PARAMETER REDUCTION FOR LPV SYSTEMS VIA PRINCIPAL COMPONENTS ANALYSIS

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Abstract: This paper is concerned with the reduction of the number of parameters of LPV and quasi LPV models for the synthesis of LPV gain scheduling controllers. The number of parameters is reduced by the principal component analysis of typical scheduling trajectories. This method enables a systematic trade-off between the number of reduced parameters and the desired accuracy. The approach is illustrated with a quasi LPV model of an arm driven inverted pendulum

Keywords: Nonlinear control, Lyapunov methods, Scheduling algorithms

1. INTRODUCTION

In recent years, LMI-based design of gain-scheduled controllers for linear parameter-varying (LPV) systems has been developed into an efficient tool for the control of nonlinear MIMO systems. An LPV system is a linear system whose state space data depend explicitly on a time-varying external parameter vector $\theta(t)$. If parameters in $\theta(t)$ depend on measured system outputs, the system is called a *quasi LPV system*. The attractiveness of LPV systems lies in the fact that they allow to extend the use of well-known linear optimal control tools to nonlinear controller design. However, even though there exists a variety of publications on controller synthesis, see e.g. (Apkarian *et al.*, 1996), (Apkarian and Tuan, 2000), (Rugh and Shamma, 2000) and references therein, there are significantly fewer results on applications, among those (Kajiwara *et al.*, 1999), (Dettori and Scherer, 2001), (Bruzelius *et al.*, 2002), (Yu *et al.*, 2002). One reason for this is the fact that with an increasing number of parameters, the design problem quickly becomes intractable; another reason is the conservatism due to overbounding the parameter range of the plant, when modelled as LPV system.

Number of parameters: For the standard H_∞ LPV gain scheduling approach with polytopic models (Apkarian *et al.*, 1996), the number N of Linear Matrix Inequalities (LMIs) to be solved depends exponentially on the number l of parameters according to

$$N = 2^{l+1} + 1 \ ,$$

so that even simple problems become untractable if l exceeds a number of $3\sim4$. For less conservative approaches like parameterized LMIs (Apkarian and Tuan, 2000), the computational burden is even higher. Approaches that deal with the design of LPV models with few parameters (Kajiwara *et al.*, 1999), (Yu *et al.*, 2002), use subsidiary controllers, neglect physical feedbacks or set terms in the LPV models to zero or fixed values to reduce the complexity. Although these procedures are motivated by the control task and guided experience, they appear not to be very intuitive.

Conservatism due to overbounding: For quasi-LPV systems usually the parameter range is a superset of the region that is spanned by the real plant parameters, if the same scheduling outputs appear in more than one parameter function. Therefore, the LPV system includes behavior, that cannot be displayed by the real plant, resulting in conservatism. One way to reduce this conservatism, is to determine operational trajectories of the real plant and to 'reshape' the hyperbox representing the parameter range, such that it matches the given operating points as closely as possible (Azuma *et al.*, 2000), (Bruzelius *et al.*, 2002). For polytopic LPV models, this method often results in an increasing number of vertices of the polytope and with that, in computational burden. In the proposed approach, the operation trajectories are used to reduce the conservatism in modelling, while using these data to reduce the number of parameters at the same time. This is done by principal components analysis (PCA) of the data.
The paper is organized as follows. The next section defines the problem, followed by the presentation of the parameter reduction in Section 3. Section 4 presents an inverted pendulum as an example. The LPV model is derived from the nonlinear model and the number of parameters is then reduced. In section 4.3, the quality of the approximated model is examined in detail.

2. PROBLEM FORMULATION

Suppose we are given the quasi-LPV system

$$\begin{aligned}\dot{x}(t) &= A(\theta(t))x(t) + B(\theta(t))u(t)\,,\\ y(t) &= C(\theta(t))x(t) + D(\theta(t))u(t)\,,\end{aligned} \tag{1}$$

where $\theta(t) \in \mathbb{R}^l$ represents a time-varying parameter vector, and the mappings $A(.)$, $B(.)$, $C(.)$ and $D(.)$ are continuous functions of θ. The parameter vector $\theta(t)$ depends on a vector of measured signals $y_s(t) \in \mathbb{R}^k$, referred to as *scheduling outputs*, according to

$$\theta(t) = f(y_s(t))\,, \tag{2}$$

where $f : \mathbb{R}^k \rightarrow \mathbb{R}^l$ is a continuous mapping. Here it is assumed that $y_s(t)$ is a subvector of the plant measurement vector $y(t)$. The problem considered in this section is the following. Find a mapping $g : \mathbb{R}^k \rightarrow \mathbb{R}^m$ such that $m < l$, and

$$\phi(t) = g(y_s(t))\,, \tag{3}$$

yields a model

$$\begin{aligned}\dot{x}(t) &= \hat{A}(\phi(t))x(t) + \hat{B}(\phi(t))u(t)\,,\\ y(t) &= \hat{C}(\phi(t))x(t) + \hat{D}(\phi(t))u(t)\,,\end{aligned} \tag{4}$$

that provides a satisfactory approximation of the system (1). Moreover, find the smallest integer m for which a satisfactory approximation is possible.

3. PARAMETER REDUCTION

The first step is to generate 'typical' output trajectories by simulation or by experiments. The output trajectories should roughly span the expected range of operation of the controlled plant. This can be the entire operating region or can be used to restrict the possible operating region considerably, as illustrated in the example below. With output data sampled at times $t = kT$, $k = 0, \cdots, N$, $N \gg l$ one obtains the data matrix

$$\begin{aligned}\Theta &= [\theta(0)\;\theta(T)\;\ldots\;\theta(NT)]\\ &= [f(y_s(0))\;f(y_s(T))\;\ldots\;f(y_s(NT))]\,,\end{aligned} \tag{5}$$

whose i^{th} row Θ_i represents the trajectory of parameter θ_i. The rows Θ_i are normalized by an operation $\mathcal{N}$ to achieve zero mean data with unity deviation:

$$\Theta_i^n = \mathcal{N}_i(\Theta_i) = (\Theta_i - m_i)/c_i\,, \tag{6}$$

$$\Theta_i = \mathcal{N}_i^{-1}(\Theta_i^n) = c_i\Theta_i^n + m_i\,, \tag{7}$$

$$\text{s.t.}\ \sum_{k=0}^{N}\Theta_i^n = 0, \quad \sigma(\Theta_i^n) = 1\,, \tag{8}$$

resulting in a normalized data matrix Θ^n

$$\Theta_n = \mathcal{N}(\Theta),\ \Theta = \mathcal{N}^{-1}(\Theta_n)\,, \tag{9}$$

where $\mathcal{N}$ and $\mathcal{N}^{-1}$ indicate the row-wise normalization and re-normalization, respectively. Now, one applies the Principal Components Analysis (PCA), a standard tool in probability and statistics (Jackson, 1991), to the data (9). Introduce a singular value decomposition of Θ^n

$$\Theta^n = [U_s\;U_n]\begin{bmatrix}\Sigma_s & 0 & 0\\ 0 & \Sigma_n & 0\end{bmatrix}\begin{bmatrix}V_s^T\\ V_n^T\end{bmatrix}\,,$$

and assume that U_s, Σ_s and V_s correspond to the m significant singular values, where $m < l$, so that

$$\hat{\Theta}^n = U_s\Sigma_sV_s^T = U_s\Phi \approx \Theta^n\,, \tag{10}$$

is a reasonable approximation of the given data. Note that the rows of $\Phi = \Sigma_sV_s^T$ represent the principal components of the normalized data matrix Θ^n, while the matrix $U_s \in \mathbb{R}^{l\times m}$ represents a basis of the significant column space of Θ^n and can be used to obtain a mapping from the data onto the principal components:

$$\Phi = U_s^T\hat{\Theta}^n\,. \tag{11}$$

An interesting feature of this approach is the possibility to adjust the accuracy of the model against the number of principal components and with that, the number of parameters.

Up to now, the approximation has been applied to data only. To extend the use of PCA from data approximation to model approximation, the PC of Θ^n need to be functionally related to the scheduling outputs $y_s(t)$. To do so, the transformation matrix in equations (10), (11) and the normalization are applied to (2). Firstly, the parameters in (2) are normalized using the values (m_i, c_i) from (6). With that, the transformation matrix U_s relates normalized parameter vector $\theta^n(t)$ to the desired mapping $\phi(t)$ in (3) by

$$\theta_i^n(t) = (f_i(y_s(t)) - m_i)/c_i\,, \qquad (12)$$

$$\phi(t) = U_s^T \theta^n(t) = U_s^T \mathcal{N}(f(y_s(t))). \qquad (13)$$

Thus, the functions g_i in (3) can be derived as

$$\begin{bmatrix} g_1(y_s(t)) \\ \vdots \\ g_m(y_s(t)) \end{bmatrix} = U_s^T \begin{bmatrix} (f_1(y_s(t)) - m_1)/c_1 \\ \vdots \\ (f_l(y_s(t)) - m_l)/c_l \end{bmatrix}.$$

The approximate mappings $\hat{A}(.),\ \hat{B}(.),\ \hat{C}(.),\ \hat{D}(.)$ in (4) are related to (1) by

$$\begin{bmatrix} \hat{A}(\phi(t)) & \hat{B}(\phi(t)) \\ \hat{C}(\phi(t)) & \hat{D}(\phi(t)) \end{bmatrix} = \begin{bmatrix} A(\hat{\theta}(t)) & B(\hat{\theta}(t)) \\ C(\hat{\theta}(t)) & D(\hat{\theta}(t)) \end{bmatrix} \qquad (14)$$

where

$$\hat{\theta}(t) = \mathcal{N}^{-1}(U_s \phi(t)) \qquad (15)$$

$$= \mathcal{N}^{-1}(U_s U_s^T \mathcal{N}(\theta(t)))\,. \qquad (16)$$

At any given time, (13) can be used to compute the reduced parameter vector $\phi(t)$, while (14) together with (15) can be used to generate the approximate LPV model. The method can also be applied to LPV systems with external parameters, when the parameter data Θ are measured or simulated directly. To summarize, the parameter reduction is executed in the following steps:

1. Determine parameter trajectories Θ_i from measurements or simulations. For quasi LPV systems use output data $y_s(k)$ and (5).
2. Compute normalization terms c_i, m_i and principal components of the parameter data.
3. Choose the number of significant principal components, obtain U_s.
4. Use transformation U_s and normalization terms of step 3 to apply the PCA to the mapping f in (2).

To illustrate the approach, the parameter reduction is applied to an arm driven inverted pendulum in the following section.

4. EXAMPLE: INVERTED PENDULUM

In (Kajiwara *et al.*, 1999) the authors investigate the benefit of LPV gain scheduling compared to robust control by applying different techniques to an arm driven inverted pendulum, as shown in Figure 1.

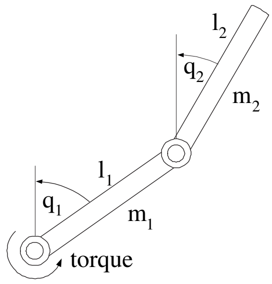

Fig. 1. Arm driven inverted pendulum

The control task is to stabilize the upper link (pendulum) in an upright position, while following given trajectories with the lower link (arm), using the torque τ as control input and measuring $q = [\,q_1\ q_2\,]^T$.

4.1 Derivation of the LPV Model

The equation of motion is

$$M(q)\,\ddot{q} + C(q,\dot{q}) + g(q) = \tau \quad \text{with} \qquad (17)$$

$$M = \begin{pmatrix} a & b\cos(q_1 - q_2) \\ b\cos(q_1 - q_2) & c \end{pmatrix} \qquad (18)$$

$$g = \begin{pmatrix} -d\sin(q_1) \\ -e\sin(q_2) \end{pmatrix} \qquad (19)$$

$$C = \begin{pmatrix} b\sin(q_1 - q_2)\dot{q}_2^2 + R_1\dot{q}_1 \\ -b\sin(q_1 - q_2)\dot{q}_1^2 - R_2\dot{q}_1 + R_2\dot{q}_2 \end{pmatrix} \qquad (20)$$

where

$a = m_1(l_{c1}^2 + l_1^2/12) + m_2 l_1^2 \qquad b = m_2 l_1 l_{c2}$

$c = m_2(l_{c2}^2 + l_2^2/12) \qquad d = g(m_1 l_{c1} + m_2 l_1)$

$e = g m_2 l_{c2}$

and with l_i, l_{ci} describing the length of the links and the center of gravity respectively, m_i and R_i describing the masses and friction in the joints for $(i = 1, 2)$ and g being the gravity constant. A nonlinear state space model can be written (Yu *et al.*, 2002) as

$$\dot{x} = F(x)x + G(x)\tau\,, \text{with } x = [\,q_1\ q_2\ \dot{q}_1\ \dot{q}_2\,]^T\,, \qquad (21)$$

where

$$F(x) = \begin{bmatrix} [O_{2\times 2}\ \ I_2]x \\ -M^{-1}(x)[C(x) + g(x)] \end{bmatrix}, \qquad (22)$$

$$G(x) = \begin{bmatrix} O_{2\times 1} \\ M^{-1}(x) \end{bmatrix}, \qquad (23)$$

with the zero matrix O and the identity matrix I. Thus, an LPV model (1) that describes the nonlinear system completely is given by

$$A(\theta) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ cd\theta_3 & -be\theta_4 & \theta_5 & -b\theta_6 \\ -bd\theta_7 & ae\theta_8 & \theta_9 & \theta_{10} \end{bmatrix}, \tag{24}$$

$$B(\theta) = [\ 0\ \ 0\ \ c\theta_1\ \ -b\theta_2\ ]^T,\ C = I,\ D = O, \tag{25}$$

$$\begin{aligned}
&h = ac - b^2\cos^2_\Delta,\ \theta_1 = 1/h,\ \theta_2 = \cos_\Delta/h\,,\\
&\theta_3 = \mathrm{si}(q_1)/h,\ \theta_4 = \cos_\Delta \mathrm{si}(q_2)/h\,,\\
&\theta_5 = (-cR_1 - b^2\sin_\Delta\cos_\Delta\dot{q}_1 - bR_2\cos_\Delta)/h\,,\\
&\theta_6 = (c\sin_\Delta\dot{q}_2 + R_2\cos_\Delta)/h,\ \theta_7 = \cos_\Delta\mathrm{si}(q_1)/h\,,\\
&\theta_8 = \mathrm{si}(q_2)/h,\ \theta_{10} = (b^2\sin_\Delta\cos_\Delta\dot{q}_2 - R_2 a)/h\,,\\
&\theta_9 = (R_1 b\cos_\Delta + ab\sin_\Delta\dot{q}_1 + R_2 a)/h,
\end{aligned}$$

with $\cos_\Delta = \cos(q_1 - q_2)$, $\sin_\Delta = \sin(q_1 - q_2)$ and $\mathrm{si}(q_i) = \sin(q_i)/q_i$. This LPV model has 10 parameters depending on $q_i(t)$ and $\dot{q}_i(t)$.

4.2 Parameter Reduction

To determine the data of the scheduling outputs ($y_s = x$), one needs to run simulations. Because the plant is unstable, local H_∞ loop-shaping controllers have been designed for several local models linearized at operating points $q_2 = 0^o$, $q_1 \in \{-45^o;\ -30^o;\ 0^o;\ 30^o;\ 45^o\}$. The resulting output trajectories are denoted by y_{s,q_1}. Figure 2 displays some results. The states vary in the following intervals:

$$\begin{aligned} &q_1 \in [-55\ \ 55],\ q_2 \in [-1.3\ \ 1.3],\\ &\dot{q}_1 \in [-27\ \ 27],\ \dot{q}_2 \in [-13\ \ 13]\,. \end{aligned} \tag{26}$$

The state trajectories have been appended to the scheduling output $y_s(k) = [y_{s,-45}(k) \cdots y_{s,45}(k)]$. The PCA of the resulting normalized parameter matrix Θ^n leads to the principal components Φ. Figure 3 shows the fractions of the total variance of the data represented by the single PC. One can see that more than 90% of the data can be represented by the first principal component. With that, 9 of the 10 PC are neglected, leading to m=1.

4.3 Validation of the Reduced Model

In this section, the original model (1) is compared with the parameter reduced model (4). Firstly, the quality of the parameter approximation is examined. Figure 4 shows the parameter trajectories $\theta(t)$ and the approximation $\hat{\theta}(t)$ for the parameters θ_1, θ_5 and θ_6, evaluated with the scheduling outputs at operating points $q_1 = 0^o$ and $q_1 = 45^o$. These parameters have been chosen because they are typical for the set of all parameter trajectories.

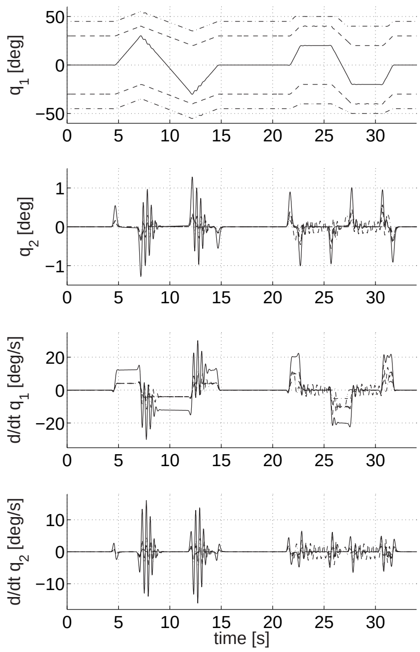

Fig. 2. Scheduling outputs, $y_{s,0}$ (solid), $y_{s,\pm 30}$ (dashed), $y_{s,\pm 45}$ (dash-dotted)

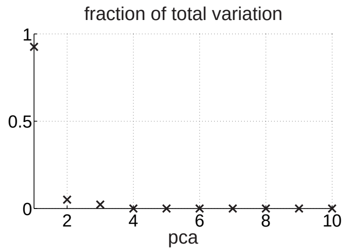

Fig. 3. Significance of principal components for upright position

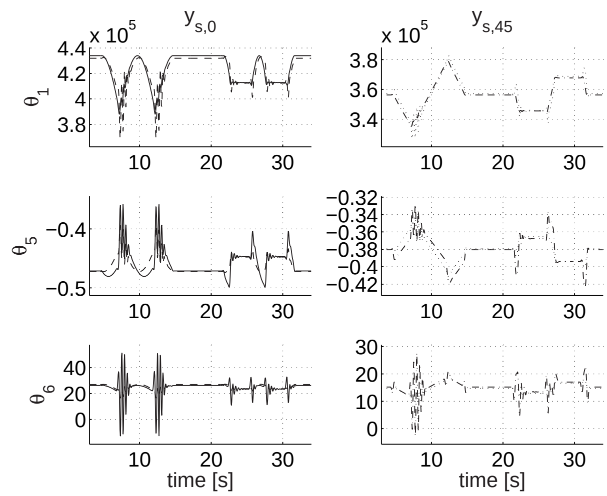

Fig. 4. Parameter approximation. Original: $y_{s,0}$ (solid), $y_{s,45}$ (dash-dot), approximations: $y_{s,0}$ (dashed), $y_{s,45}$ (dotted)

One can see clearly, that the reduced model with only m=1 parameter ϕ approximates the parameters in θ quite well. Because small deviations in the elements of the system matrices can result in con-

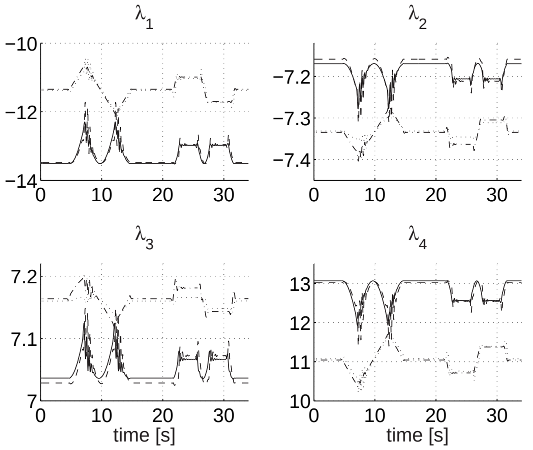

Fig. 5. Approximation of the eigenvalues. Original: $y_{s,0}$ (solid), $y_{s,45}$ (dash-dot), approximations: $y_{s,0}$ (dashed), $y_{s,45}$ (dotted)

siderable variations of the system, the parameterized models are compared in the following. First, the eigenvalues of the parameterized state matrices $\lambda_i(A(\theta(y_{s,q_1})))$ are compared with those of the approximated state matrices $\lambda_i(A(\hat{\theta}(y_{s,q_1})))$ at operating points $q_1 = 0^o$ and $q_1 = 45^o$; they are shown in Figure 5. At both operating points the eigenvalues match quite well. Finally, the input-to-state gain of the parameterized models

$$\begin{aligned} k_s &= [\, k_{s,1}\; k_{s,2}\; k_{s,3}\; k_{s,4}\,]^T = -A(\theta)^{-1}B(\theta) \\ \hat{k}_s &= [\, \hat{k}_{s,1}\; \hat{k}_{s,2}\; \hat{k}_{s,3}\; \hat{k}_{s,4}\,]^T = -A(\hat{\theta})^{-1}B(\hat{\theta}) \end{aligned}$$

is examined. Because $k_{s,3} = k_{s,4} = \hat{k}_{s,3} = \hat{k}_{s,4} = 0$, only the static gains of the angles are shown in Figure 6 for the scheduling outputs $y_{s,0}$ and $y_{s,45}$.

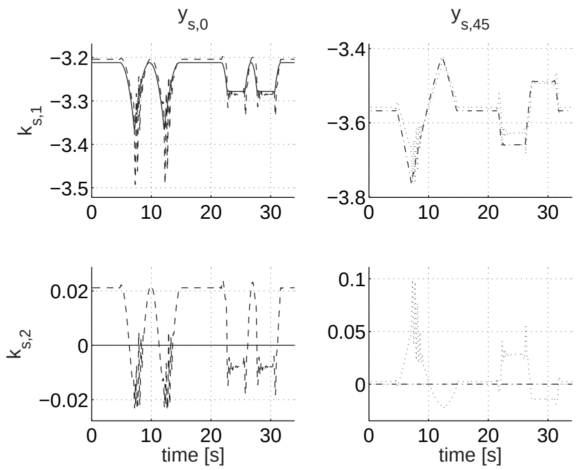

Fig. 6. Approximation of the static gain. Original: $y_{s,0}$ (solid), $y_{s,45}$ (dash-dot), Approximations: $y_{s,0}$ (dashed), $y_{s,45}$ (dotted)

While the static gain from input to the angle q_1 is perfectly matched, there are deviations in the other channel. The original model has zero static gain $k_{s,2} = k_{s,2} = 0$, but this is not displayed by the reduced model. Finally, it needs to be checked, whether the reduced model can approximate operating points that are not part of the trajectories of the scheduling outputs. To do so, the velocities are set to zero $\dot{q}_1 = \dot{q}_2 = 0$, and a grid of angles has been chosen between their extreme values. For every operating point, the parameter $\theta_i(q)$ and its approximation is calculated. Figure 7 shows the relative errors $e_{\mathrm{rel}}(\theta_i)$ for the parameter with the lowest and the highest mean relative errors

$$e_{\mathrm{rel}}(\theta_i) = \frac{|\,\theta_i(q) - \hat{\theta}_i(q)\,|}{\theta_i(q)} \tag{27}$$

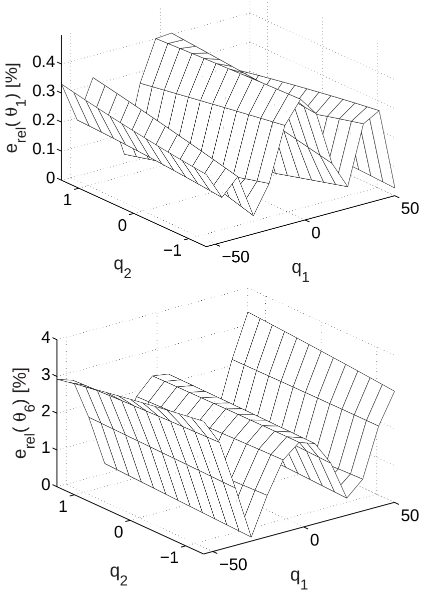

Fig. 7. Relative errors of approximation for θ_1 (upper) and θ_6 (lower)

The highest relative error in the approximation of all parameters is 3.5%. The average relative error about all operating points and all parameters is about 0.7%. Thus, the approximations for the equilibria are satisfactory.

Finally, it is shown how the quality of the parameter approximation reflects the quality of the input/output behavior of the approximated plant. Because it is difficult to compare the input/output behavior for the unstable upwards position of the pendulum, the procedure of parameter reduction has been repeated for the stable downwards position of arm and pendulum and for excitation with sinusoidal signals. The states operate in the range

$$\begin{aligned} &q_1 \in [\,90 \;\; 274\,], \qquad q_2 \in [\,89 \;\; 260\,], \\ &\dot{q}_1 \in [\,-507 \;\; 778\,], \; \dot{q}_2 \in [\,-635 \;\; 797\,]\,. \end{aligned}$$

The operating range is greater than in the upright case (26), therefore one needs approximately 5 parameter to display 95% of the system's behavior, see Figure 8.

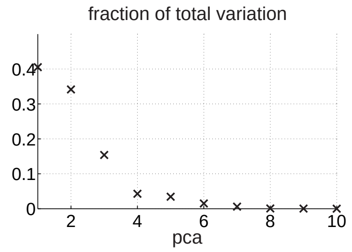

Fig. 8. Significance of principal components for downright position

This demonstrates, that the reduction depends strongly on the chosen parameter range. Figure 9 shows the state trajectories for an open-loop simulation and an excitation $\tau = 0.1 \sin(3t)$. The model with seven parameters approximates the system quite well, while five parameters appear to be required for acceptable performance.

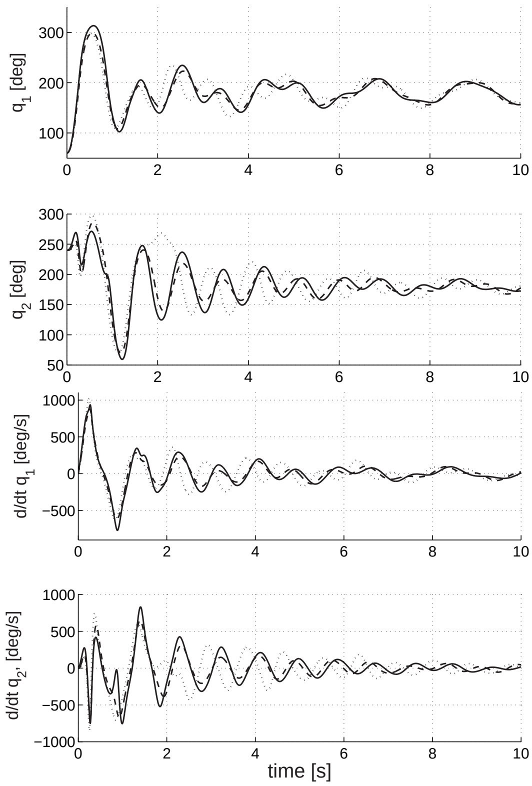

Fig. 9. Comparison of input/output behavior. original (solid), m=7 (dashed), m=5 (dotted)

5. CONCLUSION AND OUTLOOK

This paper presents a method for the reduction of the number of parameters for LPV models, that allows a systematic trade-off between model accuracy and the number of parameters. The parameter reduced model can be used to design polytopic LPV representations with less overbounding. The main idea lies in the application of Principal Components Analysis, applied along typical parameter trajectories. The example of an inverted pendulum illustrates that it is possible to neglect nine of ten parameters while still approximating with reasonable accuracy. It needs to be further investigated, however, how the choice of the scheduling outputs influences the approximation and how the quality of the parameter approximation affects the quality of the overall model behavior.
Moreover, utilization of this approach for the design of gain-scheduling state feedback and output feedback controllers is currently under investigation, and the applicability of parameter reduction for controller synthesis is being examined.

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