DISCRETE-TIME IDA-PBC DESIGN FOR SEPARABLE HAMILTONIAN SYSTEMS

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Abstract: We develop a discrete-time counterpart of IDA-PBC design for separable Hamiltonian systems. Euler approximate models are used to obtain the discrete-time control laws, by replacing "differential" in the continuous-time design with "difference" in the discrete-time design. This approach results in a type of discrete-time controller that can be seen as a simple modification of an emulation controller obtained by sample and hold of the continuous-time IDA-PBC controller. However, due to the crucial issue of Hamiltonian conservation in IDA-PBC design, this simple modification, which results in an "almost" Hamiltonian conservation in discrete-time, yields a significant improvement to the performance of the sampled-data control system.

Keywords: Hamiltonian systems; Discrete-time controller; Stabilization; Nonlinear sampled-data systems; Passivity-based control.

1. INTRODUCTION

The success in a model-based direct discrete-time design for nonlinear sampled-data control systems depends on the availability of a good discrete-time plant model to use for the design. Unfortunately, even if the continuous-time model of a plant is known, we cannot in general compute the exact discrete-time model of the plant, since it requires computing an explicit analytic solution of a nonlinear differential equation. One way to solve the problem of finding a good model is by using an approximate model of the plant.

A general framework for stabilization of sampled-data nonlinear systems via their approximate discrete-time model was presented in (Nešić and Teel, 2004; Nešić *et al.*, 1999*b*). It is suggested that approximate discrete-time models can be obtained using various numerical algorithms, such as Runge-Kutta and multistep methods. Consistency properties are used to measure the discrepancies between the approximate and the exact models (see (Nešić and Teel, 2004; Nešić *et al.*, 1999*b*; Stuart and Humphries, 1996)). Yet, to the best of the authors knowledge, almost all available results on this direction view the systems as *dissipative* systems, whereas for design purpose, there are many systems that are better modeled as Hamiltonian *conservative* systems.

The issues of constructing a discrete-time model for Hamiltonian conservative systems are in general more complicated than those for dissipative systems. The discretization is usually directed to preserving two important properties, symplectic mapping and Hamiltonian conservation. Many researchers have been putting a lot of effort to study the discretization of Hamiltonian conservative systems, focusing mainly on obtaining algorithms that are computationally robust to model the dynamic of a Hamiltonian systems for long time simulations (see (Gonzalez, 1996; Marsden and West, 2001; Sanz-Serna and Calvo, 1994; Stuart and Humphries, 1996) and references therein). Unfortunately, although many of the algorithms are

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satisfactory for numerical modeling, they are often too complicated to use for control design purposes.

A preliminary study on discrete-time stabilization for Hamiltonian systems is presented in (Laila and Astolfi, 2004), where an automatic Hamiltonian conserving algorithm for port-controlled Hamiltonian systems is proposed, generalizing the algorithm presented in (Gonzalez, 1996; Stuart and Humphries, 1996). In the current paper another approach is used to construct a discrete-time counterpart of the interconnection and damping assignment−passivity based control (IDA-PBC) design (Ortega *et al.*, 2002*b*). Instead of using a Hamiltonian conserving algorithm as in (Laila and Astolfi, 2004), we use simply Euler approximation to build the discrete-time model of the plant to *almost* conserve the Hamiltonian indirectly using the designed controller. Our approach applies a very simple idea, replacing a *derivative* with a *difference* (see similar ideas used in (Ilchmann and Sangwin, 2004; Nešić and Teel, 2001)), which helps avoiding loss of stability (due to the loss of Hamiltonian conservation) that inevitably occurs when we use emulation controller. We show by examples that our design results in significantly better closed-loop performance than emulation.

2. PRELIMINARIES

In this section we introduce notation and definitions, and present a brief review on continuous-time IDA-PBC. We focus on a subclass of Hamiltonian systems, namely separable Hamiltonian systems.

2.1 *Continuous-time IDA-PBC design*

IDA-PBC is a powerful design tool for Hamiltonian systems (Ortega *et al.*, 2002*a*; Ortega *et al.*, 2002*b*). Although IDA-PBC is applicable to a broader class of systems (see (Acosta *et al.*, 2004; Ortega *et al.*, 2001; Ortega *et al.*, 2002*a*)), it applies naturally to Hamiltonian control design due to the special structure of this class of systems.

Consider continuous-time Hamiltonian systems

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} \nabla_q H \\ \nabla_p H \end{bmatrix} + \begin{bmatrix} 0 \\ G(q) \end{bmatrix} u, \qquad (1)$$

where $p \in \mathbb{R}^n$ and $q \in \mathbb{R}^n$ are the states, and $u \in \mathbb{R}^m$, $m \leq n$, is the control action [2]. The Hamiltonian function of the system is defined as the sum of the kinetic and the potential energy, i.e.

$$H(q,p) = K(q,p) + V(q) = \frac{1}{2} p' M^{-1}(q) p + V(q),$$

where $M(\cdot)$ is the symmetric inertia matrix. System (1) is a *separable Hamiltonian system* if M is constant, so that the kinetic energy and the potential energy of the system are decoupled, i.e.

$$H(q,p) = K(p) + V(q) = \frac{1}{2} p' M^{-1} p + V(q). \qquad (2)$$

The idea of IDA-PBC design is to construct a controller for system (1) so that stabilization is achieved assigning a desired energy function

$$H_d(q,p) = K_d(p) + V_d(q) = \frac{1}{2} p' M_d^{-1} p + V_d(q), \qquad (3)$$

that has an isolated minimum at the desired equilibrium point $(q^e, 0)$ of the closed-loop system. IDA-PBC design consists of two steps. First, design the energy shaping controller u_{es} to shape the total energy of the system to obtain the target dynamic; second, design the damping injection controller u_{di} to achieve asymptotic stability. Hence, an IDA-PBC controller is of the form

$$u = u_{es} + u_{di}. \qquad (4)$$

The energy shaping controller u_{es} is obtained by solving the equation

$$\begin{aligned} &\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} \nabla_q H \\ \nabla_p H \end{bmatrix} + \begin{bmatrix} 0 \\ G(q) \end{bmatrix} u_{es} \\ &\qquad = \begin{bmatrix} 0 & M^{-1} M_d \\ -M_d M^{-1} & 0 \end{bmatrix} \begin{bmatrix} \nabla_q H_d \\ \nabla_p H_d \end{bmatrix}. \end{aligned} \qquad (5)$$

The first row of (5) is directly satisfied, and the second row can be written as

$$G u_{es} = \nabla_q H - M_d M^{-1} \nabla_q H_d. \qquad (6)$$

If G is full column rank (not necessarily invertible), the following set of constraint equations must be satisfied:

$$G^{\perp} \{ \nabla_q H - M_d M^{-1} \nabla_q H_d \} = 0, \qquad (7)$$

where $G^{\perp}$ is a full rank left annihilator of G, i.e. $G^{\perp} G = 0$. If this PDE is solvable, then u_{es} is obtained as

$$\begin{aligned} u_{es} &= (G'G)^{-1} G' (\nabla_q H - M_d M^{-1} \nabla_q H_d) \\ &= G^{+} (\nabla_q H - M_d M^{-1} \nabla_q H_d). \end{aligned} \qquad (8)$$

Moreover, the damping injection controller u_{di} is constructed as

$$u_{di} = -k_v G' \nabla_p H_d, \quad k_v > 0. \qquad (9)$$

For more details about IDA-PBC design for continuous-time systems, we refer to (Ortega *et al.*, 2002*b*; Ortega *et al.*, 2001; Ortega *et al.*, 2002*a*).

2.2 *Discrete-time model*

As shown in Section 2.1, the objective of designing the energy shaping controller u_{es} is to shape the total energy of the system while keeping the closed-loop system conservative. For this, we are free to choose a method to construct the discrete-time model of the system that would lead to *almost*

[2] The set of real and natural numbers (including 0) are denoted respectively by $\mathbb{R}$ and $\mathbb{N}$. We denote the transpose and the Moore-Penrose inverse of a matrix A by A' and A^{+}, respectively. The identity matrix and the zero matrix are denoted by I and O, respectively.

conserving the desired energy function with the discrete-time energy shaping controller. We choose to use the Euler model, which is not Hamiltonian conserving, but better preserves the Hamiltonian structure of the plant, i.e.

$$\begin{aligned} q(k+1) &= q(k) + T\dot{q}(k) \\ p(k+1) &= p(k) + T\dot{p}(k), \end{aligned} \tag{10}$$

with $q(k) := q(kT)$ and $p(k) := p(kT)$, $k \in \mathbb{N}$, and $T > 0$ the sampling period.

3. MAIN RESULT

In this section, we present our main result, namely a discrete-time IDA-PBC controller design for separable Hamiltonian systems. We present first an example to illustrate the main ideas.

3.1 *Motivating example*

Given the dynamic model of a nonlinear pendulum

$$\dot{q} = p, \quad \dot{p} = -\sin(q) + u. \tag{11}$$

The Hamiltonian of this system is

$$H = K(p) + V(q) = \frac{1}{2}p^2 - \cos(q). \tag{12}$$

The equilibrium point to be stabilized is the origin. We assign the desired energy function

$$H_d = K_d(p) + V_d(q) = \frac{1}{2}p^2 - \cos(q) + \frac{k_1}{2}q^2 + 1 \tag{13}$$

to system (11), and apply IDA-PBC design to the system. The energy shaping and the damping injection controller for system (11) are obtained as

$$u_{es}(t) = \nabla_q V - M_d M^{-1} \nabla_q V_d = -k_1 q, \tag{14}$$

$$u_{di}(t) = -k_v G' \nabla_p K_d = -k_v p, \tag{15}$$

with $k_1, k_v > 0$. Applying $u(t) := u_{es}(t) + u_{di}(t)$ to (11), yields $\dot{H}_d(p,q) = -k_v p^2$, which, by LaSalle Invariance Principle, shows that $u(t)$ globally asymptotically stabilizes system (11). It can also be shown that the emulation controller $u(k) := u_{es}(k) + u_{di}(k)$ obtained by sample and hold of the continuous-time controller $u(t)$ is a SP-AS controller for (11) (Laila *et al.*, 2002). Now, we replace the controller (14) with the following discrete-time controller

$$\begin{aligned} u_{es}^T(k) &= \nabla_q V(q(k)) \\ &\quad - M_d M^{-1} \frac{V_d(q(k+1)) - V_d(q(k))}{q(k+1) - q(k)} \\ &= u_{es}(k) - \frac{k_1}{2} T p(k) \\ &\quad + \sin(q(k)) + \frac{\cos(q(k+1)) - \cos(q(k))}{T p(k)} \\ &= u_{es}(k) - \frac{k_1}{2} T p(k) + O(T^2), \end{aligned} \tag{16}$$

while keeping $u_{di}^T(k) = u_{di}(k)$.

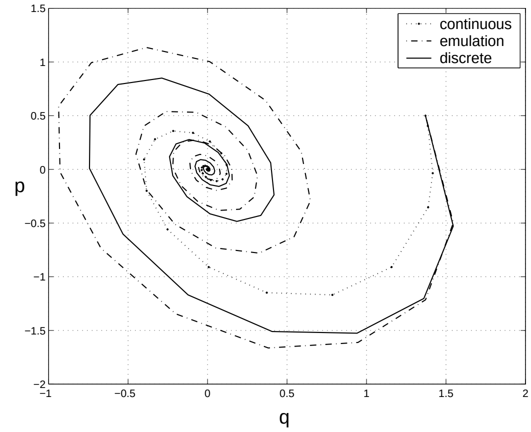

Figure 1. States trajectory of the pendulum.

Figure 1 [3] shows that taking the continuous-time system trajectory as reference, applying the discrete-time controller $u^T(k) := u_{es}^T(k) + u_{di}^T(k)$ (while omitting the $O(T^2)$ term from $u_{es}^T(k)$) keeps the trajectory of the closed-loop system closer to the reference than using the emulation controller $u(k)$. This phenomenon is explained by Figure 2,

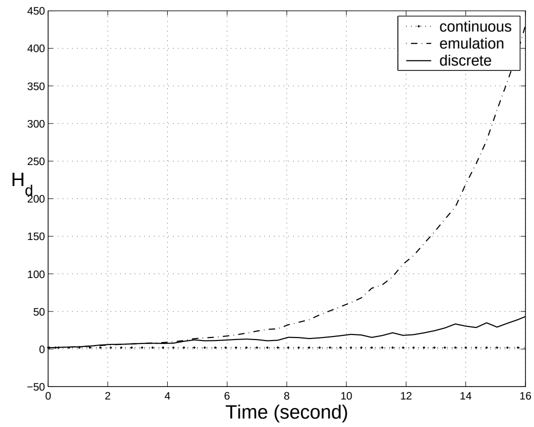

Figure 2. Desired energy function H_d with $k_v = 0$.

displaying the time history of the desired Hamiltonian H_d when applying only the energy shaping controller to the plant. In continuous-time IDA-PBC, $u_{es}(t)$ yields Hamiltonian conservation in closed-loop and hence the closed-loop system is *critically* stable. Applying the emulation controller $u_{es}(k)$ immediately destroys closed-loop stability. On the other hand, the discrete-time controller $u_{es}^T(k)$ tries to recover Hamiltonian conservation, making the closed-loop system *less unstable* than with $u_{es}(k)$. Applying each controller to the Euler model of (11) and then computing the desired energy function difference, we obtain that

$$\Delta H_d^{u_{es}^T} - \Delta H_d^{u_{es}} = -\frac{k_1}{2} T^2 p^2 + O(T^3), \tag{17}$$

which shows that $\Delta H_d^{u_{es}^T}$ is more negative than $\Delta H_d^{u_{es}}$ in a practical sense. Overall, after adding the same damping injection controller, we see in

[3] In the simulation we have used the initial state $(q_\circ, p_\circ) = (\pi/2 - 0.2, 0.5)$, $k_1 = 1$, $k_v = 1$ and $T = 0.35$.

Figure 1 that the discrete-time controller $u^T(k)$ outperforms the emulation controller $u(k)$.

Following the ideas illustrated in this example, we state Proposition 1 below, which is a constructive result for a discrete-time IDA-PBC design.

3.2 *Discrete-time IDA-PBC design*

3.2.1. *Underactuated case* We consider a class of separable Hamiltonian systems (1) with Hamiltonian (2), and the Euler discrete-time model of the system, i.e.

$$\begin{aligned} q(k+1) &= q(k) + TM^{-1}p(k) \\ p(k+1) &= p(k) - T\Big(\nabla_q V(q(k)) - Gu(k)\Big). \end{aligned} \tag{18}$$

Suppose all conditions of the continuous-time design hold, and we have assigned the desired energy function (3) for the system. We denote $O_G := I - I_G$ and $I_G := G(G'G)^{-1}G' = GG^+$. If G is invertible, then $I_G = I$, the identity matrix, and $O_G = O$, the zero matrix. We are now ready to state our main result.

Proposition 1. Consider the Euler model (18) of the separable Hamiltonian system (1). Then the discrete-time controller $u^T = u^T_{es} + u^T_{di}$ where

$$u^T_{es} = G^+\Big(\nabla_q V(q(k)) - M_d M^{-1}\left[\frac{\Delta V_d}{\Delta q}\right]\Big) \tag{19}$$

$$u^T_{di} = -k_v G'\nabla_p K_d(p(k)) = -k_v G' M_d^{-1} p(k), \tag{20}$$

where $k_v > 0$ and

$$\left[\frac{\Delta V_d}{\Delta q}\right] := \nabla_q V_d(q) + T\kappa L_V M^{-1} p \ ,$$

with L_V such that $M_d^{-1} I_G M_d M^{-1} L_V M^{-1}$ is positive semidefinite and $\kappa > 0$, is a SP-AS controller for the Euler model (18). ■

Proof of Proposition 1: Suppose K_d and V_d have been obtained by assigning the desired energy function H_d for the system, in the same way as in the continuous-time design. Comparing the formulae of the discrete-time controller and the formulae of the continuous-time controller, the only different is the term $\left[\frac{\Delta V_d}{\Delta q}\right]$ that appears in (19), replacing $\nabla_q V_d(q)$ of (8). The difference between the two controllers can be written as

$$\begin{aligned} u^T(k) - u(k) &= u^T_{es}(k) + u^T_{di}(k) - u_{es}(k) - u_{di}(k) \\ &= u^T_{es}(k) - u_{es}(k) \\ &= -G^+ M_d M^{-1}\left(\left[\frac{\Delta V_d}{\Delta q}\right] - \nabla_q V_d(q)\right) \\ &= -T\kappa G^+ M_d M^{-1} L_V M^{-1} p \ . \end{aligned}$$

Hence, the controller (19), (20) is of the form

$$\begin{aligned} u^T(k) &= u(k) - T\kappa G^+ M_d M^{-1} L_V M^{-1} p \\ &=: u(k) + T\tilde{u}(k). \end{aligned}$$

We use the fact that

$$\lim_{T\to 0} u^T(kT) \to u(t), \tag{21}$$

where $u(t)$ is the AS controller (4), to conclude practical asymptotic stability of the closed-loop system. The semiglobal property comes from the fact that T is also dependent on the set in which the initial states are defined, and this completes the proof of the proposition. ■

3.2.2. *Discussion* The proof of Proposition 1 provides the qualitative argument that the discrete-time controller (19), (20) is an SP-AS controller for the system (18). In fact we observe in the motivating example that our discrete-time controller significantly outperforms the emulation controller. To understand this, we concentrate of the effect of the energy shaping controller, on which the two controllers differ.

The desired Hamiltonian difference for the discrete-time model is

$$\begin{aligned} \Delta H_d &= H_d(k+1) - H_d(k) \\ &= \frac{1}{2}p(k+1)' M_d^{-1} p(k+1) + V_d(q(k+1)) \\ &\quad - \frac{1}{2}p(k)' M_d^{-1} p(k) - V_d(q(k)). \end{aligned} \tag{22}$$

By direct calculation involving the use of Mean Value Problem, applying the controller (19) to the Euler model (18) gives

$$\begin{aligned} \Delta H_d^{u^T_{es}} &= H_d^{u^T_{es}}(k+1) - H_d^{u^T_{es}}(k) \\ &= Tp' M_d M^{-1} O_G M_d M^{-1} \nabla_q V_d(q^*) \\ &\quad - Tp' M_d^{-1} O_G \nabla_q V(q) \\ &\quad - T^2\kappa p' M_d^{-1} I_G M_d M^{-1} L_V M^{-1} p + O(T^2), \end{aligned} \tag{23}$$

with $\kappa > 0$ and $q^* = \theta q(k) + (1-\theta) q(k+1)$, $\theta \in (0,1)$. Replacing (19) with the emulation of controller (8) and repeating the calculation, we obtain

$$\begin{aligned} \Delta H_d^{u_{es}} &= H_d^{u_{es}}(k+1) - H_d^{u_{es}}(k) \\ &= Tp' M_d M^{-1} O_G M_d M^{-1} \nabla_q V_d(q^*) \\ &\quad - Tp' M_d^{-1} O_G \nabla_q V(q) + O(T^2), \end{aligned} \tag{24}$$

Subtracting $\Delta H_d^{u_{es}}$ from $\Delta H_d^{u^T_{es}}$ (taking into account the $O(T^2)$ terms that turn up to coincide for both Hamiltonian differences), we obtain that

$$\begin{aligned} &\Delta H_d^{u^T_{es}} - \Delta H_d^{u_{es}} \\ &\quad = -T^2\kappa p' M_d^{-1} I_G M_d M^{-1} L_V M^{-1} p + O(T^3) \\ &\quad = -T^2\kappa p' \overline{M} p + O(T^3) \ . \end{aligned}$$

Note that with L_V as in Proposition 1, it is guaranteed that $\overline{M} := M_d^{-1} I_G M_d M^{-1} L_V M^{-1}$ is positive semidefinite. The fact that $\Delta H_d^{u^T_{es}}$ is more negative than $\Delta H_d^{u_{es}}$ explains why $u^T(k)$ performs better than the emulation controller $u(k)$.

3.2.3. *Fully actuated case* In the case when the system is fully actuated, i.e. when the matrix G is full rank and invertible, we have $I_G = I$ and $O_G = O$ and hence, the matrix $\bar{M}$ is simplified to $\bar{M} = M^{-1} L_V M^{-1}$. If M is a diagonal matrix, and assuming that the desired potential energy V_d is positive definite and convex, then we can always choose $L_V = \nabla_{qq} V_d(q)$. Hence, the term $\left[\frac{\Delta V_d}{\Delta q}\right]$ of the energy shaping controller is nothing but the coordinate increment discrete gradient of V_d (Itoh and Abe, 1988). We can then state the following corollary, which is a special case of Proposition 1.

Corollary 3.1. Consider the Euler model (18) of the separable Hamiltonian system (1) with matrix G invertible. Then the discrete-time controller $u^T = u^T_{es} + u^T_{di}$, where u^T_{es} and u^T_{di} satisfy (19) and (20) respectively, with

$$\left[\frac{\Delta V_d}{\Delta q}\right] := \nabla_q V_d(q) + T\kappa L_V(q) M^{-1} p \ ,$$

$L_V = \nabla_{qq} V_d(q)$ and $k_v, \kappa > 0$, is a SP-AS controller for the Euler model (18). ■

Remark 3.1. Proposition 1 solves the stabilization problem for the Euler model (18), whereas in application we are interested in the stability of the sampled-data system, i.e. when we implement the controller (19), (20) to stabilize the continuous-time plant (1). Using results of (Nešić *et al.*, 1999*a*; Nešić and Teel, 2004), we can conclude the SP-AS of the sampled-data system from the SP-AS of the closed-loop approximate model, together with the consistency property of the Euler model (18) with respect to the exact discrete-time model of (1), uniform boundedness of the controller (19), (20) and uniform boundedness of the sampled-data solutions. ■

Remark 3.2. Note that all linear Hamiltonian systems belong to the class of separable Hamiltonian systems with quadratic Hamiltonian. Therefore, Proposition 1 is also applicable to general nonseparable port-controlled Hamiltonian systems, when the controller is designed using linearized model. ■

4. EXAMPLE

We consider the stabilization problem of an inertia wheel pendulum that can be modeled as a separable Hamiltonian system (1) with

$$\begin{aligned} G &= \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad M = \begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix}, \\ V(q) &= mgL(\cos q_1 - 1) = m_3(\cos q_1 - 1). \end{aligned} \tag{25}$$

The control objective is to bring the states to the origin (Ortega *et al.*, 2002*b*).

4.1 *Continuous-time controller design*

A continuous-time IDA-PBC controller for system (25) has been designed in (Ortega *et al.*, 2002*b*). The desired inertia matrix M_d is chosen as

$$M_d = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix}, \quad a_1 > 0; \quad a_1 a_3 > a_2^2. \tag{26}$$

Solving the potential energy PDE (7), namely

$$\left(\frac{a_1 + a_2}{I_1}\right)\frac{\partial V_d}{\partial q_1} + \left(\frac{a_2 + a_3}{I_2}\right)\frac{\partial V_d}{\partial q_2} = -m_3 \sin(q_1),$$

gives the newly shaped potential energy

$$V_d(q) = \frac{I_1 m_3}{a_1 + a_2}(\cos q_1) + \Phi(z(q)), \tag{27}$$

with $z(q) = q_2 - \frac{I_1(a_2+a_3)}{I_2(a_1+a_2)} q_1 =: q_2 + \gamma_2 q_1$.

The function Φ is chosen as $\Phi(z) = \frac{k_1}{2} z^2$ so that V_d attains its minimum at the origin. To simplify calculations and for the purpose of simulation, we set the parameters $I_1 = 0.1$, $I_2 = 0.2$, $m_3 = 10$, $a_1 = 2$, $a_2 = -3$, $a_3 = 5$. Having M_d and V_d, the continuous-time IDA-PBC controller is obtained as follows. The energy shaping controller is

$$\begin{aligned} u_{es} &= G^+(\nabla_q V - M_d M^{-1} \nabla_q V_d) \\ &= 30\sin(q_1) + 5k_1(q_2 + q_1), \end{aligned} \tag{28}$$

and the damping injection controller is

$$u_{di} = -k_v G' \nabla_p K_d = -k_v(-2p_1 - p_2). \tag{29}$$

It has been shown in (Ortega *et al.*, 2002*b*) that

$$u = u_{es} + u_{di} \tag{30}$$

is an *almost globally* asymptotically stabilizing controller for the inertia wheel pendulum (26).

4.2 *Discrete-time controller design*

Using the same parameters as those used in the continuous-time design, we apply Proposition 1 to design a discrete-time controller for the inertia wheel pendulum. The Euler model of the plant follows (18). From (25) we compute

$$\nabla_{q_1} V(q(k)) = -m_3 \sin(q_1) = -10\sin(q_1). \tag{31}$$

Moreover, substituting all the parameters, the gradient of the desired potential energy is obtained as

$$\nabla_q V_d(q(k)) = \begin{pmatrix} \sin(q_1) + k_1(q_1 + q_2) \\ k_1(q_1 + q_2) \end{pmatrix}. \tag{32}$$

We choose a matrix $L = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$, such that with $L_V = MLM$ we have that

$$\begin{aligned} \bar{M} &= M_d^{-1} I_G M_d M^{-1} L_V M^{-1} \\ &= M_d^{-1} I_G M_d L = \begin{bmatrix} 10 & -16 \\ 5 & -8 \end{bmatrix} L = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \end{aligned} \tag{33}$$

is positive semidefinite. Substituting (31) and L_V into (19) results in the energy shaping controller

$$\begin{aligned} u^T_{es} &= 30\sin(q_1) + 5k_1(q_2 + q_1) + \frac{T}{2}\kappa(2p_1 + p_2) \\ &= u_{es} + T\kappa(p_1 + 0.5p_2). \end{aligned} \tag{34}$$

Moreover, the damping injection controller u^T_{di} is the same as the emulation of the controller (29). Hence the discrete-time IDA-PBC controller is

$$u^T = u^T_{es} + u^T_{di} = u_{es} + u_{di} + T\kappa(p_1 + 0.5p_2). \tag{35}$$

We compare the performance of the controller (35) with the emulation controller, using the continuous-time controller (30) as reference, to control the continuous-time plant. For relatively small sampling period both controllers are stabilizing the plant, although the response of the closed-loop system exhibits more oscillations with the emulation controller. Figure 3 shows the responses of the closed-loop system using each of the three controllers, when we use parameters $k_1 = 1$, $k_v = 5$, $\kappa = 700$, $T = 0.033$ and $(q_\circ, p_\circ)' = (2, 0, 0, 0)'$ in the simulation. It is observed that the closed-loop system with the discrete-time controller (35) is stable and the response stays close to the closed-loop response of the continuous-time system, whereas the closed-loop system is unstable with the emulation controller.

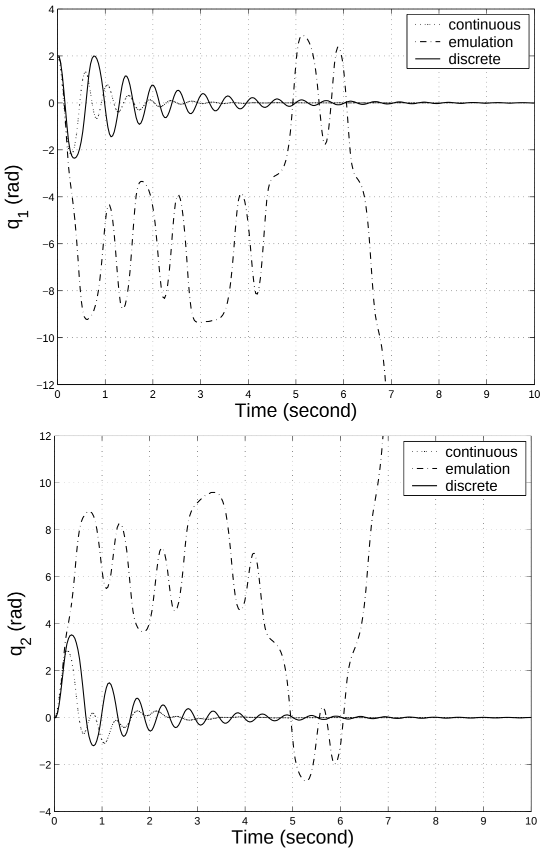

Figure 3. Response of the inertia wheel pendulum.

5. SUMMARY

We have presented a result on discrete-time control design for Hamiltonian systems. We have proposed a discrete-time counterpart of IDA-PBC design for separable Hamiltonian systems. It is proved that our discrete-time design yields a SP-AS controller for the discrete-time model, which can further be shown to be a SP-AS controller for the sampled-data system. It has been shown that our proposed discrete-time IDA-PBC outperforms the emulation controller obtained by sample and hold of the continuous-time IDA-PBC controller. This result gives a strong motivation for further investigation on discrete-time and sampled-data design for Hamiltonian conservative systems.

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