DIRECT ADAPTIVE CONTROL FOR NONLINEAR UNCERTAIN SYSTEMS WITH TIME DELAY

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Abstract: A direct adaptive control framework for nonlinear uncertain delay dynamical systems is developed. The proposed framework is Lyapunov-Krasovskii-based and guarantees asymptotic stability with respect to the plant states. Specifically, if the nonlinear system is represented in normal form, then it is shown that nonlinear adaptive controllers can be constructed without requiring knowledge of the system dynamics except the system delay amount. Furthermore, in the case where the system is particularly given in a multivariable second-order form, the adaptive control law is shown to be simplified and constructed without even requiring the information of the delay amount. Finally, a numerical example is provided to demonstrate the efficacy of the proposed approach.

Keywords: Adaptive control, delay dynamical systems, adaptive stabilization, unknown time delay, Lyapunov-Krasovskii method, multivariable second-order systems

1. INTRODUCTION

The presence of time delay effects in complex, modern controlled systems can severely degrade closed-loop system performance, and in some cases drive the system to instability. Furthermore, it is unavoidable that there exist discrepancies between real-world systems and their system models that are constructed for control purposes. It is easily surmised that the applying controls to a physical system involving coupled sources of these effects may produce highly undesirable system response such as oscillatory behavior, actuator failure, and even chaos.

In the face of such system uncertainties as well as time delays, research on adaptive control methodologies is still far from complete. Specifically, even though recent notable results concerning adaptive controllers is given in Foda & Mahmoud (1998), Wu (2000), Wu (2002), and Niculescu & Annaswamy (2003), these approaches can handle either linear or a very special class of nonlinear systems with *known* system delays to show *ultimate boundedness* (practical stability) rather than Lyapunov stability.

In this paper we develop an adaptive control framework for *nonlinear* uncertain systems in the presence of system time delays. In particular, in the first part of the paper, a Lyapunov-Krasovskii-based direct adaptive control framework is developed that requires the knowledge of the system delay amount and guarantees *partial asymptotic stability* of the closed-loop system; that is, Lyapunov stability of the overall closed-loop systems states and attraction with respect to the plant states. As a consequence, the adaptive gain states are shown to be bounded. In the case where the nonlinear system is represented in normal form (Isidori 1995) with input-to-state stable internal dynamics (Sontag 1989, Isidori 1995), we construct nonlinear adaptive controllers *without* requiring knowledge of the system dynamics except the delay amount. In addition, the proposed nonlinear adaptive controllers also guarantee asymptotic stability of the system state if the system dynamics are unknown *and* the input matrix function is parameterized by an unknown constant sign-definite matrix. Finally, in the second part of the paper, we specialize the aforementioned results to multivariable second-order uncertain nonlinear systems. In this case, we remove the assumption that the system delay amount is known. This implies that the adaptive control framework becomes *delay-independent*.

The notation used in this paper is fairly standard. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^{n\times n}$ denotes the set of $n \times n$ real matrices, $(\ )^{\mathrm{T}}$ denotes transpose, and I_n denotes the $n\times n$ identity matrix. Furthermore, we write $\mathrm{tr}(\cdot)$ for the trace operator, $\|\cdot\|$ for the Euclidean vector norm, and $\|\cdot\|_{\mathrm{F}}$ for the Frobenius matrix norm. Finally, $M \otimes N$ denotes the Kronecker product of matrices M and N.

2. DIRECT ADAPTIVE CONTROL FOR DELAY DYNAMICAL SYSTEMS

In this section we consider the problem of characterizing direct adaptive feedback control laws for nonlinear uncertain systems with time delay. Specifically, consider the nonlinear uncertain delay dynamical system $\mathcal{G}$ of the form

$$\dot{x}(t) = f(x(t)) + f_{\mathrm{d}}(x(t), x(t-\tau)) + G(x(t))u(t),$$
$$x(\theta) = \eta(\theta), \quad -\tau \le \theta \le 0, \quad t \ge 0, \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, $f : \mathbb{R}^n \to \mathbb{R}^n$ and satisfies $f(0) = 0$, $f_{\mathrm{d}} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and satisfies $f_{\mathrm{d}}(0,0) = 0$, $\tau \ge 0$ is a system delay amount, $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $\eta(\cdot) \in \mathcal{C} = \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ is a continuous vector-valued function specifying the initial state of the system, and $\mathcal{C}([-\tau, 0], \mathbb{R}^n)$ denotes a Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^n$ equipped with the topology of uniform convergence.

Note that the state of (1) at time t is the piece of trajectories x between $t-\tau$ and t, or, equivalently, the element x_t in the space of continuous functions defined on the interval $[-\tau, 0]$ and taking values in $\mathbb{R}^n$; that is, $x_t \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$, where $x_t(\theta) \triangleq x(t+\theta)$, $\theta \in [-\tau, 0]$. Furthermore, since for a given time t the piece of the trajectories x_t is defined on $[-\tau, 0]$, the uniform norm $|||x_t||| = \sup_{\theta\in[-\tau,0]} \|x(t+\theta)\|$ is used for the definitions of Lyapunov and asymptotic stability of (1) with $u(t) \equiv 0$. For further details see Krasovskii (1963) and Hale & Verduyn Lunel (1993). The control $u(\cdot)$ in (1) is restricted to the class of *admissible controls* consisting of measurable functions such that $u(t) \in \mathbb{R}^m$, $t \ge 0$. Furthermore, for the nonlinear uncertain system $\mathcal{G}$ we assume that the required properties for the existence and uniqueness of solutions are satisfied; that is, $f(\cdot)$, $f_{\mathrm{d}}(\cdot)$, $G(\cdot)$, and $u(\cdot)$ satisfy sufficient regularity conditions such that (1) has a unique solution forward in time.

Theorem 2.1. Consider the nonlinear uncertain delay dynamical system $\mathcal{G}$ given by (1). Assume there exist matrices $K_{\mathrm{g}} \in \mathbb{R}^{m\times s}$, $K_{\mathrm{dg}} \in \mathbb{R}^{m\times s_{\mathrm{d}}}$, a continuously differentiable function $V_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}$, and continuous functions $V_{\mathrm{sd}} : \mathbb{R}^n \to \mathbb{R}$, $\hat{G} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$, $\hat{G}_{\mathrm{d}} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$, $F : \mathbb{R}^n \to \mathbb{R}^s$, $F_{\mathrm{d}} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{s_{\mathrm{d}}}$, and $\ell : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$ such that $V_{\mathrm{s}}(\cdot)$ and $V_{\mathrm{ds}}(\cdot)$ are positive definite, radially unbounded, $V_{\mathrm{s}}(0) = 0$, $V_{\mathrm{sd}}(0) = 0$, $\ell(0,0) = 0$, $F(0) = 0$, $F_{\mathrm{d}}(0,0) = 0$, and, for all $x \in \mathbb{R}^n$ and $x_{\mathrm{d}} \in \mathbb{R}^n$,

$$0 = V_{\mathrm{s}}'(x) f_{\mathrm{s}}(x) + V_{\mathrm{s}}'(x) f_{\mathrm{ds}}(x, x_{\mathrm{d}}) + V_{\mathrm{sd}}(x) - V_{\mathrm{sd}}(x_{\mathrm{d}}) + \ell^{\mathrm{T}}(x, x_{\mathrm{d}})\ell(x, x_{\mathrm{d}}), \quad (2)$$

where

$$f_{\mathrm{s}}(x) \triangleq f(x) + G(x)\hat{G}(x)K_{\mathrm{g}}F(x), \quad (3)$$
$$f_{\mathrm{ds}}(x, x_{\mathrm{d}}) \triangleq f_{\mathrm{d}}(x, x_{\mathrm{d}}) + G(x)\hat{G}_{\mathrm{d}}(x)K_{\mathrm{dg}}F_{\mathrm{d}}(x, x_{\mathrm{d}}). \quad (4)$$

Firthermore, let $Q \in \mathbb{R}^{m\times m}$, $Q_{\mathrm{d}} \in \mathbb{R}^{m\times m}$, $Y \in \mathbb{R}^{s\times s}$, and $Y_{\mathrm{d}} \in \mathbb{R}^{s_{\mathrm{d}}\times s_{\mathrm{d}}}$ be positive definite. Then the adaptive feedback control law

$$u(t) = \hat{G}(x(t))K(t)F(x(t)) + \hat{G}_{\mathrm{d}}(x(t))K_{\mathrm{d}}(t)F_{\mathrm{d}}(x(t), x(t-\tau)), \quad (5)$$

where $K(t) \in \mathbb{R}^{m\times s}$ and $K_{\mathrm{d}}(t) \in \mathbb{R}^{m\times s_{\mathrm{d}}}$, with update laws

$$\dot{K}(t) = -\tfrac{1}{2}Q\hat{G}^{\mathrm{T}}(x(t))G^{\mathrm{T}}(x(t))V_{\mathrm{s}}'^{\mathrm{T}}(x(t)) \cdot F^{\mathrm{T}}(x(t))Y, \quad K(0) = K_0, \quad (6)$$
$$\dot{K}_{\mathrm{d}}(t) = -\tfrac{1}{2}Q_{\mathrm{d}}\hat{G}_{\mathrm{d}}^{\mathrm{T}}(x(t))G^{\mathrm{T}}(x(t))V_{\mathrm{s}}'^{\mathrm{T}}(x(t)) \cdot F_{\mathrm{d}}^{\mathrm{T}}(x(t), x(t-\tau))Y_{\mathrm{d}}, \quad K(0) = K_{\mathrm{d0}}, \quad (7)$$

guarantees that the solution $(x(t), K(t), K_{\mathrm{d}}(t)) \equiv (0, K_{\mathrm{g}}, K_{\mathrm{dg}})$ of the closed-loop system given by (1), (5)–(7) is Lyapunov stable and $\ell(x(t), x(t-\tau)) \to 0$ as $t \to \infty$. If, in addition, $\ell^{\mathrm{T}}(x, x_{\mathrm{d}})\ell(x, x_{\mathrm{d}}) > 0$, $(x, x_{\mathrm{d}}) \in \mathbb{R}^n \times \mathbb{R}^n$, $(x, x_{\mathrm{d}}) \ne (0,0)$, then $|||x_t||| \to 0$ as $t \to \infty$ for all $\eta(\cdot) \in \mathcal{C}$.

Proof. Note that with $u(t)$, $t \ge 0$, given by (5) it follows from (1) that

$$\dot{x}(t) = f(x(t)) + f_{\mathrm{d}}(x(t), x(t-\tau)) + G(x(t))\hat{G}(x(t))K(t)F(x(t)) + G(x(t))\hat{G}_{\mathrm{d}}(x(t))K_{\mathrm{d}}(t)F_{\mathrm{d}}(x(t), x(t-\tau)),$$
$$x(\theta) = \eta(\theta), \quad -\tau \le \theta \le 0, \quad t \ge 0, \quad (8)$$

or, equivalently,

$$\dot{x}(t) = f_{\mathrm{s}}(x(t)) + f_{\mathrm{ds}}(x(t), x(t-\tau)) + G(x(t))\hat{G}(x(t))(K(t) - K_{\mathrm{g}})F(x(t)) + G(x(t))\hat{G}_{\mathrm{d}}(x(t))(K_{\mathrm{d}}(t) - K_{\mathrm{dg}}) \cdot F_{\mathrm{d}}(x(t), x(t-\tau)),$$
$$x(\theta) = \eta(\theta), \ -\tau \le \theta \le 0, \ t \ge 0. \quad (9)$$

To show Lyapunov stability of the closed-loop system (6), (7), and (9) consider the Lyapunov-Krasovskii functional candidate $V : \mathcal{C} \times \mathbb{R}^{m\times s} \times \mathbb{R}^{m\times s_{\mathrm{d}}} \to \mathbb{R}$ given by

$$V(\psi, K, K_{\mathrm{d}}) = V_{\mathrm{s}}(\psi(0)) + \int_{-\tau}^{0} V_{\mathrm{sd}}(\psi(\theta))\mathrm{d}\theta + \mathrm{tr}\, Q^{-1}(K - K_{\mathrm{g}})Y^{-1}(K - K_{\mathrm{g}})^{\mathrm{T}} + \mathrm{tr}\, Q_{\mathrm{d}}^{-1}(K_{\mathrm{d}} - K_{\mathrm{dg}})Y_{\mathrm{d}}^{-1}(K_{\mathrm{d}} - K_{\mathrm{dg}})^{\mathrm{T}}, \quad \psi(\cdot) \in \mathcal{C}, \quad (10)$$

where $\psi(\theta) \triangleq x_{(\cdot)}(\theta)$. Note that $V(\psi_{\mathrm{e}}, K_{\mathrm{g}}, K_{\mathrm{dg}}) = 0$, where $\psi_{\mathrm{e}}(\theta) = 0$, $\theta \in [-\tau, 0]$. Furthermore, note that there exist class $\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, $\alpha_3(\cdot)$ such that

$$V(\psi, K, K_{\mathrm{d}}) \ge \alpha_1(\|\psi(0)\|) + \alpha_2(\|K - K_{\mathrm{g}}\|_{\mathrm{F}}) + \alpha_3(\|K_{\mathrm{d}} - K_{\mathrm{dg}}\|_{\mathrm{F}}). \quad (11)$$

Now, letting $x(t)$ denote the solution to (9) and using (2), (6), and (7), it follows that the Lyapunov-Krasovskii directional derivative along the closed-loop system trajectories is given by

$$\begin{aligned}
&\dot{V}(x_t, K(t), K_{\mathrm{d}}(t)) \\
&= V_{\mathrm{s}}'(x(t))\Big[f_{\mathrm{s}}(x(t)) + f_{\mathrm{ds}}(x(t), x(t-\tau)) \\
&\quad + G(x(t))\hat{G}(x(t))(K(t) - K_{\mathrm{g}})F(x(t)) \\
&\quad + G(x(t))\hat{G}_{\mathrm{d}}(x(t))(K_{\mathrm{d}}(t) - K_{\mathrm{dg}}) \\
&\quad \cdot F_{\mathrm{d}}(x(t), x(t-\tau))\Big] + V_{\mathrm{sd}}(x(t)) - V_{\mathrm{sd}}(x(t-\tau)) \\
&\quad + 2\mathrm{tr}\, Q^{-1}(K(t) - K_{\mathrm{g}})Y^{-1}\dot{K}^{\mathrm{T}}(t) \\
&\quad + 2\mathrm{tr}\, Q_{\mathrm{d}}^{-1}(K_{\mathrm{d}}(t) - K_{\mathrm{dg}})Y_{\mathrm{d}}^{-1}\dot{K}_{\mathrm{d}}^{\mathrm{T}}(t) \\
&= -\ell^{\mathrm{T}}(x(t), x(t-\tau))\ell(x(t), x(t-\tau)) \\
&\quad + \mathrm{tr}\Big[(K(t) - K_{\mathrm{g}})F(x(t))V_{\mathrm{s}}'(x(t))G(x(t)) \\
&\quad \cdot\hat{G}(x(t))\Big] + \mathrm{tr}\Big[(K_{\mathrm{d}}(t) - K_{\mathrm{dg}})F_{\mathrm{d}}(x(t))V_{\mathrm{s}}'(x(t)) \\
&\quad \cdot G(x(t))\hat{G}_{\mathrm{d}}(x(t))\Big] - \mathrm{tr}\Big[(K(t) - K_{\mathrm{g}})F(x(t)) \\
&\quad \cdot V_{\mathrm{s}}'(x(t))G(x(t))\hat{G}(x(t))\Big] - \mathrm{tr}\Big[(K_{\mathrm{d}}(t) - K_{\mathrm{dg}}) \\
&\quad \cdot F_{\mathrm{d}}(x(t))V_{\mathrm{s}}'(x(t))G(x(t))\hat{G}_{\mathrm{d}}(x(t))\Big] \\
&= -\ell^{\mathrm{T}}(x(t), x(t-\tau))\ell(x(t), x(t-\tau)) \\
&\le 0, \quad t \ge 0, \qquad (12)
\end{aligned}$$

which proves that the solution $(x(t), K(t), K_{\mathrm{d}}(t)) \equiv (0, K_{\mathrm{g}}, K_{\mathrm{dg}})$ to (6), (7), and (9) is Lyapunov stable. Furthermore, since the positive orbit $\gamma^+(\eta(\theta), K_0, K_{\mathrm{d0}})$ is bounded and $\gamma^+(\eta(\theta), K_0, K_{\mathrm{d0}})$ belongs to a compact subset of $\mathcal{C} \times \mathbb{R}^{m\times s} \times \mathbb{R}^{m\times s_{\mathrm{d}}}$ (Hale 1969), it follows from Theorem 3.1 of Hale & Verduyn Lunel (1993, p. 143) that $\ell(x(t), x(t-\tau)) \to 0$ as $t \to \infty$ for all $\eta(\cdot) \in \mathcal{C}$. If, in addition, $\ell^{\mathrm{T}}(x, x_{\mathrm{d}})\ell(x, x_{\mathrm{d}}) > 0$, $(x, x_{\mathrm{d}}) \in \mathbb{R}^n \times \mathbb{R}^n$, $(x, x_{\mathrm{d}}) \ne (0, 0)$, then $\|x_t\| \to 0$ as $t \to \infty$ for all $\eta(\cdot) \in \mathcal{C}$. □

Remark 2.1. Note that in the case where $\ell^{\mathrm{T}}(x, x_{\mathrm{d}}) \cdot\ell(x, x_{\mathrm{d}}) > 0$, $(x, x_{\mathrm{d}}) \in \mathbb{R}^n \times \mathbb{R}^n$, $(x, x_{\mathrm{d}}) \ne (0, 0)$, the conditions in Theorem 2.1 imply that $x(t) \to 0$ as $t \to \infty$ and hence it follows from (6) and (7) that $\dot{K}(t) \to 0$ and $\dot{K}_{\mathrm{d}}(t) \to 0$ as $t \to \infty$.

Remark 2.2. In the case where $\ell^{\mathrm{T}}(x, x_{\mathrm{d}})\ell(x, x_{\mathrm{d}}) = \hat{\ell}^{\mathrm{T}}(x)\hat{\ell}(x) > 0$, $(x, x_{\mathrm{d}}) \in \mathbb{R}^n \times \mathbb{R}^n$, $x \ne 0$, or $\ell^{\mathrm{T}}(x, x_{\mathrm{d}})\ell(x, x_{\mathrm{d}}) = \ell_{\mathrm{d}}^{\mathrm{T}}(x_{\mathrm{d}})\ell_{\mathrm{d}}(x_{\mathrm{d}}) > 0$, $(x, x_{\mathrm{d}}) \in \mathbb{R}^n \times \mathbb{R}^n$, $x_{\mathrm{d}} \ne 0$, where $\hat{\ell} : \mathbb{R}^n \to \mathbb{R}^{\hat{p}}$, we can also conclude that $\|x_t\| \to 0$ as $t \to \infty$ for all $\eta(\cdot) \in \mathcal{C}$.

It is important to note that the adaptive control law (5)–(7) does *not* require explicit knowledge of the gain matrices K_{g} and K_{dg}; Theorem 2.1 simply requires the existence of K_{g} and K_{dg} along with the construction of $F(x)$, $F_{\mathrm{d}}(x, x_{\mathrm{d}})$, $\hat{G}(x)$, $\hat{G}_{\mathrm{d}}(x)$, $V_{\mathrm{s}}(x)$, and $V_{\mathrm{sd}}(x)$ such that (2) holds. Furthermore, no specific structure on the nonlinear dynamics $f(x)$ is required to apply Theorem 2.1. However, if (1) is in normal form with asymptotically stable internal dynamics (Isidori 1995), then we can always construct functions $F : \mathbb{R}^n \to \mathbb{R}^s$ and $F_{\mathrm{d}} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{s_{\mathrm{d}}}$, with $F(0) = 0$ and $F_{\mathrm{d}}(0, 0) = 0$, such that the condition (2) is satisfied. To see this assume that the nonlinear uncertain system $\mathcal{G}$ is generated by

$$\begin{aligned}
q_i^{(r_i)}(t) &= f_{\mathrm{u}i}(q(t)) + f_{\mathrm{du}i}(q(t), q(t-\tau)) \\
&\quad + \sum_{j=1}^{m} G_{\mathrm{s}(i,j)}(q(t))u_j(t), \quad t \ge 0, \\
&\qquad i = 1, \cdots, m, \qquad (13)
\end{aligned}$$

where $q = [q_1, \cdots, q_1^{(r_1-1)}, \cdots, q_m, \cdots, q_m^{(r_m-1)}]^{\mathrm{T}}$, $q(\theta) = \eta(\theta)$, $-\tau \le \theta \le 0$, $q_i^{(r_i)}$ denotes the r_ith derivative of q_i, and r_i denotes the relative degree with respect to the output q_i. Here we assume that the square matrix function $G_{\mathrm{s}}(q)$ composed of the entries $G_{\mathrm{s}(i,j)}(q)$, $i, j = 1, \cdots, m$, is such that $\det G_{\mathrm{s}}(q) \ne 0$, $q \in \mathbb{R}^{\hat{r}}$, where $\hat{r} = r_1 + \cdots + r_m$ is the (vector) relative degree of (13). Furthermore, since (13) is in a form where it does not possess internal dynamics, it follows that $\hat{r} = n$. The case where (13) possesses input-to-state stable internal dynamics can be handled as shown in Hayakawa *et al.* (June 2005).

Next, define $x_i \triangleq \left[q_i, \cdots, q_i^{(r_i-2)}\right]^{\mathrm{T}}$, $i = 1, \cdots, m$, $x_{m+1} \triangleq \left[q_1^{(r_1-1)}, \cdots, q_m^{(r_m-1)}\right]^{\mathrm{T}}$, and $x \triangleq \left[x_1^{\mathrm{T}}, \cdots, x_{m+1}^{\mathrm{T}}\right]^{\mathrm{T}}$, so that (13) can be described as (1) with

$$f(x) = \tilde{A}x + \tilde{f}_{\mathrm{u}}(x), \quad f_{\mathrm{d}}(x, x_{\mathrm{d}}) = \begin{bmatrix} 0_{(n-m)\times m} \\ f_{\mathrm{du}}(x, x_{\mathrm{d}}) \end{bmatrix}, \quad G(x) = \begin{bmatrix} 0_{(n-m)\times m} \\ G_{\mathrm{s}}(x) \end{bmatrix}, \qquad (14)$$

where

$$\tilde{A} = \begin{bmatrix} A_0 \\ 0_{m\times n} \end{bmatrix}, \quad \tilde{f}_{\mathrm{u}}(x) = \begin{bmatrix} 0_{(n-m)\times 1} \\ f_{\mathrm{u}}(x) \end{bmatrix},$$

$A_0 \in \mathbb{R}^{(n-m)\times n}$ is a known matrix of zeros and ones capturing the multivariable controllable canonical form representation (Chen 1984), $f_{\mathrm{u}} : \mathbb{R}^n \to \mathbb{R}^m$ and $f_{\mathrm{du}} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ are unknown functions such that $f_{\mathrm{u}}(0) = 0$ and $f_{\mathrm{du}}(0, 0) = 0$, and x_{d} denotes the delayed value of x. Here, we assume that $f_{\mathrm{u}}(x)$ and $f_{\mathrm{du}}(x, x_{\mathrm{d}})$ are unknown and are parameterized as $f_{\mathrm{u}}(x) = \Theta f_{\mathrm{n}}(x)$ and $f_{\mathrm{du}}(x, x_{\mathrm{d}}) = \Theta_{\mathrm{d}} f_{\mathrm{dn}}(x, x_{\mathrm{d}})$, where $f_{\mathrm{n}} : \mathbb{R}^n \to \mathbb{R}^q$ and satisfies $f_{\mathrm{n}}(0) = 0$, $f_{\mathrm{dn}} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{q_{\mathrm{d}}}$ and satisfies $f_{\mathrm{dn}}(0, 0) = 0$, and $\Theta \in \mathbb{R}^{m\times q}$ and $\Theta_{\mathrm{d}} \in \mathbb{R}^{m\times q_{\mathrm{d}}}$ are matrices of uncertain constant parameters.

Next, to apply Theorem 2.1 to the uncertain system (1) with $f(x)$, $f_{\mathrm{d}}(x, x_{\mathrm{d}})$, and $G(x)$ given by (14), let $K_{\mathrm{g}} \in \mathbb{R}^{m\times s}$ and $K_{\mathrm{dg}} \in \mathbb{R}^{m\times s_{\mathrm{d}}}$, where $s = q + r$ and $s_{\mathrm{d}} = q_{\mathrm{d}} + r_{\mathrm{d}}$, be given by

$$K_{\mathrm{g}} = [\,\Theta_{\mathrm{n}} - \Theta,\ \Phi_{\mathrm{n}}\,], \quad K_{\mathrm{dg}} = [\,\Theta_{\mathrm{dn}} - \Theta_{\mathrm{d}},\ \Phi_{\mathrm{dn}}\,], \qquad (15)$$

where $\Theta_{\mathrm{n}} \in \mathbb{R}^{m\times q}$, $\Theta_{\mathrm{dn}} \in \mathbb{R}^{m\times q_{\mathrm{d}}}$, $\Phi_{\mathrm{n}} \in \mathbb{R}^{m\times r}$, and $\Phi_{\mathrm{dn}} \in \mathbb{R}^{m\times r_{\mathrm{d}}}$ are known matrices, and let

$$F(x) = \begin{bmatrix} f_{\mathrm{n}}(x) \\ \hat{f}_{\mathrm{n}}(x) \end{bmatrix}, \quad F_{\mathrm{d}}(x, x_{\mathrm{d}}) = \begin{bmatrix} f_{\mathrm{dn}}(x, x_{\mathrm{d}}) \\ \hat{f}_{\mathrm{dn}}(x, x_{\mathrm{d}}) \end{bmatrix}, \qquad (16)$$

where $\hat{f}_{\mathrm{n}} : \mathbb{R}^n \to \mathbb{R}^r$ and $\hat{f}_{\mathrm{dn}} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{r_{\mathrm{d}}}$, with $\hat{f}_{\mathrm{n}}(0) = 0$ and $\hat{f}_{\mathrm{dn}}(0, 0) = 0$, are arbitrary functions. In this case, it follows that, with $\hat{G}(x) = \hat{G}_{\mathrm{d}}(x) = G_{\mathrm{s}}^{-1}(x)$,

$$\begin{aligned} f_{\rm s}(x) &= f(x) + G(x)\hat{G}(x)K_{\rm g}F(x) \\ &= \tilde{A}x + \tilde{f}_{\rm u}(x) + \begin{bmatrix} 0_{(n-m)\times m} \\ G_{\rm s}(x) \end{bmatrix} G_{\rm s}^{-1}(x) \\ &\quad \cdot \left[\, \Theta_{\rm n} f_{\rm n}(x) - \Theta f_{\rm n}(x) + \Phi_{\rm n}\hat{f}_{\rm n}(x) \,\right] \\ &= \tilde{A}x + \begin{bmatrix} 0_{(n-m)\times 1} \\ \Theta_{\rm n} f_{\rm n}(x) + \Phi_{\rm n}\hat{f}_{\rm n}(x) \end{bmatrix} \end{aligned} \tag{17}$$

and

$$\begin{aligned} f_{\rm ds}(x, x_{\rm d}) &= f_{\rm d}(x, x_{\rm d}) + G(x)\hat{G}_{\rm d}(x)K_{\rm dg}F_{\rm d}(x, x_{\rm d}) \\ &= f_{\rm d}(x, x_{\rm d}) + \begin{bmatrix} 0_{(n-m)\times m} \\ G_{\rm s}(x) \end{bmatrix} G_{\rm s}^{-1}(x) \\ &\quad \cdot \big[\Theta_{\rm dn} f_{\rm dn}(x, x_{\rm d}) - \Theta f_{\rm dn}(x, x_{\rm d}) \\ &\quad + \Phi_{\rm dn}\hat{f}_{\rm dn}(x, x_{\rm d})\big] \\ &= \begin{bmatrix} 0_{(n-m)\times 1} \\ \Theta_{\rm dn} f_{\rm dn}(x, x_{\rm d}) + \Phi_{\rm dn}\hat{f}_{\rm dn}(x, x_{\rm d}) \end{bmatrix}. \end{aligned} \tag{18}$$

Now, since $\Theta_{\rm n} \in \mathbb{R}^{m\times q}$, $\Theta_{\rm dn} \in \mathbb{R}^{m\times q_{\rm d}}$, $\Phi_{\rm n} \in \mathbb{R}^{m\times r}$, and $\Phi_{\rm dn} \in \mathbb{R}^{m\times r_{\rm d}}$ are arbitrary constant matrices and $\hat{f}_{\rm n} : \mathbb{R}^n \to \mathbb{R}^r$ and $\hat{f}_{\rm dn} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{r_{\rm d}}$ are arbitrary functions, we can always construct $K_{\rm g}$, $K_{\rm dg}$, $F(x)$, and $F_{\rm d}(x, x_{\rm d})$ such that (2) holds without knowledge of $f(x)$ and $f_{\rm d}(x, x_{\rm d})$. In particular, choosing $\Theta_{\rm n} f_{\rm n}(x) + \Phi_{\rm n}\hat{f}_{\rm n}(x) = \hat{A}x$ and $\Theta_{\rm dn} f_{\rm dn}(x, x_{\rm d}) + \Phi_{\rm dn}\hat{f}_{\rm dn}(x, x_{\rm d}) = \hat{A}_{\rm d}x_{\rm d}$, where $\hat{A} \in \mathbb{R}^{m\times n}$ and $\hat{A}_{\rm d} \in \mathbb{R}^{m\times n}$, it follows that (17) and (18) have the form $f_{\rm s}(x) = A_{\rm s}x$ and $f_{\rm ds}(x, x_{\rm d}) = A_{\rm ds}x_{\rm d}$, where $A_{\rm s} = [A_0^{\rm T}, \hat{A}^{\rm T}]^{\rm T}$ is in multivariable controllable canonical form and $A_{\rm ds} = [0_{n\times(n-m)}, \hat{A}_{\rm d}^{\rm T}]^{\rm T}$. Hence, choosing $f_{\rm s}(x) = A_{\rm s}x$, where $A_{\rm s}$ is asymptotically stable and in multivariable controllable canonical form, it follows that if there exists a positive-definite matrix P that solves the linear matrix inequality (LMI) feasibility problem

$$0 > \begin{bmatrix} A_{\rm s}^{\rm T}P + PA_{\rm s} + R & PA_{\rm ds} \\ A_{\rm ds}^{\rm T}P & -R \end{bmatrix}, \tag{19}$$

where R is a positive-definite matrix, then (2) is satisfied with functions $V_{\rm s}(x) = x^{\rm T}Px$ and $V_{\rm sd}(x) = x^{\rm T}Rx$. Alternatively, choosing $\hat{A}_{\rm d} = 0$, any solution $P > 0$ to $0 = A_{\rm s}^{\rm T}P + PA_{\rm s} + R$ satisfies the condition in Remark 2.2. In these cases, the update laws for the adaptive controller (5) is given in the form

$$\dot{K}(t) = -Q\hat{G}^{\rm T}(x(t))G^{\rm T}(x(t))Px(t)F^{\rm T}(x(t))Y, \tag{20}$$

$$\begin{aligned} \dot{K}_{\rm d}(t) &= -Q_{\rm d}\hat{G}_{\rm d}^{\rm T}(x(t))G^{\rm T}(x(t))Px(t) \\ &\quad \cdot F_{\rm d}^{\rm T}(x(t), x(t-\tau))Y_{\rm d}. \end{aligned} \tag{21}$$

Next, we consider the case where $f(x)$, $f_{\rm d}(x, x_{\rm d})$, and $G(x)$ are uncertain. Specifically, we assume that $G(x)$ is such that $G_{\rm s}(x)$ is unknown and is parameterized as $G_{\rm s}(x) = B_{\rm u}G_{\rm n}(x)$, where $G_{\rm n} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ is known and satisfies $\det G_{\rm n}(x) \neq 0$, $x \in \mathbb{R}^n$, and $B_{\rm u} \in \mathbb{R}^{m\times m}$, with $\det B_{\rm u} \neq 0$, is an unknown symmetric sign-definite matrix but the sign definiteness of $B_{\rm u}$ is known; that is, $B_{\rm u} > 0$ or $B_{\rm u} < 0$. For the statement of the next result define $B_0 \triangleq \left[0_{m\times(n-m)}, I_m\right]^{\rm T}$ for $B_{\rm u} > 0$, and $B_0 \triangleq \left[0_{m\times(n-m)}, -I_m\right]^{\rm T}$ for $B_{\rm u} < 0$.

Corollary 2.1. Consider the nonlinear delay dynamical system $\mathcal{G}$ given by (1) with $f(x)$, $f_{\rm d}(x, x_{\rm d})$, and $G(x)$ given by (14) and $G_{\rm s}(x) = B_{\rm u}G_{\rm n}(x)$, where $B_{\rm u}$ is an unknown symmetric matrix and the sign definiteness of $B_{\rm u}$ is known. Assume there exist matrices $K_{\rm g} \in \mathbb{R}^{m\times s}$, $K_{\rm dg} \in \mathbb{R}^{m\times s_{\rm d}}$, a continuously differentiable function $V_{\rm s} : \mathbb{R}^n \to \mathbb{R}$, and continuous functions $V_{\rm sd} : \mathbb{R}^n \to \mathbb{R}$, $F : \mathbb{R}^n \to \mathbb{R}^s$, $F_{\rm d} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{s_{\rm d}}$, and $\ell : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$ such that $V_{\rm s}(\cdot)$ and $V_{\rm ds}(\cdot)$ are positive definite, radially unbounded, $V_{\rm s}(0) = 0$, $V_{\rm sd}(0) = 0$, $\ell(0, 0) = 0$, $F(0) = 0$, $F_{\rm d}(0, 0) = 0$, and, for all $x \in \mathbb{R}^n$ and $x_{\rm d} \in \mathbb{R}^n$, (2) holds. Finally, let $Y \in \mathbb{R}^{s\times s}$ and $Y_{\rm d} \in \mathbb{R}^{s_{\rm d}\times s_{\rm d}}$ be positive definite. Then the adaptive feedback control law

$$\begin{aligned} u(t) &= G_{\rm n}^{-1}(x(t))K(t)F(x(t)) \\ &\quad + G_{\rm n}^{-1}(x(t))K_{\rm d}(t)F_{\rm d}(x(t), x(t-\tau)), \end{aligned} \tag{22}$$

where $K(t) \in \mathbb{R}^{m\times s}$ and $K_{\rm d}(t) \in \mathbb{R}^{m\times s_{\rm d}}$, with update laws

$$\dot{K}(t) = -\tfrac{1}{2}B_0^{\rm T}V_{\rm s}'^{\rm T}(x(t))F^{\rm T}(x(t))Y, \tag{23}$$

$$\dot{K}_{\rm d}(t) = -\tfrac{1}{2}B_0^{\rm T}V_{\rm s}'^{\rm T}(x(t))F_{\rm d}^{\rm T}(x(t), x(t-\tau))Y_{\rm d}, \tag{24}$$

guarantees that the solution $(x(t), K(t), K_{\rm d}(t)) \equiv (0, K_{\rm g}, K_{\rm dg})$ of the closed-loop system given by (1), (22)–(24) is Lyapunov stable and $\ell(x(t), x(t-\tau)) \to 0$ as $t \to \infty$. If, in addition, $\ell^{\rm T}(x, x_{\rm d})\ell(x, x_{\rm d}) > 0$, $(x, x_{\rm d}) \in \mathbb{R}^n \times \mathbb{R}^n$, $(x, x_{\rm d}) \neq (0, 0)$, then $\|x_t\| \to 0$ as $t \to \infty$ for all $\eta(\cdot) \in \mathcal{C}$.

Proof. The result is a direct consequence of Theorem 2.1. First, let $\hat{G}(x) = \hat{G}_{\rm d}(x) = G_{\rm n}^{-1}(x)$ so that $G(x)\hat{G}(x) = G(x)\hat{G}_{\rm d}(x) = [0_{m\times(n-m)}, B_{\rm u}]^{\rm T}$. Next, since Q and $Q_{\rm d}$ are arbitrary positive-definite matrices, Q in (6) and $Q_{\rm d}$ in (7) can be replaced by $q|B_{\rm u}|^{-1}$ and $q_{\rm d}|B_{\rm u}|^{-1}$, respectively, where q and $q_{\rm d}$ are positive constants and $|B_{\rm u}| = (B_{\rm u}^2)^{\frac{1}{2}}$, where $(\cdot)^{\frac{1}{2}}$ denotes the (unique) positive-definite square root. Now, since $B_{\rm u}$ is symmetric and sign definite it follows from the Schur decomposition that $B_{\rm u} = UD_{B_{\rm u}}U^{\rm T}$, where U is orthogonal and $D_{B_{\rm u}}$ is real diagonal. Hence, $|B_{\rm u}|^{-1}\hat{G}^{\rm T}(x)G^{\rm T}(x) = |B_{\rm u}|^{-1}\hat{G}_{\rm d}^{\rm T}(x)G^{\rm T}(x) = [0_{m\times(n-m)}, \mathcal{I}_m] = B_0^{\rm T}$, where $\mathcal{I}_m = I_m$ for $B_{\rm u} > 0$ and $\mathcal{I}_m = -I_m$ for $B_{\rm u} < 0$. Now, (6) and (7), with qY and $q_{\rm d}Y_{\rm d}$ replaced by Y and $Y_{\rm d}$, respectively, imply (23) and (24), respectively. □

3. DIRECT ADAPTIVE CONTROL FOR SECOND-ORDER SYSTEMS WITH UNKNOWN TIME DELAY

In this section we present a result that does not require knowledge of the delay amount τ. Specifically, in this section, we consider the nonlinear uncertain matrix second-order delay dynamical system given by (13) with the relative degree given by $r_1 = \cdots = r_m = 2$. With $x_1 \triangleq [q_1, \cdots, q_m]^{\rm T}$, $x_2 \triangleq [\dot{q}_1, \cdots, \dot{q}_m]^{\rm T}$, and $x \triangleq [x_1^{\rm T}, x_2^{\rm T}]^{\rm T}$, it follows that the state space representation is equivalently given by (1) with $n = 2m$, $f(x)$, $f_{\rm d}(x, x_{\rm d})$, and $G(x)$ given by (14). Note that $\tilde{A}$ in (14) is given by $\tilde{A} = \begin{bmatrix} 0_m & I_m \\ 0_m & 0_m \end{bmatrix}$. Here, we assume that $f(x)$, $f_{\rm d}(x, x_{\rm d})$, and $G(x)$ are uncertain, $f_{\rm u}(x)$ is parameterized as $f_{\rm u}(x) = \Theta f_{\rm n}(x)$, where $f_{\rm n} : \mathbb{R}^{2m} \to \mathbb{R}^q$ and satisfies $f_{\rm n}(0) = 0$, $\Theta \in \mathbb{R}^{2m\times q}$ is a matrix of uncertain constant parameters, and $f_{\rm d}(\cdot, \cdot)$ belongs to $\mathcal{F}_{\rm d}$, where

$$\mathcal{F}_{\rm d} \triangleq \{f_{\rm d} : \mathbb{R}^{2m} \times \mathbb{R}^{2m} \to \mathbb{R}^m : f_{\rm du}(0,0) = 0, \quad f_{\rm du}^{\rm T}(x, x_{\rm d}) f_{\rm du}(x, x_{\rm d}) \le \gamma^{-2} x_{\rm d}^{\rm T} x_{\rm d}\}, \qquad (25)$$

and $\gamma > 0$. Furthermore, as in Section 2 we similarly assume that $G(x)$ is such that $G_{\rm s}(x)$ is unknown and is parameterized as $G_{\rm s}(x) = B_{\rm u} G_{\rm n}(x)$, where $G_{\rm n} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ is known and satisfies $\det G_{\rm n}(x) \neq 0$, $x \in \mathbb{R}^n$, and $B_{\rm u} \in \mathbb{R}^{m\times m}$, with $\det B_{\rm u} \neq 0$, is an unknown symmetric sign-definite matrix but the sign definiteness of $B_{\rm u}$ is known; that is, $B_{\rm u} > 0$ or $B_{\rm u} < 0$. For the statement of the next result define $\mathrm{sgn}(B_{\rm u}) = 1$ for $B_{\rm u} > 0$, and $\mathrm{sgn}(B_{\rm u}) = -1$ for $B_{\rm u} < 0$.

Corollary 3.1. Consider the nonlinear uncertain matrix-second order delay dynamical system $\mathcal{G}$ given by (1) with $f(\cdot)$, $f_{\rm d}(\cdot,\cdot)$, and $G(\cdot)$ given by (14) and $G_{\rm s}(x) = B_{\rm u} G_{\rm n}(x)$, where $f_{\rm d}(\cdot,\cdot) \in \mathcal{F}_{\rm d}$ and $B_{\rm u}$ is an unknown symmetric matrix and the sign definiteness of $B_{\rm u}$ is known. Let $F(x) \triangleq [f_{\rm n}^{\rm T}(x), x^{\rm T}]^{\rm T}$, $Y \in \mathbb{R}^{s\times s}$ be a positive-definite matrix, and $p \triangleq [p_{12}, p_2]^{\rm T} \in \mathbb{R}^2$ be a positive vector; that is, $p_{12}, p_2 > 0$. Then the adaptive feedback control law

$$u(t) = G_{\rm n}^{-1}(x(t)) K(t) F(x(t)), \qquad (26)$$

where $K(t) \in \mathbb{R}^{m\times(q+m)}$, with update law

$$\dot{K}(t) = -\mathrm{sgn}(B_{\rm u})(p^{\rm T} \otimes I_m) x(t) F^{\rm T}(x(t)) Y, \qquad (27)$$

guarantees that the solution $(x(t), K(t)) \equiv (0, K_{\rm g})$, where $K_{\rm g} \in \mathbb{R}^{m\times(q+m)}$, of the closed-loop system given by (1), (26), (27) is Lyapunov stable and $\|x_t\| \to 0$ as $t \to \infty$ for all $\eta(\cdot) \in \mathcal{C}$, $f_{\rm d}(\cdot) \in \mathcal{F}_{\rm d}$, and $\tau \in [0,\infty)$.

Proof. First, let $a_{\rm s} = [a_{\rm s1}, a_{\rm s2}]^{\rm T}$, $a_{\rm s1} < -\frac{1}{2p_{12}}(1 + \gamma^{-2}(p_{12}^2 + p_2^2))$ (< 0), $a_{\rm s2} < \min\{-\frac{1}{2p_{12}p_2} \cdot (1 + 4p_{12}^2 + \gamma^{-2}(p_{12}^2 + p_2^2)), -\frac{1}{p_{12}p_2}(a_{\rm s1}p_2^2 + p_{12}^2)\}$ (< 0), and $p_1 = -a_{\rm s1}p_2 - a_{\rm s2}p_{12}$ (> 0). Furthermore, define $P \triangleq \begin{bmatrix} p_1 I_m & p_{12} I_m \\ p_{12} I_m & p_2 I_m \end{bmatrix}$, $\tilde{A}_{\rm s} \triangleq \begin{bmatrix} 0_m & I_m \\ a_{\rm s1} I_m & a_{\rm s2} I_m \end{bmatrix}$, $B_0 \triangleq [0_m, I_m]^{\rm T}$, and $K_{\rm g} \triangleq B_{\rm u}^{-1}[-\Theta, a_{\rm s}^{\rm T} \otimes I_m]$. Then it follows that

$$\begin{aligned} R &\triangleq -(\tilde{A}_{\rm s}^{\rm T} P + P\tilde{A}_{\rm s} + I_{2m} + \gamma^{-2} P B_0 B_0^{\rm T} P) \\ &\ge \begin{bmatrix} -2a_{\rm s1}p_{12} I_m & -(p_1 + a_{\rm s1}p_2 + a_{\rm s2}p_{12}) I_m \\ -(p_1 + a_{\rm s1}p_2 + a_{\rm s2}p_{12}) I_m & -(2p_{12} + a_{\rm s2}p_2) I_m \end{bmatrix} \\ &\quad + \lambda_{\max}(I_{2m} + \gamma^{-2} P B_0 B_0^{\rm T} P) I_{2m} \\ &\ge \begin{bmatrix} -2a_{\rm s1}p_{12} I_m & 0 \\ 0 & -(2p_{12} + a_{\rm s2}p_2) I_m \end{bmatrix} \\ &\quad + \lambda_{\max}(1 + \gamma^{-2}(p_{12}^2 + p_2^2)) I_{2m} \\ &> 0. \end{aligned} \qquad (28)$$

Next, note that with $u(t)$ given by (26) it follows from (1) that

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + f_{\rm d}(x(t), x(t-\tau)) \\ &\quad + G(x(t)) G_{\rm n}^{-1}(x(t)) K(t) F(x(t)) \\ x(\theta) &= \eta(\theta), \quad -\tau \le \theta \le 0, \quad t \ge 0, \end{aligned} \qquad (29)$$

or, equivalently,

$$\begin{aligned} \dot{x}(t) &= \tilde{A}_{\rm s} x(t) + B_0 f_{\rm du}(x(t), x(t-\tau)) \\ &\quad + B_0 B_{\rm u}(K(t) - K_{\rm g}) F(x(t)), \\ x(\theta) &= \eta(\theta), \quad -\tau \le \theta \le 0, \quad t \ge 0. \end{aligned} \qquad (30)$$

To show Lyapunov stability of the closed-loop system (27) and (30) consider the Lyapunov-Krasovskii functional candidate $V : \mathcal{C} \times \mathbb{R}^{m\times m} \to \mathbb{R}$ given by

$$\begin{aligned} V(\psi, K) &= \psi^{\rm T}(0) P \psi(0) + \int_{-\tau}^{0} \psi^{\rm T}(\theta)\psi(\theta)\,{\rm d}\theta \\ &\quad + \mathrm{tr}\,|B_{\rm u}|(K - K_{\rm g}) Y^{-1} (K - K_{\rm g})^{\rm T}, \\ &\qquad \psi(\cdot) \in \mathcal{C}, \end{aligned} \qquad (31)$$

where $|B_{\rm u}| = (B_{\rm u}^2)^{\frac{1}{2}}$ and $(\cdot)^{\frac{1}{2}}$ denotes the (unique) positive-definite square roots. Note that $V(\psi_{\rm e}, K_{\rm g}) = 0$, where $\psi_{\rm e}(\theta) = 0$, $\theta \in [-\tau, 0]$. Furthermore, since $p_1 > 0$ and $\det P = -a_{\rm s1}p_2^2 - a_{\rm s2}p_{12}p_2 - p_{12}^2 > -a_{\rm s1}p_2^2 + \frac{1}{p_{12}p_2}(a_{\rm s1}p_2^2 + p_{12}^2)p_{12}p_2 - p_{12}^2 = 0$, P is positive definite and thus it follows that there exist class $\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ such that

$$V(\psi, K) \ge \alpha_1(\|\psi(0)\|) + \alpha_2(\|K - K_{\rm g}\|_{\rm F}). \qquad (32)$$

Now, letting $x(t)$ denote the solution to (30) and using (25), (27), and (28), it follows that the Lyapunov-Krasovskii directional derivative along the closed-loop system trajectories is given by

$$\begin{aligned} &\dot{V}(x_t, K(t)) \\ &= 2x^{\rm T}(t) P \Big[\tilde{A}_{\rm s} x(t) + B_0 f_{\rm du}(x(t), x(t-\tau)) \\ &\quad + B_0 B_{\rm u}(K(t) - K_{\rm g}) F(x(t))\Big] + x^{\rm T}(t) x(t) \\ &\quad - x^{\rm T}(t-\tau) x(t-\tau) \\ &\quad + 2\mathrm{tr}\,|B_{\rm u}|(K(t) - K_{\rm g}) Y^{-1} \dot{K}^{\rm T}(t) \\ &= x^{\rm T}(t)(\tilde{A}_{\rm s}^{\rm T} P + P\tilde{A}_{\rm s}) x(t) \\ &\quad + 2x^{\rm T}(t) P B_0 f_{\rm du}(x(t), x(t-\tau)) \\ &\quad + 2\mathrm{tr}\Big[(K(t) - K_{\rm g}) F(x(t)) x^{\rm T}(t) P B_0 B_{\rm u}\Big] \\ &\quad + x^{\rm T}(t) x(t) - x^{\rm T}(t-\tau) x(t-\tau) \\ &\quad - 2\mathrm{tr}\Big[(K(t) - K_{\rm g}) F(x(t)) x^{\rm T}(t) (p^{\rm T} \otimes I_m)^{\rm T} \\ &\quad \cdot B_0[\mathrm{sgn}(B_{\rm u})|B_{\rm u}|]\Big] \\ &\le x^{\rm T}(t)(\tilde{A}_{\rm s}^{\rm T} P + P\tilde{A}_{\rm s}) x(t) \\ &\quad + \gamma^{-2} x^{\rm T}(t) P B_0 B_0^{\rm T} P x(t) \\ &\quad + \gamma^2 f_{\rm du}^{\rm T}(x(t), x(t-\tau)) f_{\rm du}(x(t), x(t-\tau)) \\ &\quad + x^{\rm T}(t) x(t) - x^{\rm T}(t-\tau) x(t-\tau) \\ &\le x^{\rm T}(t)(\tilde{A}_{\rm s}^{\rm T} P + P\tilde{A}_{\rm s} + I_{2m} + \gamma^{-2} P B_0 B_0^{\rm T} P) x(t) \\ &\quad + x^{\rm T}(t-\tau) x(t-\tau) - x^{\rm T}(t-\tau) x(t-\tau) \\ &= -x^{\rm T}(t) R x(t) \\ &\le 0, \quad t \ge 0, \end{aligned} \qquad (33)$$

which proves that the solution $(x(t), K(t)) \equiv (0, K_{\rm g})$ to (27) and (30) is Lyapunov stable. Furthermore, since $R > 0$, it follows from Theorem 3.1 of Hale & Verduyn Lunel (1993, p. 143) that $\|x_t\| \to 0$ as $t \to \infty$ for all $\eta(\cdot) \in \mathcal{C}$, $f_{\rm d}(\cdot,\cdot) \in \mathcal{F}_{\rm d}$, and $\tau \in [0,\infty)$. □

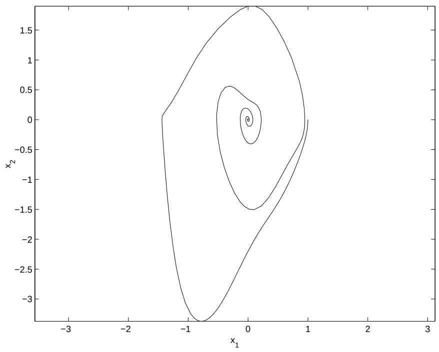

Fig. 1. Phase portrait of controlled system

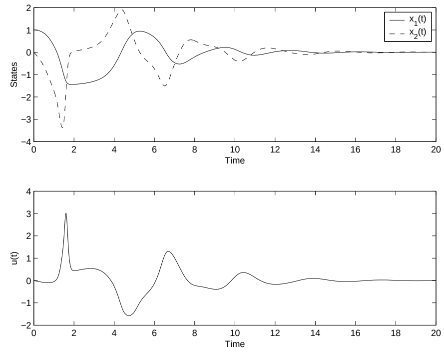

Fig. 2. State trajectories and control signal versus time

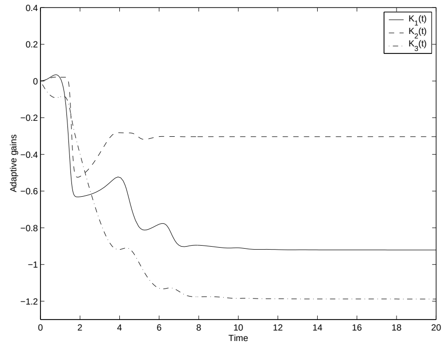

Fig. 3. Adaptive gain history versus time

4. ILLUSTRATIVE NUMERICAL EXAMPLES

In this section we present a numerical example to demonstrate the utility of the proposed direct adaptive control framework for adaptive stabilization of nonlinear uncertain delay dynamical systems. Specifically, consider the nonlinear uncertain delay dynamical system given by

$$\ddot{z}(t) + \mu(z^4(t) - \alpha)\dot{z}(t) + \beta \sin(\lambda z(t-\tau))z(t)$$
$$= bu(t), \quad z(\theta) = \eta(\theta), \quad -\tau \le \theta \le 0, \quad t \ge 0, \quad (34)$$

where $\mu, \alpha, \beta, \lambda, b \in \mathbb{R}$ are unknown. Note that with $x_1 = z$ and $x_2 = \dot{z}$, (34) can be written in state space form (1) with $x = [x_1, x_2]^{\mathrm{T}}$, $f(x) = [x_2, -\mu(x_1^4 - \alpha)x_2]^{\mathrm{T}}$, $f_{\mathrm{d}}(x, x_{\mathrm{d}}) = [0, -\beta \sin(\lambda x_1)x_{\mathrm{d}1}]^{\mathrm{T}}$, and $G(x) = [0, b]^{\mathrm{T}}$. Here, we assume that $f(x)$ and $f_{\mathrm{d}}(x)$ are unknown and $f(x)$ can be parameterized as $f(x) = [x_2, \theta_1 x_2 + \theta_2 x_1^4 x_2]^{\mathrm{T}}$, where θ_1 and θ_2 are unknown constants. Note that $f_{\mathrm{d}}(x, x_{\mathrm{d}}) \in \mathcal{F}_{\mathrm{d}}$ with $\gamma = \beta^{-1}$. Furthermore, we assume that $\operatorname{sign} b$ is known and $\tau > 0$ is unknown. Next, let $F(x) = \left[x_2, x_1^4 x_2, \hat{x}_1\right]^{\mathrm{T}}$ and $K_{\mathrm{g}} = \frac{1}{b}\left[a_{\mathrm{s}1} - \theta_1, -\theta_2, a_{\mathrm{s}2}\right]$, where $a_{\mathrm{s}1}$, $a_{\mathrm{s}2}$ are arbitrary scalars, so that $\tilde{A}_{\mathrm{s}} = \begin{bmatrix} 0 & 1 \\ a_{\mathrm{s}1} & a_{\mathrm{s}2} \end{bmatrix}$. Now, with the proper choice of $a_{\mathrm{s}1}$ and $a_{\mathrm{s}2}$ for a given positive vector $p \in \mathbb{R}^2$, it follows from Theorem 3.1 that the adaptive feedback controller (26) guarantees that $|||x_t||| \to 0$ as $t \to \infty$ for all $\tau \in [0, \infty)$. With $\mu = 2$, $\alpha = 1$, $\beta = -1$, $b = 3$, $\tau = 3$, $p = [2, 1]^{\mathrm{T}}$, $Y = 0.1 I_4$, and initial conditions $[\eta(\theta), \dot{\eta}(\theta)]^{\mathrm{T}} = [1, 0]^{\mathrm{T}}$, $-3 \le \theta \le 0$, and $K(0) = [0, 0, 0]$, Figure 1 shows the phase portrait of the controlled and uncontrolled system. Figure 2 shows the state trajectories versus time and the control signal versus time. Finally, Figure 3 shows the adaptive gain history versus time.

5. CONCLUSION

A direct adaptive nonlinear control framework for adaptive stabilization of multivariable nonlinear uncertain delay dynamical systems was developed. Using Lyapunov-Krasovskii functionals the proposed framework was shown to guarantee partial asymptotic stability of the closed-loop system. Furthermore, in the case where the nonlinear system is represented in normal form with input-to-state stable zero dynamics, the nonlinear adaptive controllers were constructed without knowledge of the system dynamics. Specifically, in the case of matrix second-order nonlinear systems, the adaptive controller does not even require the knowledge of the delay amount. Finally, an illustrative numerical example was presented to show the utility of the proposed adaptive stabilization scheme.

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