HIERARCHICAL MODELS OF NONSTATIONARY FLOWS IN COMPLEX GAS TRANSMISSION NETWORKS

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Abstract: Consideration was given to simulation of the nonstationary flows in a complex configuration of gas transmission systems. Simulation is defined as a generalized solution of the initial boundary problem for nonlinear partial derivative equation with discontinuous coefficients describing the nonstationary distributions of pressures and flows in the system. General solution is defined by the finite-volume methods with the use of multigrid approximation. The proposed methods have no counterparts and urgent for solving the problems involved in scheduling the flows in gas transmission systems. *Copyright ©2005 IFAC.*

Keywords: gas transmission network, hierarchical or multilevel: models &decomposition, multigrid &parallel computation methods.

1. INTRODUCTION

The problems of controlling the processes of gas transmission in the Gas Transmission Systems (GTS) as a whole cannot be resolved without universal adequate models of the distributions of pressures and flows in them. In the general case, development of these models encounters difficulties due to high dimensionality of the problem and existence in the motion equation of discontinuous parameters, inertial and nonlinear terms. Dimensionality is defined by the chosen step of the grid approximation of the initial boundary for the initial nonlinear equation. The discontinuous parameters admit existence only of the generalized solution. Dimensionality of the problem increases dramatically with allowance for the nonlinear dependences because solution of the system of nonlinear equations usually comes down to solution of a sequence of linear equation systems, the other methods being either inadmissible or unadvisable. At abrupt jumps of the boundary conditions in emergencies, the inertial term can give rise to the wave pressure fronts, which generates the need for much finer steps of the grid approximation, that is, for even higher increase in dimensionality. Therefore, the challenge of the grid (finite-difference, finite-volume, and others) approximations, that is, simulation, lies in the high dimensionality of the grid space of solution approximation. For the physical GTS's, it is, therefore, the necessary computations are impossible without their parallelization using the multiprocessor computer systems.

In the normal conditions, the gas flow in the pipeline actually is not affected by the inertia forces because the wave components lack in the solution of (1). Consequently, a simplified system without the inertial term can be considered instead (1).

For the isothermal $(T = const)$ flow of the real gas $(p/\rho = zRT)$, the nonstationary pressure distribution $p(x, t)$ along a unbranched pipeline segment at time t is described by the nonlinear partial derivative equations

$$(2\lambda \, | \, q \, | \, / \, d)\partial P / \partial t - \partial^2 P / \partial x^2 = 0,$$
$$0 \le x \le l, t \ge 0,$$
(1)

where $P = p^2$ is the squared pressure, $q = \rho w$ is the mass flow through the unit area of the pipe hydraulic section, $\rho = p/c^2$ is the gas density, z is the gas compressibility

coefficient, R is the gas constant, T is the absolute temperature, c is the acoustic speed in gas, λ is the coefficient of hydraulic resistance, d is the diameter of the pipe hydraulic section, and l is the pipeline segment length.

The initial boundary conditions are:

$$P(0,t) = P_{\text{lt}}, \; p(l,t) = P_{\text{rt}}(t); P(0,t) = P_{\text{lt}},$$
$$q(l,t) = q_{\text{rt}}(t); q(0,t) = q_{\text{lt}}(t), q(l,t) = q_{\text{rt}}(t), \quad (2)$$
$$P(x,0) = P_0(x),$$

$P_0(x)$ is the pressure distribution along the section at the initial time instant $t = 0, P_{\text{lt}}(t), P_{\text{rt}}(t), q_{\text{lt}}(t), q_{\text{rt}}(t)$ are the given functions of pressure and flow variations at the ends of the given segment over the time interval $[0, T]$ under consideration. Under these conditions, the two-point initial boundary problems for the nonlinear parabolic equation (1) are posed correctly. Stated differently, continuous variations of the solution of (1) correspond to the continuous variations of the boundary conditions $x = 0$ and $x = l$ (2).

The initial boundary problem that models the nonstationary gas flow needs the boundary conditions of the first, second, and third kinds at the end and intermediate nodes corresponding to the compressor plants (pressure jump) and runoffs with abrupt changes in pressure and mass flow, that is, the balance either of pressures, $\Delta P(t, x) = P(t, x+0) - P(t, x-0)$, or flows, $\Delta q(t, x) = q(t, x+0) - q(t, x-0)$, or both. In these conditions, the quasilinear uniform equation (1) becomes a nonuniform one with piecewise discontinuous functions $f(t, x)$ in the right-hand side, that is,

$$\chi(P)\partial P/\partial t + \text{divgrad}\, P = f,$$
$$\chi = \lambda |q(P)|/d, 0 \le x \le l, t \ge 0. \quad (1')$$

Consequently, the problem of modeling a nonstationary gas flow in the trunk pipelines of the type at hand is possible only in the generalized case. For example, in the Sobolev space $L_2^1 \supset L_2$, that is, in the subspace of functions which are once differentiable and summed with the squares together with their first derivatives.

2. METHODS OF PERTURBATION FOR MODELING NONSTATIONARY GAS FLOWS

Under the normal network operational conditions, the nonstationary models of gas flow distribution in the scheduling problems are defined by the generalized solution of the initial boundary problem for the quasilinear equation (1') with the boundary conditions of the first, second, and third kinds. The following modification of the perturbation method (Marchuk *et al.*, 1996) is the most effective for the problem of this sort.

Let us represent $\chi(P) = \delta(P)u(P)$, $u(P) = 1 - \beta P$, $\beta \in R$, and after the change of variable rearrange (1') in

$$\delta(P)u(P)(dP/du)(\partial u/\partial t) -$$
$$- \text{div}((dP/du)\,\text{grad}\,u) = f(t, x). \quad (1'')$$

Assume now that $\psi(P) = \delta(P)u(P)dP/du - 1/\alpha$, $\alpha \in R, \xi = \alpha/\beta > 0$, substitute $(1/\alpha) + \varepsilon\psi(P)$ for $\delta(P)u(P)dP/du$ in (1''), where $0 \le \varepsilon \le 1$, and construct the perturbed equation $\partial u/\partial t - \varepsilon F(u) = \alpha f$, where $F(u) = (\xi\chi((u-1)/\beta)-1)\partial u/\partial t$. If $\varepsilon = 1$, then the perturbed equation is reduced to the initial equation (1'), and if $\varepsilon = 0$, then to the nonperturbed equation $\partial u_0/\partial t - \xi\,\text{divgrad}\,u_0 = \alpha f(t, x)$.

By subtracting the nonperturbed equation from the perturbed one, we obtain the equation in the difference $r = u - u_0, \partial r/\partial t - \xi\,\text{divgrad}\,r + \varepsilon\varphi(r + u_0)\partial u_0/\partial t = 0$, where $\varphi(u) = \xi\chi((u-1)/\beta)-1$.

For sufficiently smooth initial data, the initial boundary problem has a unique solution for the nonperturbed equation, and the relation $0 < \xi < 2/\chi_1$, where $\chi_1 = \max_q\{\lambda |q|/d\}$, is the sufficient condition for uniqueness of the solution of the initial boundary problem for the equation in the correction r. If the solution of the nonperturbed problem u_0, is known, then for calculation of the corrections r_j $(j = 1, \ldots, J)$, we get the system of equations $\partial r_1/\partial t - \xi\,\text{divgrad}\,r_1 = -\varphi(u_0)\partial u_0/\partial t$, $\partial r_2/\partial t - \xi\,\text{divgrad}\,r_2 = -\varphi(u_0)\partial u_1/\partial t - \xi\beta^{-1}\chi'u_1\partial u_0/\partial t$, and so on. Having calculated J corrections, one can determine the corresponding approximation by the $u_J = u_0 + \sum_{i=1}^{i=J}\varepsilon^i r_i$ such that $\|u - u_J\| \le c\varepsilon^{(J+1)}$, $c > 0, r_0 = 0$.

3. BALANCE METHODS FOR MODELING THE NONSTATIONARY GAS FLOWS

At each stage of the perturbation algorithm, the operator equations that are identical to within notation are solved. First, the nonperturbed equation is solved, and u_0 is determined. Then, the equations for the corrections $r_j, j = \overline{1, J}$ are solved successively. Therefore, to describe fully the method, it suffices to consider the solution of the nonperturbed equation. At any time layer, the explicit and implicit grid approximations with the time step τ for the nonperturbed equation are defined by the difference-differential equations (the subscript is omitted, that is, $u_0 = u$)

$$(u^{(n+1)} - u^{(n)})\tau^{-1} - (\xi/\beta)\,\text{divgrad}\,u^{(n+1)} = \alpha f^{(n+1)}, \quad (3)$$

$$(u^{(n+1)} - u^{(n)})\tau^{-1} - (\xi/\beta)\,\text{divgrad}\,u^{(n)} = \alpha f^{(n)}. \quad (4)$$

The finite-volume (balance) approximations (3) and (4) in the space coordinate x obey the matrix equations

$$(\tau^{-1}\mathbf{C} + \mathbf{B})\mathbf{u}^{(n+1)} = \mathbf{G}^{(n+1)} + \tau^{-1}\mathbf{C}\mathbf{u}^{(n)}, \quad (5)$$

$$\tau^{-1}\mathbf{C}\mathbf{u}^{(n+1)} = \mathbf{G}^{(n)} + \left(\tau^{-1}\mathbf{C} - \mathbf{B}\right)\mathbf{u}^{(n)}, \qquad (6)$$

where the matrices $\mathbf{B}$ and $\mathbf{C}$ and the vector $\mathbf{G}$ correspond to the finite-volume approximations (Hackbush, 1989) of the terms $(\alpha/\beta^{-2})\,\mathrm{divgrad}\,u$, $\partial u/\partial t$, and αf, respectively. The matrix equations (5) and (6) also turn into the linear systems of the algebraic equations $\mathbf{A}\mathbf{u} = \mathbf{F}$, where $\mathbf{A} = \tau^{-1}\mathbf{C} + \mathbf{B}$, $\mathbf{F} = \mathbf{G}^{(n+1)} + \tau^{-1}\mathbf{C}\mathbf{u}^{(n)}$ and $\mathbf{A} = \tau^{-1}\mathbf{C}$, $\mathbf{F} = \mathbf{G}^{(n)} + (\tau^{-1}\mathbf{C} - \mathbf{B})\mathbf{u}^{(n)}$, respectively.

Equation (5) is solved as follows. The matrices $\mathbf{B}$ and $\mathbf{C}$ and the vector $\mathbf{G}^{(1)}$ are determined from the given initial condition $P(x,t)\big|_{t=0} = P_0(x)$, that is, $\mathbf{u}^{(0)} = \left(u_0(x_1),\dots,u_0(x_K)\right)^{\mathrm{T}}$, using the finite-volume approximation, and then $\mathbf{A} = \mathbf{B} + \mathbf{C}/\tau$ and $\mathbf{F} = \mathbf{G}^{1} + \tau^{-1}\mathbf{C}\mathbf{u}^{(0)}$ are determined form them. Next, the equation $\mathbf{A}\mathbf{u}^{(1)} = \mathbf{F}$ is solved, and the vector of the solution $\mathbf{u}^{(1)}$ on the first time layer is determined. Then, new matrices $\mathbf{B}$ and $\mathbf{C}$ are constructed, and the matrix $\mathbf{A} = \mathbf{B} + \mathbf{C}/\tau$ and the vector $\mathbf{F} = \mathbf{G}^{(2)} + \mathbf{C}\mathbf{u}^{(1)}/\tau$ are established from them, equation $\mathbf{A}\mathbf{u}^{(2)} = \mathbf{F}$ is solved, and the vector of the solution $\mathbf{u}^{(2)}$ is determined on the second time layer. By continuing this process, we get the solution of the initial boundary problem for equation (5) on all time layers t_n, $n = \overline{1,[T/\tau]}$ of the modeling time period $[0,T]$ under consideration.

Equation (6) is solved along similar lines, but the external iteration on each time layer are much simpler because for the finite-volume approximation the matrix $\mathbf{C}$ has only diagonal nonzero elements and the vectors of the solution $\mathbf{u}^{(n+1)}$ are defined explicitly by the vector of $\mathbf{u}^{(n)}$ from the preceding layer. In order to make calculations stable, however, one needs to observe certain relations between the steps in time and along the space coordinate.

For the finite-volume approximation on the n th time layer, the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ for equations (5) and (6) obey the following schemes of calculations. On each trunk branch of the gas pipeline $\Omega = [0,l]$ we take points with the coordinates x_k, $0 = x_1 \cdots \le x_k < \cdots < x_N = l$, that is, construct the cells $\Omega_k = (x_k, x_{k-1})$, $k = \overline{1,N-1}$. The left and right halves of the cell Ω_k are represented by the open intervals $\Omega'_k = (x_k, (x_k + x_{k+1})/2)$, $\Omega''_k = ((x_k + x_{k+1})/2, x_k)$. Then, construct the finite volumes $\widetilde{\Omega}_k = \Omega''_{k-1} \cup \Omega'_k$ $k = \overline{1,N}$, around each internal node (Fig. I).
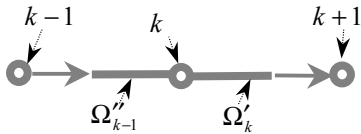

Fig. I. Definition of finite volume around internal node.

The finite volumes corresponding to the boundary nodes x_1 and x_N are defined as $\widetilde{\Omega}_1 = \Omega'_1$ and $\widetilde{\Omega}_N = \Omega''_{N-1}$. On the Ω_k, we introduce the local coordinates $x_1^{(k)} = x_k$, $x_2^{(k)} = x_{k+1}$ and notation $u_1^{(k)} = u_k$, $u_2^{(k)} = u_{k+1}$ and $f_1^{(k)} = f_k$, $f_2^{(k)} = f_{k+1}$. Then, integration of (4) over the finite volume $\widetilde{\Omega}_k$ enables us to construct the local matrices $\mathbf{B}^{(k)}$ and $\mathbf{C}^{(k)}$ and the right-hand side vector $\mathbf{G}^{(k)}$ corresponding to the finite volumes $\widetilde{\Omega}_k$, $k = \overline{1,N}$, of the simple node (Fig. I)

$$\mathbf{B}^{(k)} = \begin{pmatrix} b_{11}^{(k)} & b_{12}^{(k)} \\ b_{21}^{(k)} & b_{22}^{(k)} \end{pmatrix}, \mathbf{C}^{(k)} = \begin{pmatrix} c_{11}^{(k)} & c_{12}^{(k)} \\ c_{21}^{(k)} & c_{22}^{(k)} \end{pmatrix}$$
$$\mathbf{G}^{(k)} = \left(g_1^{(k)}, g_2^{(k)}\right)^{\mathrm{T}}.$$

The elements of the global matrices $\mathbf{B} = \{b_{ij}\}_{i,j=\overline{1,N}}$ and $\mathbf{C} = \{c_{ij}\}_{i,j=\overline{1,N}}$ and the vector $\mathbf{G} = (g_1, g_2, \dots, g_N)^{\mathrm{T}}$ of the right-hand side are determined from the elements of the local matrices $\mathbf{B}^{(k)}$ and $\mathbf{C}^{(k)}$ and the local vector $\mathbf{G}^{(k)}$ by means of assembling, that is, using the operation of assembly $\mathrm{Ass}\{\cdot\}_{k=\overline{1,N}}$ of the elements of the local matrices according to the global numeration of the finite volumes or nodes of the grid decomposition, that is,

$$\mathbf{B} = \mathrm{Ass}\left\{\mathbf{B}^{(k)}\right\}_{k=\overline{1,N}}, \mathbf{C} = \mathrm{Ass}\left\{\mathbf{C}^{(k)}\right\}_{k=\overline{1,N}},$$
$$\mathbf{G} = (g_1, g_2, \dots, g_N)^{\mathrm{T}}.$$

Where

$$b_{11} = b_{11}^{(1)},\dots, b_{kk} = b_{22}^{(k)} + b_{11}^{(k+1)},\dots, b_{NN} = b_{22}^{(N)},$$
$$b_{12} = b_{21} = b_{12}^{(1)} = b_{21}^{(1)},\dots, b_{k(k+1)} = b_{(k+1)k} =$$
$$= b_{12}^{(k)} = b_{21}^{(k)},\dots, b_{N(N-1)} = b_{(N-1)N} = b_{12}^{(N)} = b_{21}^{(N)};$$
$$c_{11} = c_{11}^{(1)},\dots, c_{kk} = c_{22}^{(k)} + c_{11}^{(k+1)},\dots, c_{NN} = c_{22}^{(N)},$$
$$c_{12} = c_{21} = c_{12}^{(1)} = c_{21}^{(1)},\dots, c_{k(k+1)} = c_{(k+1)k} =$$
$$= c_{12}^{(k)} = c_{21}^{(k)},\dots, c_{N(N-1)} = c_{(N-1)N} = c_{12}^{(N)} = c_{21}^{(N)};$$
$$g_1 = g_1^{(1)},\dots, g_k = g_2^{(k-1)} + g_1^{(k)},\dots, g_2^{(N)}.$$

For arbitrarily configured pipeline networks, the balance models may be generated in a similar way. It suffices to specify the means for constructing the finite volumes corresponding to the pipeline connecting the nodes. For the network node k (see Fig. II) where three branches meet, the finite volume can be defined by $\widetilde{\Omega}_k = \Omega''_{k-1} \cup \Omega'_k \cup \Omega'_{N+1}$, $k = \overline{1,N}$. Then, the corresponding rows of the matrices $\mathbf{B}$ and $\mathbf{C}$ and the vector $\mathbf{G}$ consist of the following nonzero elements:

$$b_{k(k-1)} = b_{12}^{(k)}, b_{kk} = b_{22}^{(k)} + b_{11}^{(k+1)} + b_{11}^{(N+1)},$$
$$b_{k(k+1)} = b_{12}^{(k)}, b_{k(N+1)} = b_{12}^{(N+1)},$$

$$c_{k(k-1)} = c_{12}^{(k)}, c_{kk} = c_{22}^{(k)} + c_{11}^{(k+1)} + c_{11}^{(N+1)}, c_{k(k+1)} = c_{12}^{(k)},$$
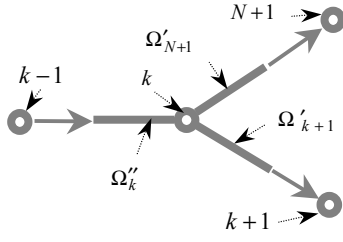$$c_{k(N+1)} = c_{12}^{(N+1)}, g_k = g_2^{(k-1)} + g_1^{(k)} + g_1^{(N+1)}.$$

Fig. II. Definition of finite volume around 3-branch node.

For arbitrarily configured GTS's, a finite (star-like) volume is constructed around each node where three or more trunk pipelines meet, and the corresponding nonzero elements $\mathbf{A}$ and $\mathbf{G}$ are calculated.

For discontinuous boundary conditions, the set of nodes of the grid decomposition must include the locations of the intermediate compressor plants, inflows, and discharges including underground storage facilities. For the finite volume $\widetilde{\Omega}_k = \Omega''_{k-1} \cup \Omega'_k$ corresponding to a compressor plant or source (inflow, runoff, underground storage facility) at the active node k, the grid approximation of equation (1) along the space coordinate must take into consideration the abrupt change in pressure or flow before and after the node under consideration.

4. STRUCTURAL SYSTEM DECOMPOSITION

Since the structure of the matrix $\mathbf{A}$ is defined by the numeration of the approximation nodes on the trunk branches of the GTS network, the need appears for establishing the optimal order of numeration making the balance mathematical models most efficient. Stated differently, the approximation nodes must be numerated so that the matrix $\mathbf{A}$ allows one to decompose the computational procedure in a way providing the minimum time losses and amount of computations. To attain this goal, one needs to construct an oriented graph with the sets of vertices and edges corresponding to the nodes and branches of the GTS at hand. Structural decomposition can be done using the heuristic methods of parallel and embedded sections with node numeration by the inverse Cuthill&McKee's algorithm. The GTS structure is distinguished for complicated subsystems with embedded loops that are connected by narrow elongated (possibly, tree-like) gas pipelines. For such systems, the aforementioned methods enable representation of $\mathbf{A}$ as a coordinated ordered block matrix with profile arrow-like structure. Therefore, the matrix $\mathbf{A}$ is represented as

$$\mathbf{A} = \begin{pmatrix} \mathbf{D}_R & \mathbf{H} \\ \mathbf{H}^{\mathrm{T}} & \mathbf{D}_B \end{pmatrix}, \mathbf{D}_R = \mathbf{diag}(\mathbf{D}_R^1, \mathbf{D}_R^1, \dots, \mathbf{D}_R^m),$$

$$\begin{pmatrix} \begin{pmatrix} \mathbf{D}_R^1 & \mathbf{H}_1 \\ \mathbf{H}_1^{\mathrm{T}} & \mathbf{D}_B^1 \end{pmatrix} & 0 & \dots & 0 & \mathbf{H}^1 \\ 0 & \begin{pmatrix} \mathbf{D}_R^2 & \mathbf{H}_2 \\ \mathbf{H}_2^{\mathrm{T}} & \mathbf{D}_B^2 \end{pmatrix} & \dots & 0 & \mathbf{H}^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \begin{pmatrix} \mathbf{D}_R^m & \mathbf{H}_m \\ \mathbf{H}_m^{\mathrm{T}} & \mathbf{D}_B^m \end{pmatrix} & \mathbf{H}^m \\ (\mathbf{H}^1)^{\mathrm{T}} & (\mathbf{H}^2)^{\mathrm{T}} & \dots & (\mathbf{H}^m)^{\mathrm{T}} & \mathbf{D}_B \end{pmatrix}, (7)$$

where $\mathbf{D}_R^i, i = \overline{1,m}$, and $\mathbf{D}_B$ и $\mathbf{D}_B^i, i = \overline{1,m}$ are the conventional or block-tridiagonal matrices if the corresponding pipeline branch has one or more runs, $\mathbf{H} = (\mathbf{H}_1, \mathbf{H}_2, \dots \dots, \mathbf{H}_m)^{\mathrm{T}}$ and $\mathbf{H}^{\mathrm{T}} = (\mathbf{H}_1^{\mathrm{T}}, \mathbf{H}_2^{\mathrm{T}}, \dots, \mathbf{H}_m^{\mathrm{T}})$ are the sparse commutative block matrices of connections, some of the block may be zero. At that, the corresponding structural decomposition and numeration of the nodes are defined by the $\Omega = (\Omega_R = \Omega_1 \cup \Omega_2 \cup \dots \cup \Omega_m) \cup \Omega_B$ and $I(\Omega_R) = = \{I(\Omega_1), I(\Omega_2), \dots, I(\Omega_m)\}$, $I(\Omega) = \{I(\Omega_R), I(\Omega_R)\}$

$$I(\Omega_i) = \{N_{i-1} + 1, \dots, N_{i-1} + N_R^i, N_{i-1} + N_R^i + 1, \dots$$
$$\dots, N_{i-1} + N_R^i + N_B^i\}, N_0 = 0, |I(\Omega_i)| = N_i, i = \overline{1,m}.$$

It deserves noting that the block-tridiagonal representation of (7) of $\mathbf{A}$ is the most effective one for modeling the multi-run corridors of the trunk pipelines with locks.

For the GTS of configuration of any complexity, the corresponding matrix $\mathbf{A}$ is representable in the block-diagonal form similar (9). It is only the number, location, and types of the embedded blocks that can vary.

5. BLOCK PARALLELIZATION OF COMPUTATIONS

To solve equation $\mathbf{AP} = \mathbf{F}$ on the n th time layer, it is advisable to use the following scheme of the method of upper relaxation:

$$\mathbf{D}_R \mathbf{u}_R^{(2s-1)} = \omega\left(\mathbf{F}_R - \mathbf{H}\mathbf{u}_B^{(2s)}\right) + (1-\omega)\mathbf{D}_R \mathbf{u}_R^{(2s-2)},$$
$$\mathbf{D}_B \mathbf{u}_B^{(2s)} = \omega\left(\mathbf{F}_B - \mathbf{H}^{\mathrm{T}}\mathbf{u}_R^{(2s-1)}\right) + (1-\omega)\mathbf{D}_B \mathbf{u}_B^{(2s-2)}, \quad (8)$$

where $1 < \omega < 2/1 + \delta$ is the relaxation parameter (δ is a constant) and the superscripts $(s+1)$ and $(2s)$ correspond to the numbers of iterations along the space coordinate. At x_{k+1} that, calculations by the first and second rows of (10) are carried out alternatively only at the odd and even iterations, respectively.

If the matrix $\mathbf{A}$ has the form (7), then the multilevel realization of the method follows the computational scheme with three levels of iteration. The first (odd) iteration is carried out using the given initial approximation $\mathbf{u}_B^{(0)}$ and $\mathbf{D}_R \mathbf{u}_R^{(1)} = \omega\left(\mathbf{F}_R - \mathbf{H}\mathbf{u}_B^{(0)}\right) + (1-\omega)\mathbf{D}_R \mathbf{u}_R^{(0)}$ and $\mathbf{u}_R^{(1)}$ is determined. Then, the second (even) iteration $\mathbf{D}_B \mathbf{u}_B^{(2)} = = \omega\left(\mathbf{F}_B - \mathbf{H}^{\mathrm{T}}\mathbf{u}_R^{(1)}\right) + (1-\omega)\mathbf{D}_B \mathbf{u}_B^{(0)}$ is performed, and $\mathbf{u}_B^{(2)}$ is determined. The following pairwise iterations

$(s := s+1)$ are carried out in a similar manner, provided that the condition $\| \mathbf{u}^{(2s+1)} - \mathbf{u}^{(2s-1)} \| > \varepsilon$ is satisfied. Realizations of the first, $\mathbf{D}_R \mathbf{u}_R^{(1)} = \omega\left(\mathbf{F}_R - \mathbf{H}\mathbf{u}_B^{(0)}\right) + (1-\omega)\mathbf{D}_R \mathbf{u}_R^{(0)}$, and subsequent odd iterations, that is, $2s+1$, $s > 1$, consist of similar embedded iterative procedures for solution of the subproblems of the middle level

$$\begin{pmatrix} \mathbf{D}_R^i & \mathbf{H}_i \\ \mathbf{H}_i^{\mathrm{T}} & \mathbf{D}_B^i \end{pmatrix} \begin{pmatrix} \mathbf{u}_R^i \\ \mathbf{u}_B^i \end{pmatrix} = \begin{pmatrix} \mathbf{F}_R^i \\ \mathbf{F}_B^i \end{pmatrix}, i = \overline{1,m}.$$

To this end, first the solutions $\mathbf{u}_B^i$, $i = \overline{1,m}$, $i = \overline{1,m}$, of the vector equations $\mathbf{D}_B^i \mathbf{u}_B^i = \mathbf{F}_B^i - \mathbf{H}_i^{\mathrm{T}} \mathbf{u}_R^i$, $i = \overline{1,m}$, are determined using (8), and then the solutions $\mathbf{u}_R^i$, $i = \overline{1,m}$, of the subproblems of the lower (third) hierarchical level, that is, the vector equations $\mathbf{D}_R^i \mathbf{u}_R^i = \mathbf{F}_R^i - \mathbf{H}_i \mathbf{u}_B^i$, $i = \overline{1,m}$, are determined for fixed values of $\mathbf{u}_B^i, i = \overline{1,m}$. By means of the resulting

$\mathbf{u}^{(2s+1)} = \left((\mathbf{u}_R^1, \mathbf{u}_R^2)^{(2s+1)}, (\mathbf{u}_B^1, \mathbf{u}_B^2, \mathbf{u}_B^3)^{(2s)} \right)^{\mathrm{T}}$ one can execute the next iteration ($s := s+1$) to solve the equation $\mathbf{A}\mathbf{u}^{(2s+1)} = \mathbf{F}$ at the upper level. Therefore, the main computational process consists of three identical procedures. For one-run and multiple-run GTS branches of the conventional and block-tridiagonal structures, one may use the effective method of simple factorization to solve the equations at any hierarchical level.

To improve precision and efficiency of the proposed approach without considerable fining of the grid (without increasing the dimensionality of the balance equations), it is necessary to employ multi-grid methods based on the relaxation scheme in particular, by the parallelized variant of the universal multi-grid for the one-dimensional, in the space coordinate, problems of modeling.

The basic algorithm of balance modeling consists of the following stages: (i) construction of the grid $\Omega = (\Omega_R = \Omega_1 \cup \Omega_2 \cup \ldots \cup \Omega_m) \cup \Omega_B$ with the optimal node ordering (10), (ii) finite-volume (balance) approximation of the boundary problem on the constructed grid corresponding to the considered GTS or UGTS as a whole, (iii) construction and ordering of the grid equations $\mathbf{A}\mathbf{u} = \mathbf{F}$, and (iv) solution of the equation system $\mathbf{A}\mathbf{u} = \mathbf{F}$ with multilevel parallelization of calculations.

6. MULTIGRID SCHEME OF PARALLELIZATION OF CALCULATIONS

If the numbers N_R^i and $N_B^i, i = \overline{1,m}$, are multiples of three, then it is possible to parallelize calculations by constructing a hierarchical structure. The set of the grid points of the balance approximation is represented as a union of the set of nodes and boundaries of the finite volumes. The zero level of roughness $\Omega(0,1) = \Omega^{\mathrm{v}}(0,1) \cup \Omega^{\mathrm{f}}(0,1)$,

$\Omega^{\mathrm{f}}(0,1) = \{x_k^{\mathrm{f}} : x_k^{\mathrm{f}} = 0,5(x_k^{\mathrm{v}} + x_{k+1}^{\mathrm{v}}), k = \overline{1,N}\}$,
$\Omega^{\mathrm{v}}(0,1) = \{x_k^{\mathrm{v}} : x_k^{\mathrm{v}} \in \Omega, k = 1, N\}$.

The initial grid $\Omega(0,1)$ is represented as a union of three more rough disjoint grids of the first level, that is, $\Omega(0,1) = \cup_{\alpha=1}^{\alpha=3} \Omega(1,\alpha)$, $\Omega(1,\alpha) \cap \Omega(1,\beta) = \phi, \alpha \neq \beta$. Each grid $\Omega(1,\alpha), \alpha = \overline{1,3}$, is considered recurrently as the initial grid for the grids $\Omega(2,\alpha), \alpha = 1,\ldots,3^2$, and the resulting more rough nine grids make up the second level, and so on. Construction of more rough grids lies in eliminating two points Ω^{v} and Ω^{f} as shown in Fig. III. The multigrid scheme of hierarchical calculations can be organized as follows. The initial equation systems like (5) or (6) that correspond to the considered GTS or UGTS are rearranged by the change of variables $\mathbf{u} = \hat{\mathbf{u}} + \mathbf{v}$ in
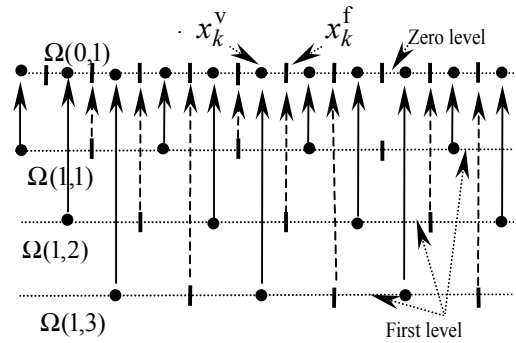

Fig. III. Operator of multigrid method (first level).

$$\left(\tau^{-1}\hat{\mathbf{C}} + \hat{\mathbf{B}}\right)\mathbf{v}^{(n+1)} = \mathbf{G}^{(n+1)} + \tau^{-1}\hat{\mathbf{C}}\hat{\mathbf{u}}^{(n)} - \left(\tau^{-1}\hat{\mathbf{C}} + \hat{\mathbf{B}}\right)\hat{\mathbf{u}}^{(n+1)} \quad (5')$$

$$\tau^{-1}\hat{\mathbf{C}}\mathbf{v}^{(n+1)} = \mathbf{G}^{(n)} + \left(\tau^{-1}\hat{\mathbf{C}} - \hat{\mathbf{B}}\right)\hat{\mathbf{u}}^{(n)} - \tau^{-1}\hat{\mathbf{C}}\hat{\mathbf{u}}^{(n)}. \quad (6')$$

With regard for the boundary conditions like (2) transformed in a similar way, systems (5′) or (6′) can be rearranged in the zero level system

$$\hat{\mathbf{A}}_0 \mathbf{V}_0 = \mathbf{J}_0, \mathbf{J}_0 = (J_1,\ldots,J_N)^{\mathrm{T}}, \\ \mathbf{v}_0 = (v_1^0,\ldots,v_N^0)^{\mathrm{T}}, \quad (9)$$

where the matrix $\hat{\mathbf{A}}_0$ has the characteristic block-arrow structure as above. The equation systems of the first, second, and so on levels are constructed according to the schemes depicted in Figs. III and IV.
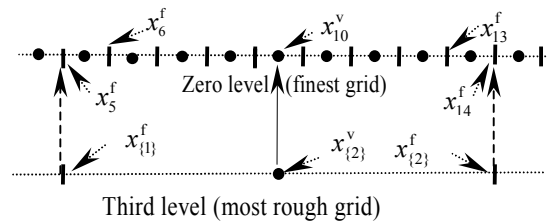

Fig. IV. Rough set of the third level ($L^+ = 3$) .

For the multi-grid structure with three hierarchical levels, the third-level grids corresponding to the nodes must comply with the scheme of Fig. III, which means that the integral $J_{\{2\}}^{L=3}$ in the right-hand side of the equation corresponding to the finite volume $[x_{\{1\}}^f, x_{\{2\}}^f] \equiv [x_5^f, x_{14}^f]$ on the rough set of the third level must be calculated as the sum of integrals over the nine finite volumes of the finest grid of the zero level because $[x_{\{1\}}^f, x_{\{2\}}^f] \equiv [x_5^f, x_{14}^f] = [x_5^f, x_6^f] \cup$

$\cup [x_6^f, x_7^f] \cup \cdots \cup [x_{12}^f, x_{13}^f] \cup [x_{13}^f, x_{14}^f]$. The row of the matrix $\widehat{\mathbf{A}}_3$ corresponding to the finite volume $[x_{\{1\}}^f, x_{\{2\}}^f]$ is calculated as usual directly on the rough grid. Systems of equations for all levels can be constructed similarly:

$$\widehat{\mathbf{A}}_L \mathbf{v}_L = \mathbf{J}_L, \ \mathbf{J}_L = (J_1^L, \ldots, J_{N_L}^L)^\tau,$$
$$\mathbf{v}_L = (v_1^L, \ldots, v_{N_L}^L)^\tau, L = 1, \ldots, L^+. \tag{10}$$

The multi-grid iterations start at the level of the roughest grids L^+. The components of the vector $\mathbf{v}_L$ calculated on the roughest grids are added to the corresponding components of the vector of approximate solution $\widehat{\mathbf{u}}$ determined at the previous iteration.

If the time of data transfer between the main processor and its modules is negligible, then according to the asymptotic values of the acceleration $S_M = T(1)/T(M)$, where $T(1)$ and $T(M)$ are, respectively, the times at using one and M processors, and the efficiency $E_M = S_M/M$ obey the following relations:

$$S_M \to M - (M-1)T_0 / \sum_{l=1}^{L^+} T_L,$$
$$E_M \to 1 - (M-1)T_0 / M \sum_{l=1}^{L^+} T_L,$$

where T_0 and T_L are the time losses at the zero and L th levels. The acceleration and efficiency may be improved if the results of computations on the rough grids of the first level are assumed to be the basic ones that are usually referred to as dynamic and the saw-shaped dynamic cycle is executed.

For particular GTS's as a whole, the saw-shaped multi grid scheme of computations with dynamic cycle is used for each level $1 < L \le L^+$ beginning from the roughest grids of the level L^+. Stated differently, if $1 < L \le L^+$ then the corresponding equation systems of the form (10) are solved for any rough grid $\Omega(L,1)$, $\Omega(L,2)$, and $\Omega(L,3)$. Therefore, for each time layer, the GTS and UGTS balance models on the whole are defined by the solutions of equations like (10), the computations consist of $L^+ + 1$ hierarchical levels of the multi-grid scheme. At any level of the multi-grid scheme, for all rough grids of a given hierarchical level the computations follows scheme (8) according to the structure of the decomposition of the matrix $\widehat{\mathbf{A}}_0$, that is,

$\mathbf{D}_R^i, i = \overline{1,m}$, $\mathbf{D}_B$ and $\mathbf{D}_B^i, i = \overline{1,m}$. Consequently, each such fragment of the internal procedure consists in turn of three hierarchical levels of block iterations of the form (8).

Convergence of the multilevel computations for the balance methods of modeling the non-stationary gas flows in gas pipelines of any complexity depends on convergence of the solutions of the sequence of equations at the external iterations according to the perturbation algorithm and the sequence of linear equation systems in the space coordinates on each time layer, that is, on the internal iterations. Repeated computational experiments with the models of complex pipeline networks corroborate convergence of the external and internal sequences to the generalized solution of the quasi-linear system.

The structure of the matrix $\mathbf{A}$ will be, obviously, defined by the order of numeration of the nodes of the grid approximation on the branch pipelines of the GTS network, which gives rise to the need for choosing an optimal order of numeration providing the highest efficiency of the variation mathematical models.

7. CONCLUSIONS

The main advantage of the multilevel balance models of the distribution of gas pressures and flows in arbitrarily configured gas pipelines is due to their extreme simplicity and universality. These distinctions are most prominent if the structural characteristics of the pipeline networks of any complexity and the corresponding grid approximations are taken into consideration with an accuracy defined actually by the source data error. Simplicity and universality of these models manifest themselves also at taking into account the impact of the control actions at the intermediate nodes where the compressor plants and other objects (sources, buffer and other consumers) with controlled gas supply and runoff are situated.

The external hierarchical multi-grid iterations with saw-shaped dynamic cycle on all rough networks of each level $L: 1 < L \le L^+$ coupled with the internal iterations (8) make the performance of the proposed computational methods hardly improvable.

The balance models of the non stationary modes of gas flow in he gas pipelines of arbitrary configuration enable one to solve the main problems of GTS scheduling on the whole and in the most general and topical formulation.

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