PRELIMINARY RESULTS ABOUT ANTI-WINDUP STRATEGY FOR SYSTEMS SUBJECT TO ACTUATOR AND SENSOR SATURATIONS

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Abstract: This paper addresses the problems of the design of anti-windup schemes for linear systems subject to amplitude limitations in actuator and sensor. The design of anti-windup loops, using both measured and estimated variables, is considered in order to enlarge the region of stability of the closed-loop system, whereas some performance in terms of bounded controlled output is guaranteed. Based on the modelling of the closed-loop saturated system including the observer loop as a linear system with dead-zone nested nonlinearities, constructive stability conditions are formulated in terms of linear matrix inequalities by using Finsler's Lemma.

Keywords: Anti-windup scheme, actuator and sensor saturations, observer, stability regions, LMI

1. INTRODUCTION

During the past years, the problem of stabilization for linear systems subject to amplitude and dynamics restricted actuators and/or sensors have attracted the attention of control community, due to its practical interest and its inherent difficulties (see, for example, (Bernstein, 2001)). The studies concerning these systems have been done in terms of modelling, properties analysis (as stability and performance analysis) or still control design problem (Gomes da Silva Jr. *et al.*, 2003). Most of these theoretical studies are motivated by the fact that physical, safety or technological constraints generally induce that the actuators and sensors cannot provide unlimited amplitude signal neither unlimited speed of reaction. The control problems of combat aircraft prototypes and launchers offer interesting examples of the difficulties due to these major constraints (Murray, 1999). Neglecting actuator and sensor limitations can be source of undesirable even catastrophic behaviors for the closed-loop system (as the lost of the stability).

Moreover, the case of actuator saturation has been addressed in much details ((Kapila and Grigoriadis, 2002) and references therein). Few results concern the case of sensor saturation. Hence among these results one can cite the studies of the effects of sensor saturation on plant observability (Koplon *et al.*, 1994). In (Kreisselmeier, 1996), the global stabilization of a linear SISO system is carried out via the use of dead beat controller. In (Cao *et al.*, 2003), the authors use the circle criterion to design an output $\mathcal{H}_\infty$ feedback controller for linear systems with sensor nonlinearity. If few results are available on systems with sensor actuator still less results concern the case of systems with both actuator and sensor limitations. In (Fliegner *et al.*, 2000), adaptive integral control design for linear systems with actuator and sensor nonlinearities is addressed by considering the asymptotic stability of the open-loop. See also (Glattfelder and Schaufelberger, 2003), in which different methods of analysis and synthesis are presented (like the PID controllers design).

The anti-windup techniques consist in taking into account the effects of saturation in a second stage after previous design performed disregarding the actuator limitations. The idea is to introduce some control modification in order to recover, as much as possible the properties induced by the previous design carried out for the unsaturated system. In particular, anti-windup schemes have been successfully applied in order to minimize the windup due the integral action in PID controllers. In this case, most of the related literature focuses on the performance improvement in the sense of avoiding large and oscillatory transient responses (see, among others, (Åström and Rundqwist, 1989)).Moreover, the influence of the anti-windup schemes in the stability and the performances of the closed-loop system has been also studied (see, for example, (Barbu *et al.*, 2000), (Kothare and Morari, 1999)). Several results on the anti-windup problem are concerned with achieving global stability properties. Since global results cannot be achieved for open-loop unstable linear systems in the presence of actuator saturation, local results have to be developed. In this context, a key issue concerns the determination of domains of stability for the closed-loop system by noting that the basin of attraction is modified by the anti-windup loop.

More recently, in (Cao *et al.*, 2002), (Gomes da Silva Jr. and Tarbouriech, 2003), (Teel, 1999), in the ACC03 Workshop "T-1: Modern Anti-windup Synthesis" or in the ACC04 (Session FrP04 "Anti-windup"), some constructive conditions are proposed both to determine suitable anti-windup loops and to quantify the closed-loop region of stability in the case of amplitude saturation actuator. Differently from the references cited above, in this paper we focus our attention on linear systems with amplitude saturation on actuator and sensor, and bounded controlled outputs. Our aim is the design of the suitable anti-windup gains in order to ensure the closed-loop stability for regions of admissible initial states as large as possible. Based on the modelling of the closed-loop system resulting from the controller plus the anti-windup loop as a linear system with dead-zone nested nonlinearities, original constructive stability conditions are directly formulated as LMI conditions. Moreover, in order to use in the anti-windup loops some unmeasured signals, a solution based on the addition of an observer step is considered. Thus, in order to address inherent structural constraints on some decision variables the Finsler's Lemma is used.

Notation. For any vector $x \in \Re^n$, $x \succeq 0$ means that all the components of x, denoted $x_{(i)}$, are nonnegative. For two vectors x, y of $\Re^n$, the notation $x \succeq y$ means that $x_{(i)} - y_{(i)} \geq 0$, $\forall i = 1,\ldots,n$. $\mathbf{1}$ and $\mathbf{0}$ denote respectively the identity matrix and the null matrix of appropriate dimensions. $A_{(i)}$, $i = 1,...,m$, denotes the ith row of matrix $A \in \Re^{m\times n}$. For two symmetric matrices, A and B, $A > B$ means that $A - B$ is positive definite. A' denotes the transpose of A. For any vector u of $\Re^m$ one defines each component of $sat_{u_0}(u)$ by $sat_{u_0}(u_{(i)}) = sign(u_{(i)})min(u_{0(i)}, |u_{(i)}|)$, with $u_{0(i)} > 0$, $i = 1,...,m$.

2. PROBLEM STATEMENT

The system under consideration is described by:

$$\begin{aligned} \dot{x} &= Ax + Bv \\ y &= sat_{y_0}(Cx) \\ v &= sat_{u_0}(u) \\ z &= C_2x + D_2v \end{aligned} \tag{1}$$

where $x \in \Re^n$, $v \in \Re^m$, $y \in \Re^p$ and $z \in \Re^l$ are the state, the input, the measured output and the controlled output vectors, respectively. The vectors u_0 and y_0 are the positive saturation levels. A, B, C, C_2 and D_2 are real constant matrices of appropriate dimensions.

Let us consider the controller dynamics in $\Re^{n_c}$:

$$\begin{aligned} \dot{\eta} &= A_c\eta + B_cu_c \\ y_c &= C_c\eta + D_cu_c \end{aligned} \tag{2}$$

where A_c, B_c, C_c, D_c are matrices of appropriate dimensions. Thus, such a controller has been designed such that the linear closed-loop system resulting from the interconnection conditions

$$v = u = y_c \ ; \ u_c = Cx \tag{3}$$

is asymptotically stable. Note that the dynamics controller (2) has been designed for system (1) without saturation (as described in (3)).

Some windup problems arise when saturations occur, that is when the previous linear interconnection is replaced by the real interconnection:

$$u = y_c \ ; \ v = sat_{u_0}(y_c) \ ; \ u_c = sat_{y_0}(Cx) \tag{4}$$

Hence, from the above description, the complete closed-loop system reads:

$$\begin{aligned} \dot{x} &= Ax + Bsat_{u_0}(C_c\eta + D_csat_{y_0}(Cx)) \\ \dot{\eta} &= A_c\eta + B_csat_{y_0}(Cx) \end{aligned} \tag{5}$$

The set of measured variables of this system for anti-windup purpose are v, y_c and y. In order to avoid the undesirable effects of the windup, or at least to mitigate them, we want to build a loop of anti-windup. Thus, the strategy consists in adding the first term

$$E_c(sat_{u_0}(y_c) - y_c) \tag{6}$$

in the dynamics of the controller. Another term that we can add is related to the output saturation, but the only available measure for the system is $y = sat_{y_0}(Cx)$. Therefore, one cannot use directly the difference $sat_{y_0}(Cx) - Cx$. A way to use the difference due to the output saturater is to build an observer in order to consider an estimate of this difference. Hence, an observer of the system can be described as:

$$\dot{\hat{x}} = A\hat{x} + Bv + L(sat_{y_0}(C\hat{x}) - sat_{y_0}(Cx)) \quad (7)$$

with the error

$$\epsilon = \hat{x} - x \quad (8)$$

From (7)-(8), the anti-windup strategy consists in adding also the term:

$$F_c(sat_{y_0}(C\hat{x}) - C\hat{x}) \quad (9)$$

in the dynamics of the controller. Thus, considering the dynamic controller and these additional anti-windup loops, the closed-loop system reads:

$$\begin{array}{rcl}\dot{x} &=& Ax + Bsat_{u_0}(C_c\eta + D_c sat_{y_0}(Cx)) \\ \dot{\eta} &=& A_c\eta + B_c sat_{y_0}(Cx) + E_c(sat_{u_0}(y_c) - y_c) \\ && +F_c(sat_{y_0}(Cx + C\epsilon) - (Cx + C\epsilon)) \\ \dot{\epsilon} &=& A\epsilon + L(sat_{y_0}(Cx + C\epsilon) - sat_{y_0}(Cx)) \\ y_c &=& C_c\eta + D_c sat_{y_0}(Cx)\end{array} \quad (10)$$

In order to deal with system (10), let us define the two connected nonlinearities ϕ_{y_0} and ϕ_{u_0}, and the nonlinearity $\phi^{\epsilon}_{y_0}$:

$$\phi_{y_0} = sat_{y_0}(Cx) - Cx \quad (11)$$

$$\begin{array}{rcl}\phi_{u_0} &=& sat_{u_0}(y_c) - y_c \\ &=& sat_{u_0}(C_c\eta + D_cCx + D_c\phi_{y_0}) \\ && -(C_c\eta + D_cCx + D_c\phi_{y_0})\end{array} \quad (12)$$

$$\phi^{\epsilon}_{y_0} = sat_{y_0}(Cx + C\epsilon) - (Cx + C\epsilon) \quad (13)$$

Thus, by defining the extended vectors

$$\begin{array}{c}\xi = \left[\, x' \;\; \eta' \;\; \epsilon' \,\right]' \in \Re^{2n+n_c} \\ \Phi = \left[\, \phi'_{y_0} \;\; \phi'_{u_0} \,\right]' \in \Re^{p+m}\end{array} \quad (14)$$

and the following matrices

$$\begin{array}{l}\mathbb{A} = \begin{bmatrix} A + BD_cC & BC_c & \mathbf{0} \\ B_cC & A_c & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A \end{bmatrix} \in \Re^{(2n+n_c)\times(2n+n_c)} \\ \mathbb{B} = \begin{bmatrix} BD_c & B \\ B_c & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \in \Re^{(2n+n_c)\times(p+m)} \\ \mathbb{R}_1 = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix} \in \Re^{(2n+n_c)\times n}; \mathbb{R}_2 = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \end{bmatrix} \in \Re^{(2n+n_c)\times n_c} \\ \mathbb{R}_3 = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix} \in \Re^{(p+m)\times m}; \mathbb{R}_4 = \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix} \in \Re^{(p+m)\times p} \\ \mathbb{C}_1 = \left[\, \mathbf{0} \;\; \mathbf{0} \;\; C \,\right] \in \Re^{p\times(2n+n_c)} \\ \mathbb{C}_2 = \left[\, C_2 + D_2D_cC \;\; D_2C_c \;\; \mathbf{0} \,\right] \in \Re^{l\times(2n+n_c)} \\ \mathbb{D}_2 = \left[\, D_2D_c \;\; D_2 \,\right] \in \Re^{l\times(m+p)} \\ \mathbb{K} = \begin{bmatrix} C & \mathbf{0} & \mathbf{0} \\ D_cC & C_c & \mathbf{0} \end{bmatrix} \in \Re^{(p+m)\times(2n+n_c)} \\ \mathbb{C}_3 = \left[\, C \;\; \mathbf{0} \;\; C \,\right] \in \Re^{p\times(2n+n_c)}\end{array} \quad (15)$$

the closed-loop system reads:

$$\begin{array}{rcl}\dot{\xi} &=& (\mathbb{A} + \mathbb{R}_1L\mathbb{C}_1)\xi + (\mathbb{B} + \mathbb{R}_2E_c\mathbb{R}'_3 - \mathbb{R}_1L\mathbb{R}'_4)\Phi \\ && +(\mathbb{R}_2F_c + \mathbb{R}_1L)\phi^{\epsilon}_{y_0} \\ z &=& \mathbb{C}_2\xi + \mathbb{D}_2\Phi\end{array} \quad (16)$$

The problem to compute E_c and F_c for enlarging the basin of attraction of the closed-loop system, whereas some performance described with respect to the controlled output is satisfied, can be summarized as follows.

Problem 1. *Determine anti-windup gains E_c, F_c, an observer gain L and a set S_0 such that:*

1. (Stability) *The asymptotic stability of the closed-loop system (16) is ensured for any $\left[\, x(0)' \;\; \eta(0)' \;\; \epsilon(0)' \,\right]' \in S_0$, where S_0 is as large as possible.*

2. (Performance) *For any $\left[\, x(0)' \;\; \eta(0)' \;\; \epsilon(0)' \,\right]' \in S_0$ the controlled output signal z is bounded and takes values in the set $\mathcal{Z}_0$ defined by:*

$$\mathcal{Z}_0 = \{z \in \Re^l; -z_0 \preceq z \preceq z_0,\; z_{0(i)} > 0\} \quad (17)$$

Note that the satisfaction of point 1 of Problem 1 will assure the convergence of the observer.

3. THEORETICAL ANTI-WINDUP CONDITIONS

Let us consider the generic nonlinearity $\varphi(v) = sat_{v_0}(v) - v$, $\varphi(v) \in \Re^m$ and the following set:

$$S(v_0) = \{v \in \Re^m, w \in \Re^m; -v_0 \preceq v - w \preceq v_0\} \quad (18)$$

Lemma 1. *(Tarbouriech* et al.*, 2004) If v and w are elements of $S(v_0)$ then the nonlinearity $\varphi(v)$ satisfies the following inequality:*

$$\varphi(v)'T(\varphi(v) + w) \leq 0 \quad (19)$$

for any diagonal positive definite matrix $T \in \Re^{m\times m}$.

Moreover, in order to treat in a potentially less-conservative framework the possibility of considering structural conditions, the technique developed in the sequel is based upon the use of the Finsler's Lemma, recalled below (de Oliveira and Skelton, 2001).

Lemma 2. *Consider a vector $\zeta \in \Re^n$, a symmetric matrix $\mathcal{P} \in \Re^{n\times n}$ and a matrix $\mathcal{B} \in \Re^{m\times n}$, such that rank$(\mathcal{B}) < n$. The following statements are equivalent:*

(i) $\zeta'\mathcal{P}\zeta < 0$, $\forall\zeta$ *such that* $\mathcal{B}\xi = 0$, $\zeta \neq 0$.
(ii) $(\mathcal{B}^{\perp})'\mathcal{P}\mathcal{B}^{\perp} < 0$.
(iii) $\exists\mu \in \Re : \mathcal{P} - \mu\mathcal{B}'\mathcal{B} < 0$.
(iv) $\exists\mathcal{F} \in \Re^{n\times m} : \mathcal{P} + \mathcal{F}\mathcal{B} + \mathcal{B}'\mathcal{F}' < 0$.

The following proposition can then be stated based upon the use of Lemmas 1 and 2.

Proposition 1. *If there exist a symmetric positive definite matrix W, matrices Y_1, Y_2, Y_3, X, Z_1, Z_2, F_{21}, F_{1j}, F_{2j}, $j = 4, ..., 9$ and two diagonal matrices S_1, S_2 satisfying*[1] *:*

$$\begin{bmatrix} M_1 + N_1 + N_1' & M_2 - F_1 + N_2' \\ \star & M_3 - F_2 - F_2' \end{bmatrix} < 0 \qquad (20)$$

$$\begin{bmatrix} W & Y_1' & W\mathbb{K}_{(i)}' - Y_{1(i)}' \\ \star & sym(S_1 + \mathbb{R}_3 Y_2 \mathbb{R}_4') & S_1 \mathbb{R}_4 D_c' \mathbb{R}_{3(i)}' - \mathbb{R}_4 Y_2' \mathbb{R}_{3(i)}' \\ \star & \star & u_{1(i)}^2 \end{bmatrix} \geq 0, i = 1, ..., m + p \qquad (21)$$

$$\begin{bmatrix} W & W\mathbb{C}_{3(i)}' - Y_{3(i)}' \\ \star & y_{0(i)}^2 \end{bmatrix} \geq 0, i = 1, ..., p \qquad (22)$$

$$\begin{bmatrix} W & Y_1' & W\mathbb{C}_{2(i)}' \\ \star & sym(S_1 + \mathbb{R}_3 Y_2 \mathbb{R}_4') + \mathbb{R}_4 Y_2' \mathbb{R}_3' & S_1 \mathbb{D}_{2(i)}' \\ \star & \star & z_{0(i)}^2 \end{bmatrix} \geq 0 \quad i = 1, ..., l \qquad (23)$$

with

$$M_1 = \begin{bmatrix} \mathbb{A}W + W\mathbb{A}' & \mathbb{B}S_1 + \mathbb{R}_2 Z_1 \mathbb{R}_3' - Y_1' & \mathbb{R}_2 Z_2 - Y_3' \\ \star & -sym(S_1 + \mathbb{R}_3 Y_2 \mathbb{R}_4') & \mathbf{0} \\ \star & \star & -2S_2 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} W\mathbb{C}_1' & \mathbf{0} & \mathbf{0} \\ \star & \mathbf{0} & \mathbf{0} \\ \star & \star & \mathbf{0} \end{bmatrix} ; \; M_3 = \begin{bmatrix} \mathbf{0} & -\mathbb{R}_4' S_1 & S_2 \\ \star & \mathbf{0} & \mathbf{0} \\ \star & \star & \mathbf{0} \end{bmatrix}$$

$$F_1 = \begin{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix} F_{21} & F_{14} & F_{17} \\ \mathbf{0} & F_{15} & F_{18} \\ \mathbf{0} & F_{16} & F_{19} \end{bmatrix} ; \; F_2 = \begin{bmatrix} F_{21} & F_{24} & F_{27} \\ \mathbf{0} & F_{25} & F_{28} \\ \mathbf{0} & F_{26} & F_{29} \end{bmatrix}$$

$$N_1 = \begin{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix} X'\mathbb{R}_1' & F_{14} & F_{17} \\ \mathbf{0} & F_{15} & F_{18} \\ \mathbf{0} & F_{16} & F_{19} \end{bmatrix} ; \; N_2 = \begin{bmatrix} X'\mathbb{R}_1' & F_{24} & F_{27} \\ \mathbf{0} & F_{25} & F_{28} \\ \mathbf{0} & F_{26} & F_{29} \end{bmatrix}$$

then the anti-windup gains $E_c = Z_1 \mathbb{R}_3' S_1^{-1} \mathbb{R}_3 = Z_1 S_{12}^{-1}$ (if we denote $S_1 = diag(S_{11}, S_{12})$), $F_c = Z_2 S_2^{-1}$, the observer gain $L = X(F_{21}')^{-1}$ and the set $S_0 = \{\xi \in \Re^{2n+n_c}; \xi' W^{-1} \xi \leq 1\}$ are solutions to Problem 1.

Proof. According to the nonlinearities Φ and $\phi_{y_0}^{\epsilon}$, Lemma 1 applies as follows.

- In the case of Φ, one considers:

$$\nu = \mathbb{K}\xi + \mathbb{R}_3 D_c \mathbb{R}_4' \Phi$$
$$w = G_1 \xi + \mathbb{R}_3 G_2 \mathbb{R}_4' \Phi$$
$$v_0 = u_1 = \begin{bmatrix} y_0' & u_0' \end{bmatrix}' \in \Re^{p+m}$$

- In the case of $\phi_{y_0}^{\epsilon}$, one considers:

$$\nu = \mathbb{C}_3 \xi; w = G_3 \xi; v_0 = y_0$$

Consider $G_1 = Y_1 W^{-1}$, $G_2 = Y_2 \mathbb{R}_4' S_1^{-1} \mathbb{R}_4 = Y_2 S_{11}^{-1}$, if we denote $S_1 = diag(S_{11}, S_{12})$, and $G_3 = Y_3 W^{-1}$. Then, one has to prove that the set $S_0 = \{\xi \in \Re^{2n+n_c}; \xi' W^{-1} \xi \leq 1\}$ is included in $S(u_1)$ and in $S(y_0)$. For this, one has first to satisfy, $\forall i = 1, ..., m + p$:

$$\begin{bmatrix} \xi \\ \Phi \end{bmatrix}' \begin{bmatrix} \mathbb{K}_{(i)}' - G_{1(i)}' \\ \mathbb{R}_4(D_c' - G_2')\mathbb{R}_{3(i)}' \end{bmatrix} \begin{bmatrix} \mathbb{K}_{(i)}' - G_{1(i)}' \\ \mathbb{R}_4(D_c' - G_2')\mathbb{R}_{3(i)}' \end{bmatrix}' \begin{bmatrix} \xi \\ \Phi \end{bmatrix} \leq u_{1(i)}^2$$

for ξ and Φ such that $\begin{cases} \xi' W^{-1} \xi \leq 1 \\ \Phi T(\Phi + G_1 \xi + \mathbb{R}_3 G_2 \mathbb{R}_4' \Phi) \leq 0 \end{cases}$.
Thus, the satisfaction of relation (21) ensures that the above condition is satisfied and therefore that the set S_0 is included in $S(u_1)$. By using the same type or arguments one can prove that the satisfaction of relation (22) guarantees that the set S_0 is included in $S(y_0)$.

Consider the quadratic Lyapunov function $V(\xi) = \xi' P \xi$ with $P = P' > 0$. Its time-derivative writes:

$$\begin{aligned} \dot{V}(\xi) = \; & \xi'((\mathbb{A} + \mathbb{R}_1 L \mathbb{C}_1)' P + P(\mathbb{A} + \mathbb{R}_1 L \mathbb{C}_1))\xi \\ & +2\xi' P(\mathbb{B} + \mathbb{R}_2 E_c \mathbb{R}_3' - \mathbb{R}_1 L \mathbb{R}_4')\Phi \\ & +2\xi' P(\mathbb{R}_2 F_c + \mathbb{R}_1 L)\phi_{y_0}^{\epsilon} \end{aligned}$$

and therefore satisfies for all $\xi \in S_0$:

$$\begin{aligned} \dot{V}(\xi) \leq \; & \dot{V}(\xi) - 2\Phi' T_1(\Phi + G_1 \xi + \mathbb{R}_3 G_2 \mathbb{R}_4' \Phi) \\ & -2\phi_{y_0}^{\epsilon}{}' T_2(\phi_{y_0}^{\epsilon} + G_3 \xi) \end{aligned}$$

Using some algebraic manipulations, this last inequality can write as

$$\begin{bmatrix} \xi' P & \Phi' T_1 & \phi_{y_0}^{\epsilon}{}' T_2 \end{bmatrix} M \begin{bmatrix} P\xi \\ T_1 \Phi \\ T_2 \phi_{y_0}^{\epsilon} \end{bmatrix}$$

with

$$\begin{aligned} M &= M_1 + N M_2' + M_2 N' + N M_3 N' \\ &= \begin{bmatrix} \mathbf{1} & N \end{bmatrix} \begin{bmatrix} M_1 & M_2 \\ M_2' & M_3 \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ N' \end{bmatrix} \end{aligned}$$

where $W = P^{-1}$, $S_1 = T_1^{-1}$, $S_2 = T_2^{-1}$, $N = \begin{bmatrix} \mathbb{R}_1 L & \mathbf{0} & \mathbf{0} \\ \star & \mathbf{1} & \mathbf{0} \\ \star & \star & \mathbf{1} \end{bmatrix}$ and matrices M_1, M_2, M_3 are previously defined. Thus, in order to satisfy $\dot{V}(\xi) < 0$ one has to satisfy $M < 0$. By applying Lemma 2, the satisfaction of $M < 0$ is equivalent to find multipliers F_1 and F_2 such that

$$\begin{bmatrix} M_1 & M_2 \\ M_2' & M_3 \end{bmatrix} + \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \begin{bmatrix} N' & -\mathbf{1} \end{bmatrix} + \begin{bmatrix} N \\ -\mathbf{1} \end{bmatrix} \begin{bmatrix} F_1' & F_2' \end{bmatrix} < 0$$

By choosing matrices F_1 and F_2 as

$$F_1 = \begin{bmatrix} F_{11} & F_{14} & F_{17} \\ \mathbf{0} & F_{15} & F_{18} \\ \mathbf{0} & F_{16} & F_{19} \end{bmatrix} ; \; F_2 = \begin{bmatrix} F_{21} & F_{24} & F_{27} \\ \mathbf{0} & F_{25} & F_{28} \\ \mathbf{0} & F_{26} & F_{29} \end{bmatrix}$$

with $F_{11} = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix} F_{21}$, and by using the change of variables $LF_{21}' = X$, the satisfaction of relation (20) means that for all $\xi \in S_0$, one gets $\dot{V}(x) < 0$. Since this reasoning is valid for all $\xi \in S_0$, $\xi \neq 0$, one can conclude that the set S_0 is a set of asymptotic stability for the closed-loop system. Moreover, the satisfaction of relation (22) guarantees that for all $\xi \in S_0$ the controlled output z remains bounded in the set Z_0. Therefore, one can conclude that the conditions of Proposition 1 allows to exhibit solutions to Problem 1. □

Proposition 1 provides solutions to Problem 1 in a local context. In the absence of controlled output

[1] The symbol $\star$ stands for symmetric blocks. Moreover, $sym(A) = A + A'$.

limitations (i.e. $z_0 \to \infty$) and provided that some open-loop stability assumptions are verified for matrix A, the global asymptotic stability can be addressed as follows.

Corollary 1. *If there exist a symmetric positive definite matrix W, matrices X, Z_1, Z_2, F_{21}, F_{1j}, F_{2j}, $j = 4, ..., 9$ and two diagonal matrices S_1, S_2 satisfying relation (20) with*

$$M_1 = \begin{bmatrix} \mathbb{A}W + W\mathbb{A}' & \mathbb{B}S_1 + \mathbb{R}_2 Z_1 \mathbb{R}_3' - W\mathbb{K}' & \mathbb{R}_2 Z_2 - W\mathbb{C}_3' \\ \star & -sym(S_1 + \mathbb{R}_3 D_c \mathbb{R}_4' S_1) & \mathbf{0} \\ \star & \star & -2S_2 \end{bmatrix}$$

then the anti-windup gains $E_c = Z_1 \mathbb{R}_3' S_1^{-1} \mathbb{R}_3 = Z_1 S_{12}^{-1}$ (if we denote $S_1 = diag(S_{11}, S_{12})$), $F_c = Z_2 S_2^{-1}$, the observer gain $L = X(F_{21}')^{-1}$ are such that the global asymptotic stability of the closed-loop system is ensured.

Proof. By considering $G_1 = \mathbb{K}$, $G_2 = D_c$, and $G_3 = \mathbb{C}_3$ one gets $S(u_1) = \Re^{2n+n_c}$ and $S(y_0) = \Re^{2n+n_c}$, and therefore, by applying Lemma 1 it follows that the sector conditions relative to the nonlinearities Φ and $\phi_{y_0}^{\epsilon}$ are globally satisfied, that is, for all $\xi \in \Re^{2n+n_c}$. □

An important feature of the proposed conditions resides in the separation between the Lyapunov matrix P, which appears through the matrix W, and the decision variable L. Furthermore the matrix F_{21} is a general (nonsingular) matrix, that is it does not present any structural constraint such that the symmetry. Such a feature allows to cope with other constraints on the variables. This results appears to be potentially less conservative than in the quadratic case (without the use of Lemma 2) due to the fact that here the Lyapunov matrix P remains structurally unconstrained (differently from the quadratic case where matrix P has to be diagonal, even to be related to diagonal matrices S_1 and S_2).

4. NUMERICAL ISSUES

4.1 Discussion

It is worth to notice that the conditions in Proposition 1 are under LMI form in the decision variables. This fact is due to use of model (16) with Lemma 1. The use of classical sector condition as in (Gomes da Silva Jr. *et al.*, 2002) or of polytopic model as in (Cao *et al.*, 2002) does not lead to LMI conditions but to BMI conditions in the case of only actuator saturation. It is not difficult to see that this same drawback will appear when we consider in plus the saturation on the sensor and the addition of an observer loop. Thus, in these cases, the exhibition of anti-windup gains associated to an observer gain maximizing the estimate of the region of stability should be done by means of iterative shemes. Such solutions are very sensitive to the initial considered guess and local sub-optimality can be guaranteed. In the current paper, the solution does not require initial guesses neither iterative schemes.

Moreover, the extension of our approach to provide global asymptotic stability condition (in the case where the open-loop system is stable and when we do not consider bounds on the controlled output signal) is direct as shown in Corollary 1. When a polytopic model is considered it is not possible to exhibit a condition ensuring global stability. This is due to the local character of such a model for representing the saturated system.

The case without observer can be considered. In this case, the closed-loop systems reads:

$$\begin{aligned} \dot{\zeta} &= \mathbb{A}_1 \zeta + (\mathbb{B}_1 + \mathbb{R}_{21} E_c \mathbb{R}_3')\Phi \\ z &= \mathbb{C}_{21}\zeta + \mathbb{D}_2 \Phi \end{aligned} \tag{24}$$

where $\zeta = \begin{bmatrix} x' & \eta' \end{bmatrix}' \in \Re^{n+n_c}$ and matrices $\mathbb{A}_1$, $\mathbb{B}_1$, $\mathbb{R}_{21}$ and $\mathbb{C}_{21}$ are the appropriate parts of matrices defined in (15).

4.2 Optimization issues

For design purposes, according to Proposition 1, a set Ξ_0 defined by its vertices:

$$\Xi_0 = co\{v_j, j = 1, ..., r, v_j \in \Re^{2n+n_c}\}$$

may be used as a shape set to serve for optimizing the size of the set S_0. Two cases can be addressed if we consider the set Ξ_0 and a scaling factor β. Thus, we want to satisfy $\beta\ \Xi_0 \subset S_0$. In case 1, this problem reduces to a feasibility problem with $\beta = 1$. In case 2, the objective consists in maximizing β, which corresponds to define through Ξ_0 the directions in which we want to maximize S_0. Thus, the problem of maximizing β can be accomplished by solving the following convex optimization problem:

$$\begin{gathered} \min \mu \\ \text{subject to relations } (20), (21), (22), (23) \\ \begin{bmatrix} \mu & v_j' \\ v_j & W \end{bmatrix} \geq 0, \quad j = 1, \ldots, r \end{gathered} \tag{25}$$

Considering $\beta = 1/\sqrt{\mu}$, the minimization of μ implies the maximization of β.

4.3 Illustrative example

Consider system (1) described by the following data:

$$\begin{gathered} A = 0.1; B = 1; C = 1; C_2 = 1; D_2 = 1 \\ u_0 = 0.5; y_0 = 5 \end{gathered}$$

with the simple PI controller defined by

$$A_c = 0; B_c = -0.2; C_c = 1; D_c = -2$$

By solving the optimization problem (25), where Ξ_0 is the hypercube centered in 0 and with vertices components equal to 1 or -1, one gets $E_c =$

0.2736, $F_c = 0.0015$, $L = -1.0001$, $\beta = 1.6491$ with a volume of the ellipsoid $\mathcal{V} = 475.2802$. In the case without anti-windup, one gets $\beta = 0.9590$ and $\mathcal{V} = 28.5379$. The size of the region of stability with the anti-windup scheme is enlarged of 72%.

5. CONCLUDING REMARKS

In this paper, LMI conditions have been proposed to treat the anti-windup gain design for linear systems subject to both actuator and sensor saturation. We have particularly focused on the problem of enlargement of the basin of attraction of the closed-loop system, whereas the controlled output signal remains bounded. The anti-windup loops use some measured varaibles (due to the actuator) and some estimated one (due to the sensor) issued from an observer step. The obtained conditions are proposed via the use of a modified sector condition and Finsler's Lemma which allows to cope with some drawback appearing when using other modelling for the closed-loop system (like classical sector condition or polytopic model).

Some discussion about the observer step and the possibility to build additional anti-windup loops (in particular dynamic anti-windup loops) will be provided in a more complete version of the paper. Moreover, in a future work the input-to-state stabilization problem for linear systems subject to $\mathcal{L}_2$-disturbances should be addressed.

REFERENCES

Åström, K. J. and L. Rundqwist (1989). Integrator windup and how to avoid it. In: *American Control Conference*. Pittsburgh, PA. pp. 1693–1698.

Barbu, C., R. Reginatto, A. R. Teel and L. Zaccarian (2000). Anti-windup for exponentially unstable linear systems with inputs limited in magnitude and rate. In: *American Control Conference*. Chicago, IL.

Bernstein, D. S. (2001). A plant taxonomy for designing control experiments. *IEEE Control Systems Magazine* pp. 7–14.

Cao, Y-Y., Z. Lin and D.G. Ward (2002). An antiwindup approach to enlarging domain of attraction for linear systems subject to actuator saturation. *IEEE Transactions on Automatic Control* **47**(1), 140–145.

Cao, Y.Y., Z. Lin and B.M. Chen (2003). An output feedback h_∞ controller design for linear systems subject to sensor nonlinearities. *IEEE Transactions on Circuits and Systems-I* **50**(7), 914–921.

de Oliveira, M. C. and R. E. Skelton (2001). *Stability Tests for Constrained Linear Systems.* pp. 241–257. in Perspectives in Robust Control, Lecture Notes in Control and Information Sciences. Springer-Verlag.

Fliegner, T., H. Logemann and E.P. Ryan (2000). Time-varying and adative integral control of linear systems with actuator and sensor nonlinearities. In: *Proc. of 15th International Symposium on Mathematical Theory of Networks and Systems (MTNS)*. Perpignan, France.

Glattfelder, A.H. and W. Schaufelberger (2003). *Control systems with input and output constraints*. Springer-Verlag, London.

Gomes da Silva Jr., J.M. and S. Tarbouriech (2003). Anti-windup design with guaranteed regions of stability: an LMI-based approach. In: *Proceedings of the 42th IEEE Conference on Decision and Control*. Hawaii, USA.

Gomes da Silva Jr., J.M., S. Tarbouriech and G. Garcia (2003). Local stabilization of linear systems under amplitude and rate saturating actuators. *IEEE Transactions on Automatic Control* **48**(5), 842–847.

Gomes da Silva Jr., J.M., S. Tarbouriech and R. Reginatto (2002). Analysis of regions of stability for linear systems with saturating inputs through an anti-windup scheme. In: *Proceedings of the IEE Conference on Control Applications (CCA)*. Glasgow, Scotland.

Kapila, V. and K. Grigoriadis (2002). *Actuator saturation control*. Marcel Dekker, Inc.

Koplon, R., M.L.J Hautus and E.D. Sontag (1994). Observability of linear systems with saturated outputs. *Linear Algebra and Applications* **205-206**, 909–936.

Kothare, M. V. and M. Morari (1999). Multiplier theory for stability analisys of anti-windup control systems. *Automatica* **35**, 917–928.

Kreisselmeier, G. (1996). Stabilization of linear systems in the presence of output measurement saturation. *scl* **29**, 27–30.

Murray, R.M. (1999). Geometric approach to control in the presence of magnitude and rate saturation. *Technical Report 99-001, Division of Eng. and Appl. Science, California Institute of Technology.*

Tarbouriech, S., C.Prieur and J.M. Gomes da Silva Jr. (2004). Stability analysis and stabilization of systems presenting nested saturations. In: *Proc. of 43rd Conference on Decision and Control (CDC'04)*. Paradise Island, Bahamas.

Teel, A. R. (1999). Anti-windup for exponentially unstable linear systems. *Int. J. Robust and Nonlinear Control* **9**(10), 701–716.