively represented by

$$\begin{aligned} \tilde{y}_k &= (B(z^{-1}) + \Delta B(k, z^{-1}))B_p(z^{-1})A_p^{-1}(z^{-1}) \\ &\quad \cdot (A(z^{-1}) + \Delta A(k, z^{-1}))^{-1}\tilde{u}_k, \\ \tilde{u}_k &= \tilde{y}_k. \end{aligned} \quad (28)$$

Using $\tilde{y}_k$ and $\tilde{u}_k$ as expressed in (27), the closed loop system (20) and (25) can be represented as

$$\begin{aligned} \tilde{y}_k &= (B(z^{-1}) + \Delta B(k, z^{-1}))B_p(z^{-1})A_p^{-1}(z^{-1}) \\ &\quad \cdot (A(z^{-1}) + \Delta A(k, z^{-1}))^{-1}\tilde{u}_k, \\ \tilde{u}_k &= \tilde{y}_k + \Delta\tilde{u}_k, \\ \Delta\tilde{u}_k &= (\hat{F}(k, z^{-1}) + \Delta\hat{F}_a(k, z^{-1}))^{-1} \\ &\quad \cdot (\Delta\hat{F}_b(k, z^{-1}) - \Delta\hat{F}_a(k, z^{-1}))\tilde{y}_k. \end{aligned} \quad (29)$$

Since (28) represents a stable closed loop system model, the close loop system (29) can be viewed as the stable system (28) subject to a disturbance $\Delta\tilde{u}_k$ which is produced by the model perturbation $(\hat{F}(k, z^{-1}) + \Delta\hat{F}_a(k, z^{-1}))^{-1}(\Delta\hat{F}_b(k, z^{-1}) - \Delta\hat{F}_a(k, z^{-1}))$ in the closed loop.

Under the condition that the periodic polynomial $\hat{F}(k, z^{-1})$ possesses some degree of stability robustness and the coefficients of the quantization error polynomials $\Delta\hat{F}_a(k, z^{-1})$ and $\Delta\hat{F}_b(k, z^{-1})$ are small, the perturbation model $(\hat{F}(k, z^{-1}) + \Delta\hat{F}_a(k, z^{-1}))^{-1}(\Delta\hat{F}_b(k, z^{-1}) - \Delta\hat{F}_a(k, z^{-1}))$ to the closed loop system (28) is stable and its magnitude is small. Hence, the stability of the closed loop control system with the pipelined and quantized controller follows immediately from the well known small gain theorem (Vidyasagar, 1993). □

Remark 1. The above result does not require that the stabilizing digital feedback controller is stable. So far there exist no methods for the design of stable look ahead systems for unstable digital systems and the proposed periodic pipelining scheme presents a novel approach to pipelining and hardware implementation of digital controllers.

5. CONCLUSION

The pipelined integrated circuit implementation of digital feedback controllers requires the design of a stable look ahead system model of the controller. This paper resolves this problem using a periodic scheme, while there has been no known LTI solution to such a problem. Analysis is carried out to show that the proposed periodic pipelining scheme for fast digital controllers can maintain closed loop system stability when subject to quantization errors in implementation.

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