PEAK CONTROLLERS FOR UNCERTAIN LINEAR DELAYED SYSTEMS UNDER INITIAL CONDITIONS

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Abstract: This paper extends peak-to-peak gain control of linear system to linear delayed systems. In the proposed method, the supremum of 2-norm is used instead of L_∞-norm, and local stabilization is used instead of global stabilization. While the reachable set of the dynamic delayed systems is bounded by an inescapable ellipsoid, a state feedback controller is determined under a given initial state condition. As main results, the state ball of initial condition for the system bearing time-delay is explicitly computed. The corresponding peak index controller is formulated in terms of linear matrix inequalities. The robustness and minimization problems of peak-to-peak gain control are also discussed.

Keywords: Performance indices, Reachable states, Initial state, Robustness, Time-delay.

1. INTRODUCTION

For some control engineering problems, it is often important to ensure that whenever the exogenous input signal has bounded amplitude, the controlled output signal should have bounded or minimum amplitude. For examples, motor control problems with electrical or mechanical restrictions and chemical process control problems with concentration restrictions, the violation of the amplitude restrictions can lead to performance degradation and possibly catastrophic system failure. In these cases, energy-based synthesis techniques, such as H_∞ control, are often inadequate for these types of performance problems.

A measure of above amplitude-based performance is often denoted by peak-to-peak gain, which is based on L_1 or induced L_∞ norms. Theoretically, L_1 optimization problem has not obtained good results for continuous time systems (Dahleh and Pearson, 1987), and it is even more difficult than l_1 optimization problem for discrete time systems (Vidyasagar, 1986; Diaz-Bobillo and Dahleh, 1993). By using l_1 methods and letting sub-optimal L_1 to approach optimal L_1 index arbitrarily, some L_1 optimization problems can be solved (Blanchini and Sznaier, 1994; Dahleh and Diaz-Bobillo, 1995). But this approach needs to deal with large-scale linear programming and the order of resulting controller is high. To avoid the complexity, one approach to directly estimate the peak-to-peak gain of continuous time linear systems is to bound the reachable set of system with an invariant ellipsoid (Abedor *et al*., 1996; Boyd *et al*., 1994; Blanchini, 1999).

Since the time-delay often occurs in many practical situations. The controller synthesis of time-delay systems has attracted much interest in the literature for several decades. Many theories and techniques of linear systems had been extended to delayed systems. On the other hand, almost all control systems operate with uncertain parameters, the robust synthesis problems, i.e. peak-to-peak gain control against the uncertainties of delayed systems are more important. However, up to now, it seems that no literature involves the topic about the peak-to-peak gain control for uncertain linear delayed systems.

In this paper, a state feedback controller for time-delay systems is determined with an associated set of initial conditions. The main results are based on the Lyapunov-Krasovskii theorem and are motivated by the results (Abedor *et al*., 1996; Boyd *et al*., 1994; Tarbouriech *et al*., 2000; Blanchini, 1999), where the local stabilization is used instead of global stabilization. By means of L_∞-norm defined as the supremum of 2-norm, the so-called peak-to-peak gain control problem is formulated in state space. While the initial state of the closed-loop system is constrained by a ball in a state space, the trajectory of the system is restricted in an invariant ellipsoid. And then the corresponding controller can be constructed in terms of linear matrix inequalities (LMIs). Furthermore, the results are expanded to cover the time-delay systems with norm-bounded uncertainties. The robustness and minimization problems are discussed.

Notations: R^n denotes the n dimensional Euclidean space, and $R^{n\times m}$ denotes the set of all $n\times m$ real matrices. R^+ is the set of nonnegative real number. $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote, respectively, the maximal and minimal eigenvalue of matrix P. $C_{\tau,n}=C([-\tau,0],R^n)$ denotes the Banach space of continuous vector functions mapping the interval $[-\tau,0]$ into R^n with the topology of uniform convergence, in which the norm is defined by $\|\phi\|_c=\sup_{\theta\in[-\tau,0]}\|\phi(\theta)\|$. $\|\cdot\|$ refers to either the Euclidean vector norm or the induced matrix 2 - norm.

2. MAIN RESULTS

Consider linear delayed systems described by following state equation:

$$\begin{cases} \dot{x}(t)=Ax(t)+A_1x(t-\tau_1)+Bw(t) \\ \qquad +B_1u(t)+B_2u(t-\tau_2), \\ z(t)=Cx(t)+Dw(t). \end{cases} \quad (1)$$

with initial condition

$$x(t_0+\theta)=\phi(\theta),\ \phi(\theta)\in C_{\tau,n}, \quad (2)$$
$$\theta\in[-\max(\tau_1,\tau_2),0],$$

where $x(t)\in R^n$ and $u(t)\in R^m$ are the state and input respectively, $x(t-\tau_1)$ and $u(t-\tau_2)$ are the delayed state and input with time delay $\tau_1,\tau_2>0$. A,A_1,B, B_1,B_2, C,D are known real constant matrices of appropriate dimensions. Let G denote the transfer function from $w(t)$ to $z(t)$, the peak-to-peak gain of system (1) is defined as

$$\|G\|_{pp}=\sup\{\|z(T)\|\mid x(0)=0,T>0,$$
$$\|w(t)\|\le 1,\forall t\ge 0\},$$

where the supremum of 2-norm of all the t is used instead of L_∞-norm. For a given peak-to-peak gain level $\gamma>0$, the state feedback control law $u(t)=Kx(t)$ is said to be a peak-to-peak gain controller, if, for the closed-loop system, peak-to-peak gain is less than γ.

Assume the energy functional follows as

$$L(x_t)=x^TPx+\int_{t-\tau_1}^{t}x^T(\omega)S_1x(\omega)d\omega$$
$$+\int_{t-\tau_2}^{t}x^T(\omega)S_2x(\omega)d\omega,$$

where $x_t=x(t+\theta)$, $\theta\in[-\max(\tau_1,\tau_2),0]$, and $0<P,S_1,S_2\in R^{n\times n}$ are symmetric matrices. For the state of closed-loop system, define an invariant ellipsoid as

$$\Omega(P,\eta)=\{x\in R^n \mid \ x^TPx<\eta^{-1},\eta>0\}.$$

Associated with above invariant ellipsoid, assume the initial condition is restricted in following state ball with radius ν

$$\Phi(\nu)=\{\phi\in C_{\tau,n}:\|\phi\|_c^2\le\nu,\nu>0\}.$$

In general, based on the Lyapunov-Krasovskii theorem, the closed-loop system is said to be global stabilization, if there exists real number $\varepsilon>0$ such that the derivative of energy functional satisfies $\dot{L}(x_t)\le-\varepsilon\|x(t)\|^2$. Intuitively, the closed-loop system is said to be local stabilization, if there exist real number $\varepsilon,\eta>0$ such that $\dot{L}(x_t)\le-\varepsilon\|x(t)\|^2$ while the state of closed-loop system beyond the invariant ellipsoid $\Omega(P,\eta)$. Furthermore, for time-delay systems, the invariant ellipsoid $\Omega(P,\eta)$ is associated with a initial state ball $\Phi(\nu)$.

Applying local stabilization, the peak-to-peak gain control problem is extended to linear delayed systems. Compared with linear systems, linear delayed systems bound the reachable set with an invariant ellipsoid. The reachable set depends on a given initial condition. As main results, in following theorem, the associated ball of initial state, which guarantees the invariant ellipsoid, is explicitly computed.

Also for a given peak-to-peak gain level $\gamma>0$, following theorem gives a method to construct state feedback controller such that the resulting time-delay closed-loop system satisfies $\|G\|_{pp}<\gamma$. To simplify the presentation for sequent discussion, the variables t on x,w et al. will be dropped, and the shorthand $x_1=x(t-\tau_1)$ and $x_2=x(t-\tau_2)$ are used.

Theorem 1: For systems (1), if there exists real number $\eta>0$ such that following LMIs (4) and (5) are solvable on positive-definite symmetric matrices $X,V_1,V_2\in R^{n\times n}$, matrix $Y\in R^{m\times n}$ and real number $\delta>0$, the peak-to-peak gain controller is given by $u=YX^{-1}x$ and the state ball $\Phi(\nu)$ of initial condition is given by (3).

$$\nu = \frac{\eta^{-1}}{\lambda_{\max}(X^{-1}) + \tau_1 \lambda_{\max}(N_1) + \tau_2 \lambda_{\max}(N_2)}. \quad (3)$$

$$\Xi(X, V_1, V_2, \eta, \delta) = \begin{bmatrix} M & A_1 X & B_2 Y & B & X \\ X A_1^T & -V_1 & 0 & 0 & 0 \\ Y^T B_2^T & 0 & -V_2 & 0 & 0 \\ B^T & 0 & 0 & -I & 0 \\ X & 0 & 0 & 0 & -\delta I \end{bmatrix} < 0, \quad (4)$$

$$\Pi(X, \eta, \gamma) = \begin{bmatrix} \eta X & 0 & X C^T \\ 0 & (\gamma - 1) I & D^T \\ C X & D & \gamma I \end{bmatrix} > 0, \quad (5)$$

where

$$N_1 = X^{-1} V_1 X^{-1},$$
$$N_2 = X^{-1} V_2 X^{-1},$$
$$M = AX + XA^T + Y^T B_1^T + B_1 Y + V_1 + V_2 + \eta X .$$

Proof: From the definition of peak-to-peak gain, while $\|w\| \le 1$, assume

$$\dot{L}(x_t) + \eta x^T P x - \|w\|^2 < -\varepsilon \|x\|^2 . \quad (6)$$

It implies $\dot{L}(x_t) < -\varepsilon \|x\|^2$ while $x^T P x \ge \eta^{-1}$, i.e., the reachable set of delayed systems (1) is bounded by invariant ellipsoid $\Omega(P, \eta)$. Considering state feedback controller, the corresponding closed-loop systems are given by

$$\dot{x} = (A + B_1 K) x + A_1 x_1 + B_2 K x_2 + B w.$$

With above assumption (6), the closed-loop systems are said to be local stabilization. After calculating $\dot{L}(x_t)$, the inequality (6) is rewritten as follows

$$\begin{aligned} & x^T (A^T P + PA + K^T B_1^T P \\ & + P B_1 K + S_1 + S_2 + \eta P + \varepsilon I) x \\ & + x_1^T A_1^T P x + x^T P A_1 x_1 + x_2^T K^T B_2^T P x \\ & + x^T P B_2 K x_2 - x_1^T S_1 x_1 - x_2^T S_2 x_2 \\ & + w^T B^T P x + x^T P B w - w^T w < 0. \end{aligned} \quad (7)$$

For nonzero vector $\begin{bmatrix} x & x_1 & x_2 & w \end{bmatrix}^T$, above inequality is then guaranteed by following inequality,

$$\begin{bmatrix} W & P A_1 & P B_2 K & P B \\ A_1^T P & -S_1 & 0 & 0 \\ K^T B_2^T P & 0 & -S_2 & 0 \\ B^T P & 0 & 0 & -I \end{bmatrix} < 0, \quad (8)$$

where

$$\begin{aligned} W = & A^T P + PA + K^T B_1^T P + P B_1 K \\ & + S_1 + S_2 + \eta P + \varepsilon I. \end{aligned}$$

Pre- and post-multiply above (8) by block-diagonal matrix $Diag\{P^{-1}\ P^{-1}\ P^{-1}\ I\}$, let $X = P^{-1}$, $Y = KX$, $V_1 = X S_1 X$, $V_2 = X S_2 X$, apply Schur complements, and at last let $\varepsilon^{-1} = \delta$, inequality (8) is equivalent to (4). On the other hand, let

$$\|z\|^2 < \gamma(\eta x^T P x + (\gamma - 1)\|w\|^2), \quad (9)$$

Keep in mind the condition $\|w\| \le 1$ and invariant ellipsoid $\Omega(P, \eta)$, the above inequality leads output signal $z(t)$ bounded by amplitude level γ. From the definition of peak-to-peak gain, obviously, it follows $\|G\|_{pp} < \gamma$. Substituting $z = Cx + Dw$, above (9) can be rewritten as following

$$\begin{aligned} & \eta x^T P x + (\gamma - 1) w^T w - \gamma^{-1} (x^T C^T C x \\ & + x^T C^T D w + w^T D^T C x + w^T D^T D w) \\ & > 0. \end{aligned} \quad (10)$$

Considering $X = P^{-1}$, obviously, above (10) is guaranteed by (5). Summing up, the inequalities (4) and (5) are guaranteed by invariant ellipsoid $\Omega(P, \eta)$ with given η. But the invariant ellipsoid depends on an associated initial state restriction. By the energy functional, following inequality can be concluded

$$L(x_t) \le \alpha \|x_t\|_c^2, \quad (11)$$

where

$$\alpha = \lambda_{\max}(X^{-1}) + \tau_1 \lambda_{\max}(N_1) + \tau_2 \lambda_{\max}(N_2) .$$

In the meantime, it is also obvious that

$$x^T(t) P x(t) \le L(x_t) . \quad (12)$$

For local stable closed-loop systems, following inequality holds

$$x^T(t) P x(t) \le L(x_t) \le L(x_{t_0}) . \quad (13)$$

Rewriting (11), letting

$$x_{t_0} = x(t_0 + \theta) = \phi(\theta) \in \Phi(\nu) ,$$

and substituting ν with (3), it follows

$$L(x_{t_0}) \le \alpha \|x_{t_0}\|_c^2 = \alpha \|\phi(\theta)\|_c^2 = \eta^{-1} . \quad (14)$$

Hence, combining (13) and (14), for any initial condition in ball $\Phi(\nu)$, $x(t) \in \Omega(P, \eta), \forall t > t_0$ is guaranteed, or $\Omega(P, \eta)$ is the invariant ellipsoid of state space. In other words, the ball which constraint the initial condition is explicitly given by (3). □

3. ROBUSTNESS AND GAIN MINIMIZATION

The robustness of peak-to-peak gain control is discussed directly from above results. Consider systems (1) with norm-bounded uncertainties described as follows

$$\begin{cases} \dot{x}(t) = Ax(t) + A_1 x(t-\tau_1) + Bw(t) \\ \quad + B_1 u(t) + B_2 u(t-\tau_2) + B_3 p(t), \\ z(t) = Cx(t) + Dw(t), \\ q(t) = Ex(t) + E_1 x(t-\tau_1) \\ \quad + F_1 u(t) + F_2 u(t-\tau_2), \\ p(t) = \Delta(t) q(t), \end{cases} \tag{15}$$

where $p(t) \in R^h$ and $q(t) \in R^l$ are the variables of the uncertain part of systems, $\Delta(t)$ is unknown real time-varying matrices with Lebesgue measurable elements satisfying $\|\Delta(t)\| \le I$, and B_3, E, E_1, F_1, F_2 are known real constant matrices of appropriate dimensions.

A feedback controller is said to be a robust peak-to-peak gain controller for uncertain systems (15), if for some scalar η, the corresponding closed-loop systems are robust local stable associated with invariant ellipsoid $\Omega(P,\eta)$ and initial ball $\Phi(\nu)$, and satisfies $\|G\|_{pp} < \gamma$ for all admissible uncertainties.

Theorem 2: For uncertain systems (15), if there exists $\eta > 0$ such that $\Pi(X,\eta,\gamma) > 0$ and following LMI (16) is jointly solvable on positive-definite symmetric matrices X, V_1, V_2, matrix Y and positive real number δ, β, the robust peak-to-peak gain controller is given by $u = YX^{-1}x$, and ν is given by (3)

$$\begin{bmatrix} \Xi(X,V_1,V_2,\eta,\delta) & \beta\overline{B}_3 & \overline{E}^T(X,Y) \\ \beta\overline{B}_3^T & -\beta I & 0 \\ \overline{E}(X,Y) & 0 & -\beta I \end{bmatrix} < 0 \tag{16}$$

where

$$\overline{E}(X,Y) = \begin{bmatrix} EX + F_1 Y & E_1 X & F_2 Y & 0 & 0 \end{bmatrix},$$

$$\overline{B}_3 = \begin{bmatrix} B_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Proof: The uncertainties of closed-loop systems satisfy $p(t) = \Delta(t)q(t)$ and $\|\Delta(t)\| \le I$, i.e.

$$\xi_1 < 0, \tag{17}$$

while letting

$$\begin{aligned} \xi_1 = {} & p^T p - x^T (E + F_1 K)^T (E + F_1 K) x \\ & - x^T (E + F_1 K)^T E_1 x_1 \\ & - x^T (E + F_1 K)^T F_2 K x_2 \\ & - x_1^T E_1^T (E + F_1 K) x - x_1^T E_1^T E_1 x_1 \\ & - x_1^T E_1^T F_2 K x_2 - x_2^T K^T F_2^T (E + F_1 K) x \\ & - x_2^T K^T F_2^T E_1 x_1 - x_2^T K^T F_2^T F_2 K x_2 . \end{aligned}$$

Similar to the proof of Theorem 1, by calculating $\dot{L}(x_t)$, the inequality (6) is rewritten as follows

$$\xi_2 < 0, \tag{18}$$

where

$$\begin{aligned} \xi_2 = {} & x^T (A^T P + PA + K^T B_1^T P \\ & + PB_1 K + S_1 + S_2 + \eta P + \varepsilon I) x \\ & + x_1^T A_1^T P x + x^T P A_1 x_1 + x_2^T K^T B_2^T P x \\ & + x^T P B_2 K x_2 - x_1^T S_1 x_1 - x_2^T S_2 x_2 \\ & + w^T B^T P x + x^T P B w - w^T w \\ & + p^T B_3^T P x + x^T P B_3 p. \end{aligned}$$

Applying S-procedure, for any real number $\beta > 0$, inequalities (17) and (18) holds, if $\xi_2 - \beta\xi_1 < 0$. Furthermore, condition (16) is guaranteed by $\xi_2 - \beta\xi_1 < 0$. The details of derivation are similar to the proof of Theorem 1 and here are omitted. □

Remark 1: Obviously, from (16), the conditions of Theorem 2 imply the conditions of Theorem 1. Above Theorem 1 and Theorem 2 are formulated the synthesis of controller as LMI form. Directly, the least peak-to-peak gain may be obtained by performing a linear search over η while minimizing γ subject to (5) and (4), or (16). On the other hand, for a given gain level γ, the largest bound ν^* on ball of initial condition which is given by (3) may be obtained by searching the least η satisfying the LMIs in Theorem 1 or Theorem 2. In fact, to get the largest bound ν^* completely, it need to minimize the parameter η and the term

$$\lambda_{\max}(X^{-1}) + \tau_1 \lambda_{\max}(X^{-1} V_1 X^{-1}) + \tau_2 \lambda_{\max}(X^{-1} V_2 X^{-1}) .$$

in the meantime. Obviously, this is very difficult and is left for further research.

4. NUMERICAL EXAMPLES

Consider uncertain delayed systems (15) with following matrices

$$A = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0.1 \\ 0.1 & 0.1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}, \ B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$
$$B_2 = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, \ B_3 = \begin{bmatrix} 0 \\ \rho \end{bmatrix},$$
$$C = [1 \quad 1], \ D = 0.5, \ E = E_1 = [1 \quad 1],$$
$$F_1 = F_2 = 1, \ \tau_1 = 0.8, \ \tau_2 = 0.5$$

where a parameter ρ is introduced to characterize the bound of the uncertainties.

Example 1: Assume there has no uncertainties, for given bound $\eta = 1.5$, applying Theorem 1, the peak-to-peak gain controller is obtained by $u = [-6.2040 \quad -3.7266]x$ with gain level $\gamma = 2.0502$, correspondingly the ball of initial condition follows with $\nu = 0.0183$. The least gain level can reach $\gamma^* = 1.2071$.

Example 2: On the other hand, if let $\gamma = 1.5$, applying Theorem 1, the maximum $\eta = 17.7669$, while let $\gamma = 5$, the maximum $\eta = 19.5511$. In other words, while the peak-to-peak gain level is allowed to be lesser, the largest bound η^{-1} on the invariant ellipsoid may be enlarged.

Example 3: Assume the parameter $\eta = 1.5$, $\rho = 0.5$, Applying Theorem 2, the robust peak-to-peak gain controller $u = -[4.6237 \quad 2.9289]x$ is obtained with gain level $\gamma = 2.2725$, correspondingly the ball of initial condition follows with $\nu = 0.0196$.

5. CONCLUSION

By bounding the reachable set of system with an invariant ellipsoid, the peak-to-peak gain control problems have been extended to a class of linear delayed systems. The controller depends on the state ball of initial conditions and is then constructed in terms of linear matrix inequalities. Based on these results, the robustness and gain minimization problems can be obtained directly.

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