OUTPUT SHAPING BY MEANS OF OUTPUT CONSTRAINTS IN CONTINUOUS-TIME PREDICTIVE CONTROL

Çağatay Yavuzyılmaz * and **Hüseyin Demircioğlu** *

* *Hacettepe University*
Department of Electrical and Electronics Engineering
Ankara, Turkey

Abstract: In many control applications, it is expected that the system output tracks a desired reference signal. This paper describes a new approach in which the tracking problem is formulated as an *output shaping problem*. The approach uses the framework of predictive control and imposes the desired reference signal as a constraint on the output. Some simulation results are also given to illustrate the effectiveness of the proposed method.

Keywords: output constraints, optimization, continuous-time predictive control, quadratic programming.

1. INTRODUCTION

The aim of many control problems is that the system output tracks a desired reference signal. In classical control, this aim is tried to achieve by using time domain specifications such as settling time, maximum overshoot and steady-state error. In optimal control, this is tried to achieve by minimizing the error between the desired reference signal and the system output. In these designs, there is always a tracking error.

Predictive control methods are based on prediction of the system output and minimization of a quadratic cost function of the error between the predicted output and the desired output (Clarke *et al.*, 1987; Demircioğlu and Gawthrop, 1991; Demircioğlu and Clarke, 1992; Camacho, 1999). In some recent works, it has been seen that the output constraints could be taken into account and satisfied exactly (Scokaert and Clarke, 1994; Scokaert and Mayne, 1998; Demircioğlu, 1999).

In this study, we propose a very different approach. The desired reference signal is used as a constraint on the output and the control problem is solved subject to this constraint. This method also aims at minimizing the predicted control energy that satisfies the output constraint. The proposed approach can be considered as *output shaping problem*. This approach is very useful as it allows us to shape the system output as desired.

Predictive control constitutes a suitable framework for the solution of the output shaping problem. In this work, a continuous-time approach is adopted and the continuous-time predictive control framework (Demircioğlu, 1999) is utilized for the solution.

2. A BRIEF REVIEW OF OUTPUT PREDICTION

Consider the following continuous-time model:

$$A(s)Y(s) = B(s)U(s) + C(s)V(s) \qquad (1)$$

where $Y(s)$, $U(s)$ and $V(s)$ are the system output, control input and disturbance input respectively. $A(s)$, $B(s)$ and $C(s)$ are polynomials in Laplace operator s. The design polynomial $C(s)$ is stable with a degree of 1 less or equal to that of $A(s)$.

In continuous-time predictive control, the output is predicted by using truncated Taylor series

$$\hat{y}(t+T) = \sum_{k=0}^{N_y} y_k(t)\frac{T^k}{k!} \tag{2}$$

where N_y is the predictor order, T is the future time variable and

$$y_k(t) = \frac{d^k y(t)}{dt^k} \tag{3}$$

Since taking the derivatives of the system output is not feasible, instead of the output derivatives the emulated values are used in equation 2. By substituting the emulated values in equation 2 and rearranging in matrix form, the T-ahead output predictor is obtained as (Demircioğlu and Gawthrop, 1991):

$$y^*(t+T) = \mathbf{T}_{N_y}\mathbf{H}\mathbf{u} + \mathbf{T}_{N_y}\mathbf{Y}^o \tag{4}$$

where

$$\mathbf{T}_{N_y} = \left[1 \; T \; \frac{T^2}{2} \; \cdots \; \frac{T^{N_y}}{N_y!} \right] \tag{5}$$

$$\mathbf{H} = \begin{bmatrix} h_0 & 0 & \cdots & 0 \\ h_1 & h_0 & \cdots & 0 \\ h_2 & h_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \cdot & \cdot & \cdots & h_0 \\ \cdot & \cdot & \cdots & h_1 \\ \vdots & \vdots & \cdots & \vdots \\ h_{N_y} & h_{N_y-1} & \cdots & h_{(N_y-N_u)} \end{bmatrix} \tag{6}$$

$$\mathbf{u} = \left[u(t) \; u_1(t) \; u_2(t) \; \cdots \; u_{N_u}(t) \right]^T \tag{7}$$

$$\mathbf{Y}^o = \left[y_0^o(t) \; y_1^o(t) \; y_2^o(t) \; \cdots \; y_{N_y}^o(t) \right]^T \tag{8}$$

where $u_k(t)$ is the k^{th} derivative of the input, N_u is the control order and the elements of the $\mathbf{H}$ matrix are the Markov parameters of the open-loop system $B(s)/A(s)$, $y_k^o(t)$ is realizable part of the emulated value of k^{th} derivative of the output. The emulated value $y_k^*(t)$ of the k^{th} derivative of the output is given in Laplace domain by

$$Y_k^*(s) = H_k(s)U(s) + Y_k^o(s) \tag{9}$$

and the realizable part $y_k^o(t)$ is,

$$Y_k^o(s) = \frac{G_k(s)}{C(s)}U(s) + \frac{F_k(s)}{C(s)}Y(s) \tag{10}$$

and the polynomials $H_k(s)$, $G_k(s)$ and $F_k(s)$ are obtained from the identities:

$$\begin{aligned} \frac{s^k C(s)}{A(s)} &= E_k(s) + \frac{F_k(s)}{A(s)} \\ \frac{E_k(s)B(s)}{C(s)} &= H_k(s) + \frac{G_k(s)}{C(s)} \end{aligned} \tag{11}$$

3. OUTPUT SHAPING BY MEANS OF OUTPUT CONSTRAINTS

Output constraints can be used to shape the system output as desired. In other words, the desired shape of the output function can be imposed on the control problem as output constraints. In constrained predictive control, output constraints are written in terms of predicted output (Demircioğlu, 1999). Let y^{min} and y^{max} denotes the lower and upper constraining functions (shaping functions) then we could write:

$$y^{min}(t+T) \leq y^*(t+T) \leq y^{max}(t+T) \tag{12}$$

where $T_1 \leq T \leq T_N$, where T_1 and T_N are lower and upper constraint horizons respectively. By the proper choice of the constraining functions y^{min} and y^{max}, the output can be shaped as desired. y^{min} and y^{max} may, for example, defines a certain band around a desired function. In this case the output will follow the desired function staying within the band defined by y^{min} and y^{max}. By narrowing this band a tighter tracking performance can be obtained. If an exact tracking or shaping is desired then this band can be set equal to zero, that is y^{min} and y^{max} will be the same functions.

Since the constraints are applied on predicted future output, the future values of the signals $y^{min}(t)$ and $y^{max}(t)$ must be known. However, in some control applications future values of the constraining functions may not be known. For these applications, the limits may be assumed as constant and equal to the value at time t in the interval $[T_1, T_N]$.

$$y^{min}(t) \leq y^*(t+T) \leq y^{max}(t) \tag{13}$$

The upper and lower limits are still functions of time, so the output shaping can still be obtained by using this assumption. In this study, the future values of the constraining functions are assumed to be known.

By using equation 4, inequality 12 can be expressed as a single matrix inequality as follows:

$$\mathbf{Q}_y(T)\mathbf{u} \leq \mathbf{p}_y(t,T) \tag{14}$$

where

$$\mathbf{Q}_y(T) = \begin{bmatrix} \mathbf{T}_{N_y} \\ -\mathbf{T}_{N_y} \end{bmatrix} \mathbf{H} \tag{15}$$

$$\mathbf{p}_y(t,T) = \begin{bmatrix} y^{max}(t+T) - \mathbf{T}_{N_y}\mathbf{y}^o \\ -y^{min}(t+T) + \mathbf{T}_{N_y}\mathbf{y}^o \end{bmatrix} \qquad (16)$$

The inequality 14 must hold for any value T, say T_i, over the interval $[T_1, T_N]$ and the number of T_i values in this interval are obviously infinite. It may be assumed that the inequality 14 holds over the entire interval $[T_1, T_N]$, if it holds at a finite number of T_i values, provided that T_i values are chosen close enough to each other. In this interval, if N number of T_i values are chosen, we then have

$$\mathbf{Q}_y(T_i)\mathbf{u} \le \mathbf{p}_y(t,T_i) \qquad (17)$$

where T_i's are time instants and $i = 1, 2, \ldots, N$.

These N number of inequalities can be rearranged as a single matrix inequality as follows:

$$\mathbf{Q}_y\mathbf{u} \le \mathbf{p}_y \qquad (18)$$

where

$$\mathbf{Q}_y = \begin{bmatrix} \mathbf{Q}_y(T_1) \\ \mathbf{Q}_y(T_2) \\ \vdots \\ \mathbf{Q}_y(T_N) \end{bmatrix}, \; \mathbf{p}_y = \begin{bmatrix} \mathbf{p}_y(t,T_1) \\ \mathbf{p}_y(t,T_2) \\ \vdots \\ \mathbf{p}_y(t,T_N) \end{bmatrix} \qquad (19)$$

4. CONTROL LAW

The control problem is to find a control signal which minimizes the predicted control energy and satisfies the output shaping constraints. The predicted control energy is obtained by using the predicted future input. At time t, the predicted future input can be written as the truncated Taylor series expansion.

$$u^*(t+T) = \mathbf{T}_{N_u}\mathbf{u} \qquad (20)$$

where

$$\mathbf{T}_{N_u} = \begin{bmatrix} 1 \; T \; \frac{T^2}{2} \; \cdots \; \frac{T^{N_u}}{N_u!} \end{bmatrix} \qquad (21)$$

For the time interval $[0, T_c]$, the predicted control energy can be calculated as:

$$E = \int_0^{T_c} u^{*T}(t+T)u^*(t+T)dT \qquad (22)$$
$$= \mathbf{u}^T\mathbf{T}_u\mathbf{u}$$

where

$$\mathbf{T}_u = \int_0^{T_c} \mathbf{T}_{N_u}^T\mathbf{T}_{N_u}dT \qquad (23)$$

and T_c is called control horizon. Equation 22 is a quadratic function that includes input and its derivatives.

As a result, the output shaping control problem can be defined as:

$$\textit{Minimize} \;\; E = \mathbf{u}^T\mathbf{T}_u\mathbf{u} \qquad (24)$$
$$\textit{Subject to} \;\; \mathbf{Q}_y\mathbf{u} \le \mathbf{p}_y$$

This is a QP (quadratic programming) problem. There exists algorithms to solve this problem (Fletcher, 1987; Bett, 2001). The first element, $u(t)$, of the vector $\mathbf{u}$ obtained from the solution of the QP problem is applied to the system as receding horizon strategy.

5. SIMULATIONS

This section presents some simulation results which illustrate the output shaping by means of output constraints in continuous-time predictive control. In all examples, the aim is to keep the output within the band defined by the shaping (constraining) functions. The shaping functions are chosen as a combination of linear, exponential and sinusoidal functions in order to include different forms of signals encountered in practice, although it is not necessary the lower and upper shaping functions are chosen to have the same form. In figures the upper graph shows the system output, $y(t)$ (solid), and the upper and the lower shaping functions, $y^{max}(t)$ and $y^{min}(t)$ (dashed), the lower graph shows the control input, $u(t)$. In simulations, sampling interval is 0.05 sec.

5.1 Example 1

A double integrator is chosen as a first example.

$$\frac{B(s)}{A(s)} = \frac{1}{s^2} \qquad (25)$$

Simulation result is obtained by using the following control parameters.

$$y^{max}(t) - y^{min}(t) = 0.5 \qquad (26)$$
$$C(s) = s + 1$$
$$N_y = 3$$
$$N_u = 1$$
$$T_c = 1$$
$$T_i = 0.05i, \quad i = 1, 2, \ldots, 20$$

The number of constraint inequalities is 20 at each sampling instant. The result is given in figure 1. As seen from the figure, the output remains within the band defined by the shaping functions y^{min} and y^{max}. Narrowing this band makes the constraints more stringent, and as a result the system output becomes more similar to

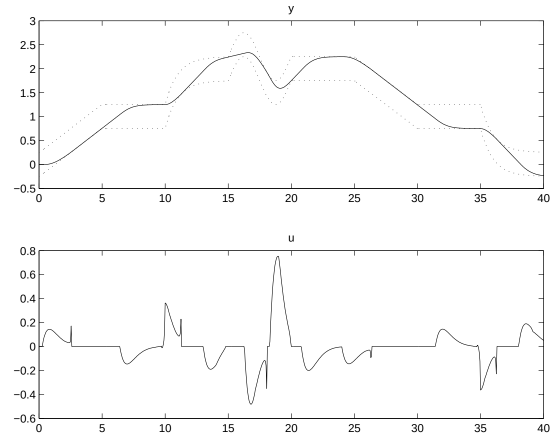

Fig. 1. Control of example 1 ($y^{max} - y^{min} = 0.5$)

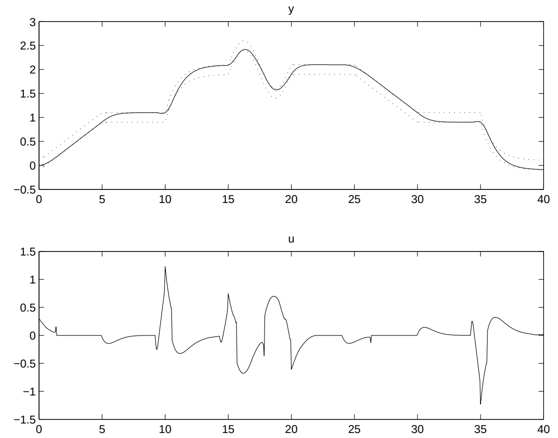

Fig. 2. Control of example 1 ($y^{max} - y^{min} = 0.2$)

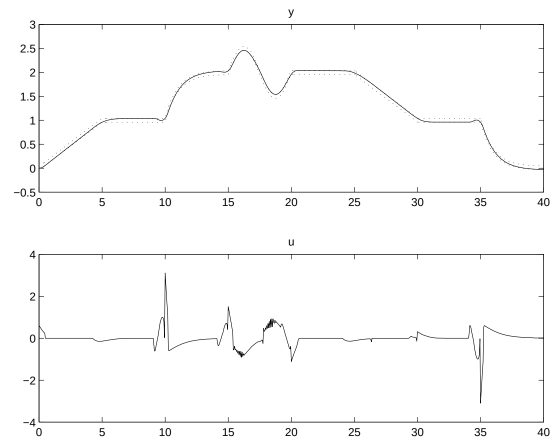

Fig. 3. Control of example 1 ($y^{max} - y^{min} = 0.08$)

the shaping functions. As this band approaches zero, the form of the system output approaches to the form of shaping functions. Simulation results illustrating this are given in figures 2 and 3. In figures 2 and 3, the difference between y^{max} and y^{min} is 0.2 and 0.08 respectively and the other control parameters are the same as before. As seen from the figures, the output is shaped more effective.

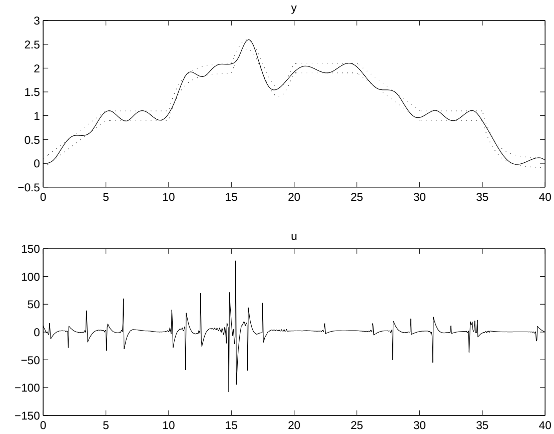

Fig. 4. Control of example 2 ($y^{max} - y^{min} = 0.2$)

5.2 Example 2

Second example is an double oscillator.

$$\frac{B(s)}{A(s)} = \frac{1}{(s^2+1)(1.5s^2+1)} \tag{27}$$

Simulation result is obtained by using the following control parameters for this more complex system.

$$\begin{aligned} y^{max}(t) - y^{min}(t) &= 0.2 \\ C(s) &= s^4 + 2s^3 + 4s^2 + 2s + 1 \\ N_y &= 20 \\ N_u &= 5 \\ T_c &= 4 \\ T_i &= 0.2i, \quad i = 1, 2, \ldots, 20 \end{aligned} \tag{28}$$

The result is given in figure 4. As seen from the figure, although the system output oscillates as a result of the complexity of the system, the system output satisfies the output constraints. Note that rapid changes in shaping functions cause the control signal became more active. This is an expected result. It is clear that the control activity can be reduced by relaxing the constraints.

Because the system is more complex, the upper constraint horizon is now chosen as 4. This is larger than that of the previous example but the number of inequalities are still 20 at each sampling instant. The difference between the maximum and minimum shaping functions is 0.2. When this difference is reduced to 0.1, the simulation result in figure 5 is obtained. As seen from the figure, the shape of the system output is very close to the desired shape. However, this causes very effective control signal.

Although reducing the difference between the limits may cause much more stringent constraints, the system output can be shaped more effective.

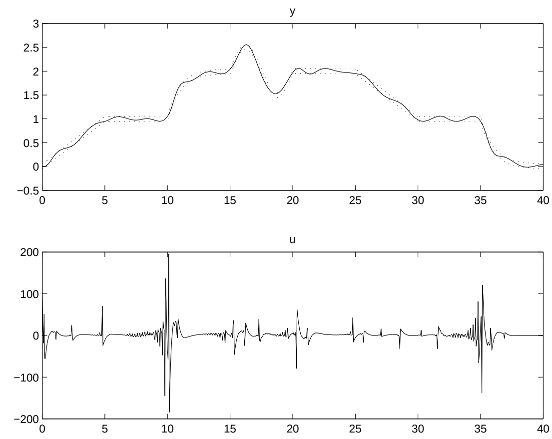

Fig. 5. Control of example 2 ($y^{max} - y^{min} = 0.1$)

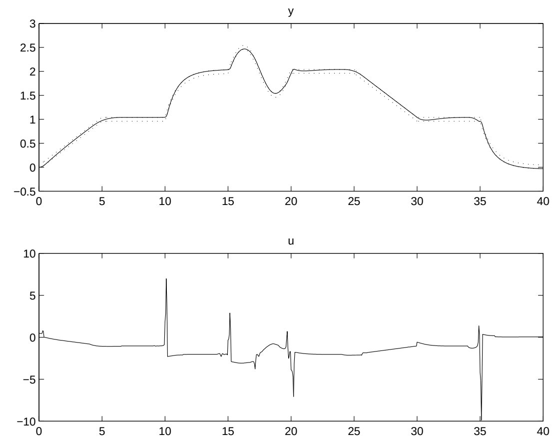

Fig. 6. Control of example 3

5.3 Example 3

An unstable system is considered in this example.

$$\frac{B(s)}{A(s)} = \frac{1}{s^2 - 1} \tag{29}$$

The system is simulated with the following design parameters.

$$\begin{aligned} y^{max}(t) - y^{min}(t) &= 0.08 \\ C(s) &= s + 1 \\ N_y &= 3 \\ N_u &= 0 \\ T_c &= 1 \\ T_i &= 0.05i, \quad i = 1, 2, \ldots, 20 \end{aligned} \tag{30}$$

The result is given in figure 6. As seen from the figure, the output remains between the constraining functions. The system output is shaped as desired.

6. CONCLUSIONS

This paper shows that output tracking problem can be formulated as output shaping problem by imposing the desired trajectory as constraint on the output. The predictive control constitutes a suitable framework for the solution of the problem. Effectiveness of the method is illustrated by the simulations using a double integrator, a double oscillator and a unstable system. In these different examples, the output is shown to remain within the band defined by the shaping functions (constraining functions). In short, the output can be forced to have any desired shape by chosen proper shaping functions.

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