PORT REPRESENTATIONS OF THE TELEGRAPHER'S EQUATIONS

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Abstract: This article studies the telegrapher's equations with boundary port variables. Firstly, a link is made between the telegrapher's equations and a skew-symmetric linear operator on a spatial domain. Associated to this linear operator is a Dirac structure which includes the port variables on the boundary of this spatial domain. Secondly, we present all partitions of the port variables into inputs and outputs for which the state dynamics is dissipative. Particularly, we recognize the possible input-outputs for which the system is impedance energy-preserving, i.e., $\frac{1}{2}\frac{d}{dt}\|x(t)\|^2 = u(t)^T y(t)$, as well as scattering energy-preserving, i.e.,$\frac{1}{2}\frac{d}{dt}\|x(t)\|^2 = \|u(t)\|^2 - \|y(t)\|^2$. Additionally, we show how to represent the corresponding system as an abstract infinite-dimensional system, i.e., $\dot{x}(t) = Ax(t) + Bu(t)$ and $y(t) = Cx(t) + Du(t)$.

Keywords: Transmission lines, distributed-parameter systems, modelling.

1. INTRODUCTION

The description of any system consists of two distinct parts: A description of the nature of the elements in the system and a definition of the geometric structure of the interconnection of these elements. In this paper, this geometric structure is described by a Dirac structure. The key point of this approach is the definition of power conjugated variables called flows (e.g., velocities, currents) and efforts (e.g., forces, torques, voltages).

In order to model any physical system it is necessary to define the system boundary, since the interaction with the environment is done through this. Here we study the telegrapher's equations with boundary ports. This paper describes how to define those port variables so that the state dynamics of the resulting system is dissipative. Also, it is shown how to select inputs and outputs from these port variables. The main idea is to parameterize the inputs and outputs by the selection of certain matrices. Corresponding to that selection one can obtain systems with different properties.

Here we use the following notation; $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)$ for $X \times Y$ and $F_{|\mathfrak{D}}$ denotes the restriction of an operator F to the subspace $\mathfrak{D}$. Also, $\mathfrak{D}'$ denotes the dual space of $\mathfrak{D}$. $H^N((a,b);\mathbb{R}^n)$ denotes the space of N-times differentiable $L_2((a,b);\mathbb{R}^n)$ functions and ∂_v denotes the partial derivative with respect to v.

2. BOUNDARY CONTROL SYSTEMS (BCS)

The class of BCS described here is based on (Curtain and Zwart, 1995, §3.3). That is, BCS of the form

$$\begin{aligned} \dot{x}(t) &= \mathfrak{A}\, x(t), \quad x(0) = x_0, \\ u(t) &= \mathcal{B}\, x(t), \end{aligned} \tag{1}$$

where $\mathfrak{A} : D(\mathfrak{A}) \subset X \rightarrow X$, $u(t) \in U$, a separable Hilbert space, the boundary operator $\mathcal{B} : D(\mathcal{B}) \subset X \rightarrow U$ satisfies $D(\mathfrak{A}) \subset D(\mathcal{B})$, and

Definition 1. The control system (1) is a *boundary control system* if the following hold:

a. The operator $A : D(A) \rightarrow X$ with $D(A) = D(\mathfrak{A}) \cap \ker(\mathcal{B})$ and
$$A\,x = \mathfrak{A}\,x \quad \text{for } x \in D(A)$$
is the infinitesimal generator of a C_0-semigroup on X.
b. There exists a $B \in \mathcal{L}(U, X)$ such that for all $u \in U$, $Bu \in D(\mathfrak{A})$, the operator $\mathfrak{A}B$ is an element of $\mathcal{L}(U, X)$, and $\mathcal{B}Bu = u$ for $u \in U$.

As an example we can consider the transmission line with length $L = b - a$. Kirchhoff's laws describing the transmission line are given by

$$\begin{aligned} f_\phi &= -\frac{\partial e_q}{\partial z} \\ f_q &= -\frac{\partial e_\phi}{\partial z}. \end{aligned} \tag{2}$$

These equations are known as telegrapher's equations. Here, $z \in [a, b]$ is the spatial variable, f_q is the rate of charge density, $e_q = \frac{Q}{C}$ is the distributed voltage, $e_\phi = \frac{\phi}{L}$ is the distributed current, f_ϕ is the rate of flux density, and Q and φ are the charge density and the flux density, respectively. The boundary variables are

$$\begin{aligned} e_\phi(a) = f_L, &\quad e_q(a) = e_L \\ e_\phi(b) = f_R, &\quad e_q(b) = e_R. \end{aligned} \tag{3}$$

The total energy (Hamiltonian) stored at time t is given by

$$\mathcal{H}(Q, \varphi) = \int_a^b \frac{1}{2}\left(\frac{Q^2(t,z)}{C(z)} + \frac{\varphi^2(t,z)}{L(z)}\right) dz, \tag{4}$$

where the energy variables are Q and φ. Observe that $f_\phi = \frac{\partial Q}{\partial t}$ and $f_q = \frac{\partial \varphi}{\partial t}$. The resulting energy-balance is, see (van der Schaft and Maschke, 2002)

$$\frac{d\mathcal{H}}{dt} = -e_\phi(t,b)\,e_q(t,b) + e_\phi(t,a)\,e_q(t,a). \tag{5}$$

Observe that the system above does not have a defined input. We have only defined boundary variables. In this paper we study how to decompose the boundary variables so that the input is a linear combination of these variables and that the resulting system is a BCS with total energy $\mathcal{H}$.

3. DIRAC STRUCTURES AND PORT HAMILTONIAN SYSTEMS (PHS)

Let the *space of flow variables*, denoted by $\mathcal{F}$, and the *space of effort variables*, denoted by $\mathcal{E}$, be real Hilbert spaces endowed with the inner products $\langle .\,, .\rangle_{\mathcal{F}}$ and $\langle .\,, .\rangle_{\mathcal{E}}$, respectively. Assume moreover that $\mathcal{F}$ and $\mathcal{E}$ are isometrically isomorphic. Define now the *space of bond variables* as the Hilbert space $\mathcal{B} = \mathcal{F} \times \mathcal{E}$ endowed with the natural inner product:

$$\langle b^1, b^2 \rangle = \langle f^1, f^2 \rangle_{\mathcal{F}} + \langle e^1, e^2 \rangle_{\mathcal{E}}$$

for all $b^1 = (f^1, e^1) \in \mathcal{B}, b^2 = (f^2, e^2) \in \mathcal{B}$. In order to define a Dirac structure, we endow the bond space $\mathcal{B}$ with a *canonical symmetrical pairing*, i.e., a bilinear form defined as follows:

$$\langle b^1, b^2 \rangle_+ = \langle R\,b^1, b^2 \rangle_{\mathcal{F}\times\mathcal{E}} \tag{6}$$

where b^1, $b^2 \in \mathcal{B}$ and R is a fundamental symmetry (Dritschel and Rovnyak, 1996, p. 4) for $\mathcal{B}$. Now we may define a Dirac structure on the bond space $\mathcal{B}$ by using this canonical pairing. Denote by $\mathcal{D}^\perp$ the orthogonal subspace to $\mathcal{D}$ with respect to the symmetrical pairing (6):

$$\mathcal{D}^\perp = \{b \in \mathcal{B} \,|\, \langle b, b' \rangle_+ = 0, \forall\, b' \in \mathcal{D}\}. \tag{7}$$

Definition 2. (van der Schaft and Maschke, 2002). A *Dirac structure* $\mathcal{D}$ on the bond space $\mathcal{B} = \mathcal{F} \times \mathcal{E}$ is a subspace of $\mathcal{B}$ which satisfies

$$\mathcal{D}^\perp = \mathcal{D}. \tag{8}$$

The definition of a port Hamiltonian system is based on the definition of two objects: the interconnection structure given by a Dirac structure and the Hamiltonian function representing the total energy of the system.

Definition 3. Let $\mathcal{B} = \mathcal{F} \times \mathcal{E}$ be defined as above and consider the Dirac structure $\mathcal{D}$ and the Hamiltonian function $\mathcal{H}(v)$, where v contains the energy variables. Define the time variation of the energy variables as the flow variables, $f \in \mathcal{F}$, and the variational derivative of $\mathcal{H}$ as the effort variables, $e \in \mathcal{E}$. Then the system

$$(f, e) \in \mathcal{D}, \tag{9}$$

is a *port Hamiltonian system* with total energy $\mathcal{H}$.

For more information on Dirac structures and PHS we refer to (van der Schaft and Maschke, 2002) or to (Le Gorrec *et al.*, 2004) and the references therein.

The transmission line is an example of a PHS, with $v = (Q, \phi)$ being the energy variables, $\mathcal{H}$ is given by (4), $f = (f_\phi,\, f_q)$ and $e = (e_\phi,\, e_q)$ with respect to a Dirac structure induced by a (skew-symmetric) differential operator defined in (2). In fact, in the next section we show how any skew-symmetric differential operator defines a Dirac structure.

4. DIRAC STRUCTURE ASSOCIATED WITH A SKEW-SYMMETRIC OPERATOR

Observe from equation (2) that the flows, f, and efforts, e, are connected through a (differential) operator. Furthermore, from Definition 3 one can see that this efforts and flows describe the dynamics of a system and at the same time they belong to a Dirac structure. That is why we associate a Dirac structure with a differential operator. This section presents some results given in (Le Gorrec *et al.*, 2004), which will be used throughout this paper. In that paper, the authors define a Dirac structure which includes the boundary port variables associated with a skew-symmetric linear operator of the form

$$\mathcal{J}e = \sum_{i=0}^{N} P_i \frac{d^i e}{dz^i}(z) \qquad z \in [a,b]\,, \tag{10}$$

where $e \in H^N((a,b);\mathbb{R}^n)$ and P_i, $i = 0,\ldots,N$, is a $n \times n$ real matrix with P_N nonsingular. Since $\mathcal{J}$ is assumed to be skew-symmetric we get

$$P_i = P_i^T(-1)^{i+1}. \tag{11}$$

Here we study the case when $N = 1$ since this case includes the telegrapher's equations. The boundary port variables can be chosen as follows.

Definition 4. The boundary *port variables* associated with the differential operator $\mathcal{J}$ with $N = 1$ are the vectors $e_\partial, f_\partial \in \mathbb{R}^n$, defined by

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = R_{\text{ext}} \begin{pmatrix} e(b) \\ e(a) \end{pmatrix}, \tag{12}$$

where the nonsingular matrix R_{ext} is given by

$$R_{\text{ext}} = \frac{\sqrt{2}}{2} \begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix}. \tag{13}$$

Consider the effort and flow space $\mathcal{E} = \mathcal{F} = L^2((a,b);\mathbb{R}^n) \times \mathbb{R}^n$ with their natural inner product. Define the bond space $\mathcal{B}$ as $\mathcal{F} \times \mathcal{E}$ with the canonical symmetrical pairing

$$\begin{aligned}&\left\langle \left(f^1, f^1_\partial, e^1, e^1_\partial\right), \left(f^2, f^2_\partial, e^2, e^2_\partial\right)\right\rangle_+ = \\ &\quad \langle e^1, f^2\rangle_{L^2} + \langle e^2, f^1\rangle_{L^2} - \langle e^1_\partial, f^2_\partial\rangle - \langle e^2_\partial, f^1_\partial\rangle,\end{aligned}$$

where $\left(f^i, f^i_\partial, e^i, e^i_\partial\right) \in \mathcal{B}$, $i = \{1,2\}$.

Theorem 5. The subspace $\mathcal{D}_{\mathcal{J}}$ of $\mathcal{B}$ defined as

$$\mathcal{D}_{\mathcal{J}} = \left\{ \begin{pmatrix} f \\ f_\partial \\ e \\ e_\partial \end{pmatrix} \,\middle|\, e \in H^1((a,b);\mathbb{R}^n), \mathcal{J}e = f, \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = R_{\text{ext}} \begin{pmatrix} e(b) \\ e(a) \end{pmatrix} \right\} \tag{14}$$

is a Dirac structure.

From the Dirac structure and the Hamiltonian we define systems such that e becomes the state, f becomes the change of the state with respect to time and the Hamiltonian is the energy of the system, see Definition 3. Moreover, inputs and outputs of the system are linear combinations of the port variables (12). Here, inputs will be chosen so that the state dynamics of the resulting system is dissipative. Note that here the term input is used in the same philosophy as in behavioral systems theory, see (Willems and Polderman, 1998). However, for technical reasons it is not exactly the same. In this paper the input is assumed to be smooth, i.e., $u \in C^2((0,\infty);\mathbb{R}^{nN})$, whereas in behavioral systems the input is usually assumed to be in L_1^{loc}. The following theorem is taken from (Le Gorrec *et al.*, 2004).

Theorem 6. Let $W = S\,(\,I + V,\ I - V\,)$, with S invertible and $V V^T \leq I$, be a full rank matrix of size $n \times 2n$. Define $\mathcal{B} : H^1((a,b),\mathbb{R}^n) \to \mathbb{R}^n$ as

$$\mathcal{B}x(t) := W \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix}. \tag{15}$$

Then the system

$$\begin{aligned} \dot{x}(t) &= \mathcal{J}x(t), \\ \mathcal{B}x(t) &= u(t) \end{aligned} \tag{16}$$

is a boundary control system, where $A_W = \mathcal{J}_{|\ker \mathcal{B}}$ is the generator of a contraction semigroup on $L_2((a,b),\mathbb{R}^n)$ with

$$D(A_W) = \{x \in H^1((a,b),\mathbb{R}^n) \mid \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} \in \ker W\}.$$

Furthermore, if we define the linear mapping $\mathcal{C} : H^1((a,b),\mathbb{R}^n) \to \mathbb{R}^n$ as

$$\mathcal{C}x(t) := S_2\left(\,I - V^T,\ -I - V^T\,\right) \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} \tag{17}$$

with S_2 invertible and the output as

$$y(t) = \mathcal{C}x(t), \tag{18}$$

then for $u \in C^2((0,\infty);\mathbb{R}^n)$ and $x(0) - Bu(0) \in D(A_W)$ the following balance equation is satisfied:

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|^2 = \left(\,u^T(t)\ y^T(t)\,\right) P_W \begin{pmatrix} u(t) \\ y(t) \end{pmatrix}, \tag{19}$$

where P_W is given by

$$\frac{1}{4}\begin{pmatrix} S^{-T}(\tilde{P}_1^2 - \tilde{P}_1 V V^T \tilde{P}_1)S^{-1} & -2S^{-T}\tilde{P}_1 V \tilde{P}_2 S_2^{-1} \\ -2S_2^{-T}\tilde{P}_2 V^T \tilde{P}_1 S^{-1} & S_2^{-T}(-\tilde{P}_2^2 + \tilde{P}_2 V^T V \tilde{P}_2)S_2^{-1} \end{pmatrix}, \tag{20}$$

and $\tilde{P}_1 = (I + VV^T)^{-1}$, $\tilde{P}_2 = (I + V^T V)^{-1}$.

Observe that for the systems considered in the theorem above we have that the Hamiltonian $\mathcal{H}$ is given by $\frac{1}{2}\|x(t)\|^2$ and the energy-balance $\frac{d}{dt}\mathcal{H}$ is the energy exchanged at the boundary (compare with equation (5)).

5. TELEGRAPHER'S EQUATIONS WITH BOUNDARY PORTS

Consider the telegrapher's equations described in Section 2, see (2)-(3). Recall that $z \in [a,b]$ is the spatial variable.

Observe that equations (2) can be rewritten as follows

$$\mathcal{J}e = \sum_{i=0}^{1} P_i \frac{d^i e}{dz^i}(z) = P_1 \frac{de}{dz}(z), \tag{21}$$

where $P_0 = 0_{2\times 2}$,

$$P_1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \text{and} \quad e = \begin{pmatrix} e_\phi \\ e_q \end{pmatrix}. \tag{22}$$

Following equations (10) and (11) we can see that the telegrapher's equations fit into the framework described in Section 4. Thus from Definition 4, the port variables can be chosen as

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix}\begin{pmatrix} e(b) \\ e(a) \end{pmatrix}$$

$$= \frac{\sqrt{2}}{2}\begin{pmatrix} e_q(a) - e_q(b) \\ e_\phi(a) - e_\phi(b) \\ \hdashline e_\phi(a) + e_\phi(b) \\ e_q(a) + e_q(b) \end{pmatrix} = \begin{pmatrix} f_{\partial_1} \\ f_{\partial_2} \\ \hdashline e_{\partial_1} \\ e_{\partial_2} \end{pmatrix}, \quad (23)$$

where equation (22) was used. The problem is now the selection of inputs and outputs from these port variables.

In (Le Gorrec *et al.*, 2004) the authors show that the kernel of the matrices W in Theorem 6 give necessary and sufficient conditions for the differential operator A_W in Theorem 6 to be the infinitesimal generator of a contractive semigroup. Observe that once W is chosen the matrices S and V can be obtained from W as follows

$$S = \frac{1}{2}(W_1 + W_2), \quad V = (W_1 + W_2)^{-1}(W_1 - W_2) \quad (24)$$

where W is partitioned as $W = \begin{pmatrix} W_1 & W_2 \end{pmatrix}$ with W_i, $i = 1, 2$, square.

Thus, for a given input there corresponds a matrix W. If this matrix satisfies the conditions in Theorem 6, then we know that the state dynamics is dissipative.

Example 7. Consider the system (2)-(3). Assume that we want to have the input $u = \begin{pmatrix} e_q(b) \\ e_\phi(a) \end{pmatrix}$. Since the input is defined from the port variables described in equation (23), we can rewrite u as

$$\begin{aligned} u &= \begin{pmatrix} e_q(b) \\ e_\phi(a) \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} -(e_q(a) - e_q(b)) + (e_q(a) + e_q(b)) \\ (e_\phi(a) - e_\phi(b)) + (e_\phi(a) + e_\phi(b)) \end{pmatrix} \\ &= \frac{\sqrt{2}}{2}\begin{pmatrix} -f_{\partial_1} + e_{\partial_2} \\ f_{\partial_2} + e_{\partial_1} \end{pmatrix} \end{aligned}$$

and since we have to formulate it as $W\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix}$, see (15), we see that W is given by

$$W = \begin{pmatrix} W_1 & W_2 \end{pmatrix} = \frac{\sqrt{2}}{2}\left(\begin{array}{cc:cc} -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right). \quad (25)$$

From (24) we get

$$S = \frac{\sqrt{2}}{4}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } V = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (26)$$

Using this V together with (23) in equation (17) we get that the possible outputs are described by

$$y = \sqrt{2}\, S_2 \begin{pmatrix} -e_\phi(b) + e_q(a) \\ -e_\phi(b) - e_q(a) \end{pmatrix},$$

where S_2 is any nonsingular matrix.

Example 8. Consider the system (2)-(3). Now assume that we have the input $u = \frac{1}{2}\begin{pmatrix} e_q(b) - e_\phi(b) \\ e_\phi(a) + e_q(a) \end{pmatrix}$. The matrix W that corresponds to this input is

$$W = \frac{\sqrt{2}}{4}\begin{pmatrix} -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

From (24) we get

$$S = \frac{\sqrt{2}}{4}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } V = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Using this together with (23) in equation (17) gives that the possible outputs are described by

$$y = S_2 \begin{pmatrix} e_q(a) - e_q(b) - (e_\phi(a) + e_\phi(b)) \\ e_\phi(a) - e_\phi(b) - (e_q(a) + e_q(b)) \end{pmatrix},$$

where S_2 is any nonsingular matrix. One can see that although S_2 gives some freedom in the selection of the output y, the set of possible outputs is now restricted.

It is easy to see that once an input is chosen, the selection of possible outputs is bounded by that choice. However, the choice of an input is not completely free, since it is restricted by the fact that W has to satisfy the condition in Theorem 6.

Note that one is not only allowed to choose an input and output, but also one can shape the energy to satisfy some conditions or relations. This will be shown in the following two sections.

6. OBTAINING AN IMPEDANCE ENERGY-PRESERVING SYSTEM

Here we use the term 'impedance energy-preserving system' in the sense of (Staffans, 2002). In that paper the author shows that an impedance energy-preserving system satisfies the relation

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|^2 = u(t)^T y(t)$$

for $u \in C^2((0,\infty); \mathbb{R}^{nN})$ and $x(0) - Bu(0) \in D(A_W)$. Following equation (19) we see that in this case we must have that

$$P_W = \frac{1}{2}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Comparing this with equation (20) one can conclude that $VV^T = I$, $\tilde{P}_1 = \tilde{P}_2 = \frac{1}{2}I$, and $S_2^{-T}V^T = -4S$. This means that W must have the form $W = \frac{1}{4}S_2^{-T}\begin{pmatrix} -I - V^T & I - V^T \end{pmatrix}$ (compare with (17)). Thus the inputs will be

$$u = \frac{1}{4}S_2^{-T}\begin{pmatrix} -I - V^T & I - V^T \end{pmatrix}\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} \quad (27)$$

and the outputs are

$$y = S_2\begin{pmatrix} I - V^T & -I - V^T \end{pmatrix}\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix}. \quad (28)$$

Furthermore, it can be shown that if the type of systems described in Theorem 6 are impedance energy-preserving, then they are always conservative, this means that the adjoint system is also impedance energy-preserving.

It can be easily shown that the matrices S and V in Example 7 satisfy the conditions above, and hence one can conclude that the system is impedance energy-preserving if the output is chosen as (28) with $S_2 = -\frac{1}{4}S^{-T}V$.

7. OBTAINING A SCATTERING ENERGY-PRESERVING SYSTEM

Here we use the term 'scattering energy-preserving system' in the sense of (Staffans, 2002). In that paper the author shows that a scattering energy-preserving system satisfies the relation

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|^2 = \|u(t)\|^2 - \|y(t)\|^2$$

for $u \in C^2((0,\infty);\mathbb{R}^{nN})$ and $x(0) - Bu(0) \in D(A_W)$. Following equation (19) we see that in this case we must have that

$$P_W = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

Comparing this with equation (20) we can conclude that $V = 0$, $\tilde{P}_1 = \tilde{P}_2 = I$, $S^{-T}S^{-1} = 4I$, and $S_2^{-T}S_2^{-1} = 4I$. This means that W has the form $W = S\,(\,I\;\;I\,)$. Thus, from Theorem 6 and equation (23) we can conclude that the set of possible inputs is described by

$$u = \frac{\sqrt{2}}{2}\,S\begin{pmatrix} e_q(a) - e_q(b) + e_\phi(a) + e_\phi(b) \\ e_\phi(a) - e_\phi(b) + e_q(a) + e_q(b) \end{pmatrix}$$

where S must satisfy $S^{-T}S^{-1} = 4I$. Observe that now the output is restricted to

$$y = S_2\begin{pmatrix} e_q(a) - e_q(b) - (e_\phi(a) + e_\phi(b)) \\ e_\phi(a) - e_\phi(b) - (e_q(a) + e_q(b)) \end{pmatrix}$$

where S_2 satisfies $S_2^{-T}S_2^{-1} = 4I$. In (Villegas *et al.*, 2005) it is shown that a system described by the telegrapher's equations with this type of input and output is exponentially stable.

Observe that in Example 8 the matrix S satisfies $S^{-T}S^{-1} = 4I$, hence choosing any matrix S_2 satisfying the same condition gives a scattering energy-preserving system.

8. REPRESENTATION AS AN INFINITE-DIMENSIONAL SYSTEM

Many finite- and infinite-dimensional linear systems can be described by the equations

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t), \quad x(0) = x_0 \\ y(t) &= C\,x(t) + D\,u(t), \quad t \ge 0, \end{aligned} \tag{29}$$

on the Hilbert spaces, namely, the input space U, the state space X and the output space Y, where $u(t) \in U$, $x(t) \in X$ and $y(t) \in Y$. The operator A is generally the generator of a C_0-semigroup. The system node (see (Staffans, 2004), (Malinen *et al.*, 2003, §2)) has been introduced as a generalization of this set of equations. Roughly speaking the system node can be thought of as the block operator $\mathcal{S} = \left(\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right)$ from $X \times U$ to $X \times Y$, which allows to replace equation (29) by

$$\begin{pmatrix} \dot{x}(t) \\ y(t) \end{pmatrix} = \mathcal{S}\begin{pmatrix} x(t) \\ u(t) \end{pmatrix}, \quad t \ge 0, \quad x(0) = x_0.$$

Hence it is more general than (29). One of the main properties of the system node is that it is a closed operator. Also,

$$Ax := A\&B\begin{pmatrix} x \\ 0 \end{pmatrix} \text{ with } D(A) := \begin{pmatrix} x \\ 0 \end{pmatrix} \in D(\mathcal{S})$$

generates a C_0-semigroup. See (Staffans, 2004) or (Malinen *et al.*, 2003, §2) for a proper definition of the system node.

In this section we show how to represent the telegrapher's equations as a system node. To do this we first need some results, which are taken from (Villegas *et al.*, 2005).

Theorem 9. Consider the system (16) with

$$\mathcal{J}e = P_0\,e(z) + P_1\frac{de}{dz}(z)$$

and output (17). Let A_e be the extension of $\mathcal{J}$ to the state space $X = L_2((a,b);\mathbb{R}^n)$ and A_W be the restriction of $\mathcal{J}$ to $D(A_W)$ (see Theorem 6). Then this system can be described as a system node with $A\&B = [A_e\,B_{\text{node}}]_{|D(\mathcal{S})}$,

$$D(\mathcal{S}) = \left\{ \begin{pmatrix} x \\ u \end{pmatrix} \in \begin{pmatrix} X \\ U \end{pmatrix} \;\middle|\; x - Bu \in D(A_W) \right\},$$

$$\begin{aligned} C\&D\begin{pmatrix} x \\ u \end{pmatrix} &= S_2\,(\,I - V^T \;\; -I - V^T\,)\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} \\ &= S_2\,(\,I - V^T \;\; -I - V^T\,)\,R_{\text{ext}}\begin{pmatrix} x(b) \\ x(a) \end{pmatrix}, \end{aligned}$$

and B_{node} is given

$$\begin{aligned} &< x \,|\, B_{\text{node}}\,u >_{D(A_W^*),D(A_W^*)'} \\ &\qquad = -\langle A_W^* x, B\,u\rangle + \langle x, \mathcal{J}B\,u\rangle, \end{aligned} \tag{30}$$

where B is introduced in Definition 1.

Example 10. Consider the transmission line (2)-(3) with boundary port variables (23). Observe, from Theorem 9, that the state vector is

$$x = e = \begin{pmatrix} e_\phi \\ e_q \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{(see (22))}. \tag{31}$$

The operator $\mathcal{J}$ is given by (see equations (21) and (22))

$$\mathcal{J} = \begin{pmatrix} 0 & -\partial_z \\ -\partial_z & 0 \end{pmatrix} \tag{32}$$

with $P_0 = 0$ and $D(\mathcal{J}) = H^1((a,b);\mathbb{R}^2)$. From Theorem 6 we know that if W has the form $W = S\,(\,I + V,\; I - V\,)$, with S invertible and $VV^T \le I$, then the operator $\mathcal{J}$ restricted to

$$D(A_W) = \{x \in H^1((a,b),\mathbb{R}^n) \mid \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} \in \ker W\} \tag{33}$$

generates a C_0-semigroup, i.e., $\dot{x}(t) = A_W x(t)$ has a unique solution.

Next we describe how to get a system for which $\frac{1}{2}\frac{d}{dt}\|x(t)\|^2 = u(t)^T y(t)$, i.e., to construct a positive real system, see (Curtain and Zwart, 1995). Let the state space be $X = L_2((a,b);\mathbb{R}^2)$ and $U = Y = \mathbb{R}^2$. Following Example 7 we choose W as given by (25), which gives the input

$$u = \begin{pmatrix} e_q(b) \\ e_\phi(a) \end{pmatrix} = \begin{pmatrix} x_2(b) \\ x_1(a) \end{pmatrix}. \qquad (34)$$

From (26) select $S_2 = -\frac{1}{4}S^{-T}V$ which in this case is equal to the matrix S given in (26). This gives the output

$$y = \begin{pmatrix} -x_1(b) \\ x_2(a) \end{pmatrix}. \qquad (35)$$

Now that we have a BCS we will represent it as a system node. Basically, that requires to find the operators A_e, B_{node}, and $C\&D$. First we find A_e. We know that the operator A_e of Theorem 9 is the extension of (32) to X. Furthermore, the domain of $A_W = A_{e\,|D(A_W)}$ is given by

$$D(A_W) = \{x \in H^1((a,b),\mathbb{R}^2) \mid x_1(a) = x_2(b) = 0\}.$$

Notice that the condition in this domain is the same as letting $u = 0$, see (34).

Next we find B_{node} from (30). To do so, we first need the operator B. It can easily be checked that the operator

$$B = \frac{1}{(b-a)} \begin{pmatrix} 0 & -(z-b) \\ (z-a) & 0 \end{pmatrix}$$

satisfies the conditions in Definition 1.b. Moreover, we also have

$$\mathcal{J}B = \frac{1}{(b-a)} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Also, in this case it is not difficult to show that $A_W^* = -A_W$ and $D(A_W^*) = D(A_W)$. Using all this in (30) and after integrating by parts gives

$$-\langle A_W^* x, B\,u\rangle + \langle x, \mathcal{J}B\,u\rangle = -x_1(b)u_1 + x_2(a)u_2.$$

Rewriting the right-hand side of the equation above and using (30) gives

$$< x \mid B_{\text{node}}\, u >_{D(A^*),D(A^*)'} = \int_a^b \begin{pmatrix} x_1(z) \\ x_2(z) \end{pmatrix}^T \left[\begin{pmatrix} -\delta_b & 0 \\ 0 & \delta_a \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right] dz,$$

which shows that

$$B_{\text{node}}\, u = \begin{pmatrix} -\delta_b & 0 \\ 0 & \delta_a \end{pmatrix} u$$

where $\delta_a : H^1 \ \to \mathbb{R}$ is defined as $\delta_a x = x(a)$ and $\delta_b : H^1 \ \to \mathbb{R}$ is defined as $\delta_b x = x(b)$.

Finally, we find $C\&D$. From the output (35) one can see directly that

$$C\&D \begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} -\delta_b & 0 \\ 0 & \delta_a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Note that the input u does not appear explicitly in the representation above, but it is embedded in the domain of the system node $\mathcal{S} = \left(\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right)$, which is described by

$$D(\mathcal{S}) = \left\{ \begin{pmatrix} x \\ u \end{pmatrix} \in \begin{pmatrix} X \\ U \end{pmatrix} \,\middle|\, x - Bu \in D(A_W) \right\}.$$

9. CONCLUSION

Here we studied the transmission line with boundary ports. The telegrapher's equations were related to a skew-symmetric differential operator. Also, it was described how to select the port variables for the transmission line and a Dirac structure was defined including these port variables.

It was also described how to choose inputs and outputs from the port variables. We showed how one can select inputs and outputs so that the energy function satisfies some desired relation. Particularly, we presented how to obtain impedance energy-preserving and scattering energy-preserving systems.

Finally, we showed how to represent the telegrapher's equations as a system node, which is a form of describing abstract infinite-dimensional systems.

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