A DELAY-DEPENDENT STABILITY CRITERIA FOR T-S FUZZY SYSTEM WITH TIME-DELAYS

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Abstract: This paper addresses the criterion of delay-dependent stability for nonlinear systems with time-delays which can be represented by Takagi–Sugeno (T–S) fuzzy model with time-delays. Based on Leibniz-Newton formula, the effect of states and the delay states to the stability is considered to construct a Lyapunov functional and to form some special equations, which facilitate proposing a new delay-dependent stability criterion with less conservatism compared with the existing methods, which breaked through the limit to the varying ratio of time-delay. An illustrative example is presented to demonstrate the validity of the proposed stability criterion.

Keywords: delay-dependent stability; Takagi–Sugeno (T–S) fuzzy system; Linear Matrix Inequality(LMI); Leibniz-Newton formula

1. INTRODUCTION

During the past several years, Takagi-Sugeno (T-S) fuzzy systems have been concerned(K. Tanaka, et al., 1996, J. Joh et al., 1998, G. Feng, 2001). The advantage of fuzzy modes, compared with conventional mathematical models, is which can represent complex nonlinear systems by fuzzy sets and fuzzy reasoning. The T-S fuzzy systems are to combine some simple local linear dynamic systems with their linguistic description to represent highly nonlinear dynamic systems. This method is feasible, since in many situations, human experts can provide linguistic descriptions of local systems in terms of IF-THEN rules. The method is quite interesting because it gives a way to smoothly connect local linear systems to form global nonlinear systems by fuzzy membership functions. T-S fuzzy model can provide an effective representation of complex nonlinear systems in terms of fuzzy sets and fuzzy reasoning applied to a set of linear input-output submodels.

On the other side, the problem of stability of time-delay systems has received considerable attention in the last two decades. It can be classified into two categories: delay-dependent and delay-independent criteria(E. Fridman, 2001). Since delay-dependent criteria make use of information on the length of delays, they are less conservative than delay-independent ones, especially, when the size of delay is small. To reduce the conservatism, many researchers have made great efforts and put forward lots of approaches. The main approaches consist of model transformation of the prototype system, the construction of new Lyapunov-Krasovskii functional and the development of

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new bounding techniques for the inner product of involved cross-terms. It has been proved that the transformed system is not equivalent to the original one, through the first-order transformation. This is because additional eigenvalues are introduced into the system characteristic equation. And the neutral transformation also yields a unequivalent system to the original one. A descriptor model transformation can transform the original system to an equivalent descriptor form representation and will not introduce additional dynamics(E. Fridman et al., 2001). In order to reduce the conservatism arisen from the inequality $-a^T b \le a^T X a + b^T X^{-1} b, a, b \in R^n, X > 0$, used to determine the stability of the system, Park(Park, et al., 1998) introduced a free matrix M, and obtain a less conservative inequality, which is the well-known Park Inequality. And Moon etc. (Y. S. Moon et al., 2001(a); Y. S. Moon et al., 2001(b)) extended it to a more general form, the Moon inequality. In their analysis, they applied the Leibniz-Newton formula, but they just replaced some of the term $x(t-\tau)$ with $x(t) - \int_{t-\tau}^{t} \dot{x}(s)\, ds$ and do not consider the effect of $x(t), x(t-\tau)$ and $\dot{x}(t)$ to the stability result. Wu etc. (M. Wu, et al., 2004) proposed a new analysis approach applying the Leibniz-Newton formula. However they only consider the linear systems, and they don't break through the limit to the varying ratio of the time-delay, so their result is somehow conservative.

Delayed fuzzy systems were first introduced and studied in(Y. Cao, et al., 2000) with developing the T-S fuzzy model. Generally speaking, the dynamic behaviors of systems with delay are more complicated than those without any delays. Fuzzy delayed systems are combined with fuzzy linguistic descriptions to achieve nonlinearity. The stability of T-S model fuzzy systems without delays has been widely studied in the recent years, and many results have been reported. However as for T–S fuzzy systems with time-delays, there are relatively seldom literatures that consider the delay-dependent stability(Y. Cao, et al., 2001; Cao, Y. Y., 2001 ;X. P. Guan, et al., 2001).We extend the method of Min Wu to T-S fuzzy system, and get new criteria of asymptotical stability. In many cases, it is difficult to know the delays exactly, but we can estimate the bounds of the delays in advance. In this paper, the upper bound of the varying ratio of time-delay is not necessary to be known.

2. PROBLEM FORMULATION

In this section, we first consider a nonlinear time-delay system represented by the T-S fuzzy model with time-delay :

Rule i : If $M_1(t)$ is F_{i1} and $M_2(t)$ is F_{i2} and$\cdots$and $M_g(t)$ is F_{ig}

Then

$$\begin{cases} \dot{x}(t) = A_i x(t) + A_{di} x(t - d(t)), \\ x(t) = \phi(t), -\tau \le t \le 0, \\ 0 \le d(t) \le \tau, \dot{d}(t) \le \mu. \end{cases} \tag{1}$$

where $x \in \mathcal{R}^n$ denotes the state vector, $M_1(t), M_2(t), ..., M_g(t)$ are the premise variables, τ and μ are positive constant upper bound for delay time and its derivative respectively, F_{ij} is the fuzzy set, $i = 1, ..., \bar{n}$, $j = 1, ..., g$. $\bar{n}$ is the number of If-Then rules. $\phi(t)$ is an initial value, A_i and A_{di} represent constant matrices with appropriate dimensions, respectively.

Using center-average defuzzier, product inference and singleton fuzzier, the dynamic fuzzy model (1) can be expressed as the following global model:

$$\dot{x}(t) = \sum_{i=1}^{\bar{n}} h_i(t) \left(A_i x(t) + A_{di} x(t - d(t)) \right), \tag{2}$$

where $h_i(t)$ is the normalized membership function which satisfies

$$h_i(t) = \frac{\mu_i(M(t))}{\sum\limits_{i=1}^{\bar{n}} \mu_i(M(t))}, \quad h_i(M(t)) \ge 0,$$
$$\sum_{i=1}^{\bar{n}} h_i(t) = 1, \mu_i(M(t)) \ge 0,$$
$$\mu_i(M(t)) = \prod_{j=1}^{g} F_{ij}(M(t)),$$

and $F_{ij}(M_j(t))$ is the grade of membership of $M_j(t)$ in the fuzzy set F_{ij}. In the next section, a delay-dependent criteria of asymptotical stability about global system (2).

3. STABILITY RESULT

Firstly, we consider the stability criterion of the global system (2). The following theorem is given.

Theorem 1. Given a constant delay upper bound τ, The global T-S fuzzy system (2) is asymptotically stable for any $0 < d(t) \le \tau$, if there exist $P = P^T > 0$, $Q = Q^T > 0$, $Z = Z^T > 0$ and appropriated real matrices N_k , P_k $(k = 1, 2)$ and X_{i11}, X_{i12},X_{i22} such that the following LMIs hold:

$$\Theta_i = \begin{bmatrix} L_{i11} & L_{i12} & L_{i13} \\ L_{i12} & L_{i22} & L_{i23} \\ L_{i13} & L_{i23} & L_{i33} \end{bmatrix} < 0, \tag{3}$$

$$\Xi_i = \begin{bmatrix} X_{i11} & X_{i12} & N_1 \\ X_{i12}^T & X_{i22} & N_2 \\ N_1^T & N_2^T & Z \end{bmatrix} \ge 0, \tag{4}$$

where

$$L_{i11} = N_1 + N_1^T + P_1 A_i + A_i^T P_1^T + \tau X_{i11} + Q,$$
$$L_{i12} = -N_1 + N_2^T + P_1 A_{di} + \tau X_{i12},$$
$$L_{i13} = -P_1 + A_i^T P_2^T + P,$$
$$L_{i22} = -N_2 - N_2^T + P_2 + P_2^T + \tau X_{i22} - (1-\mu) Q,$$
$$L_{i23} = A_{di}^T P_2^T,$$
$$L_{i33} = -P_2 - P_2^T + \tau Z.$$

Proof. Choose the delay-dependent Lyapunov functional as

$$\begin{aligned} V(x_t) &= V_1 + V_2 + V_3 \\ V_1 &= x^T(t) P x(t) \\ V_2 &= \int_{t-d(t)}^{t} x^T(s) Q x(s) ds \\ V_3 &= \int_{-\tau}^{0} \int_{t+\theta}^{t} \dot{x}^T(s) Z \dot{x}(s) ds d\theta \end{aligned} \tag{5}$$

where the matrices $P = P^T > 0$, $Q = Q^T > 0$ and $Z = Z^T > 0$ need to be determined.

Using Leibniz-Newton formula

$$x(t) - x(t-d(t)) - \int_{t-d(t)}^{t} \dot{x}(s) ds = 0 \tag{6}$$

one has the following equation

$$\begin{aligned} &2\left[x^T(t) N_1 + x^T(t-d(t)) N_2\right] \\ &\times \left[x(t) - x(t-d(t)) - \int_{t-d(t)}^{t} \dot{x}(s) ds\right] = 0 \end{aligned} \tag{7}$$

with appropriate real matrices $N_i\,(i=1,2)$.

Moreover, the following equation also holds:

$$\begin{aligned} &2\left[x^T(t) P_1 + \dot{x}^T(t) P_2\right] \times [-\dot{x}(t) \\ &+ \sum_{i=1}^{n} h_i(t)(A_i x(t) + A_{di} x(t-d(t)))] = 0 \end{aligned} \tag{8}$$

with appropriate real matrices $P_i\,(i=1,2)$.

The derivative of V_1, V_2, and V_3, are as follows, respectively:

$$\begin{aligned} \dot{V}_1 &= 2x^T(t) P \dot{x}(t) \\ &= 2x^T(t) P \dot{x}(t) + 2\left[x^T(t) P_1 + \dot{x}^T(t) P_2\right] \\ &\quad \times \Bigg[-\dot{x}(t) + \sum_{i=1}^{n} h_i(t)(A_i x(t) - A_{di} \\ &\quad \times x(t-d(t)))\Bigg] \\ &\quad + 2\left[x^T(t) N_1 + x^T(t-d(t)) N_2\right] \\ &\quad \times \left[x(t) - x(t-d(t)) - \int_{t-d(t)}^{t} \dot{x}(s) ds\right] \\ &= \sum_{i=1}^{n} h_i(t)\left(\xi^T(t) M_i \xi(t)\right) - 2\left[x^T(t) N_1\right. \\ &\quad \left. + x^T(t-d(t)) N_2\right] \int_{t-d(t)}^{t} \dot{x}(s) ds \end{aligned} \tag{9}$$

$$\begin{aligned} \dot{V}_2 &= x^T(t) Q x(t) \\ &\quad - \left(1 - \dot{d}(t)\right) x^T(t-d(t)) Q x(t-d(t)) \\ &\le x^T(t) Q x(t) \\ &\quad - (1-\mu) x^T(t-d(t)) Q x(t-d(t)) \end{aligned} \tag{10}$$

$$\begin{aligned} \dot{V}_3(t) &= \tau \dot{x}^T(t) Z \dot{x}(t) \\ &\quad - \int_{-\tau}^{0} \dot{x}^T(t+\theta) Z \dot{x}(t+\theta) d\theta \\ &= \tau \dot{x}^T(t) Z \dot{x}(t) - \int_{t-\tau}^{t} \dot{x}^T(s) Z \dot{x}(s) ds \end{aligned} \tag{11}$$

where

$$\begin{aligned} \xi(t) &= \begin{bmatrix} x(t) & x(t-d(t)) & \dot{x}(t) \end{bmatrix}^T \\ M_i &= \begin{bmatrix} M_{i11} & M_{i12} & M_{i13} \\ M_{i12}^T & M_{i22} & M_{i23} \\ M_{i13}^T & M_{i23}^T & M_{i33} \end{bmatrix} \\ M_{i11} &= N_1 + N_1^T + P_1 A_i + A_i^T P_1^T \\ M_{i12} &= -N_1 + N_2^T + P_1 A_{di} \\ M_{i13} &= -P_1 + A_i^T P_2^T + P \\ M_{i22} &= -N_2 - N_2^T \\ M_{i23} &= A_{di}^T P_2^T \\ M_{i33} &= -P_2 - P_2^T \end{aligned}$$

From $\Xi \ge 0$, so $X_i = \begin{bmatrix} X_{i11} & X_{i12} \\ X_{i12}^T & X_{i22} \end{bmatrix} \ge 0$, and $0 \le d(t) \le \tau$,then the following inequality holds:

$$\begin{aligned} &\tau \sum_{i=1}^{n} h_i(t) \eta^T(t) X_i \eta(t) \\ &- \sum_{i=1}^{n} h_i(t) \int_{t-d(t)}^{t} \eta^T(s) X_i \eta(s) ds \ge 0, \end{aligned} \tag{12}$$

where $\eta(s) = \begin{bmatrix} x(s) & x(s-d(s)) \end{bmatrix}^T$.

Then, the derivative of $V(x_t)$ is as following:

$$\dot{V}(x_t) = \dot{V}_1 + \dot{V}_2 + \dot{V}_3 \qquad (13)$$
$$\leq \xi^T(t)\Theta_i\xi(t) - \int_{t-d(t)}^{t} \varsigma^T(s)\Xi_i\varsigma(s)\,ds,$$

where

$$\varsigma(s) = \left[\, x(t)\;\; x(t-d(t))\;\; \dot{x}(s) \,\right]^T,$$

The proof is completed. ■

Remark: Because when $\mu \geq 1$, $V_2 > 0$ and $\dot{V}_2 > 0$, one can always select an appropriate positive-definite matrix Q, such that the influence of V_2 to the stability of system (1) is near zero, V_2 can be canceled. But when $\mu < 1$, the existence of V_2 can reduce the conservatism than without V_2.

4. AN ILLUSTRATIVE EXAMPLE

Consider the following fuzzy system with time-delay:

$$\dot{x}(t) = \sum_{i=1}^{3} h_i(t)\left(A_i x(t) + A_{di} x(t-d(t))\right) \qquad (14)$$

where

$$x(t) = \left[\, x_1(t)\;\; x_2(t)\;\; x_3(t) \,\right]^T$$
$$A_1 = \begin{bmatrix} -2.1 & 0.2 & -0.1 \\ 0.1 & -1.2 & 0 \\ 0 & 0.2 & -1.2 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -2.0 & -0.1 & 0.3 \\ 0.2 & -1.1 & 0.1 \\ 0.1 & 0 & -1.3 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} -1.9 & 0.1 & 0.2 \\ 0.1 & -0.9 & 0 \\ 0 & 0 & -1.1 \end{bmatrix},$$
$$A_{d1} = \begin{bmatrix} -1.1 & 0.1 & 0 \\ -1.0 & -1.1 & 0.1 \\ 0.2 & 0 & -1.5 \end{bmatrix},$$
$$A_{d2} = \begin{bmatrix} -1.0 & -0.1 & 0.2 \\ -1.1 & -1.2 & 0.3 \\ 0 & 0.1 & -1.4 \end{bmatrix},$$
$$A_{d3} = \begin{bmatrix} -0.9 & 0.1 & 0.1 \\ -0.9 & -0.9 & 0.1 \\ 0.1 & 0 & -1.3 \end{bmatrix},$$

and the membership function are given as:

$$h_1 = \begin{cases} 1, & \text{if } x_2 \leq -0.2 \\ -x_2, & \text{if } -0.2 < x_2 < 0 \\ 0, & \text{if } x_2 \geq 0 \end{cases}$$
$$h_2 = 1 - h_1 - h_3,$$
$$h_3 = \begin{cases} 0, & \text{if } x_2 \leq 0 \\ x_2, & \text{if } 0 < x_2 < 0.4 \\ 1, & \text{if } x_2 \geq 0.4 \end{cases}$$

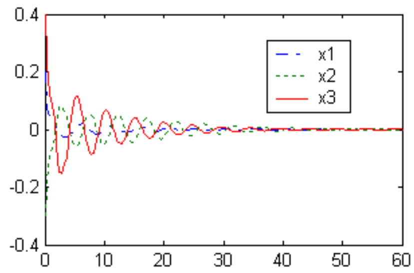

Fig. 1. The state trajectories of the T-S fuzzy system with timedelay (14).

The upper bounds on the time delay obtained from Theorem2 are displayed in the following table, with different μ.

Table 1. $\tau_{\max}$ with different μ

μ	0	0.1	0.5	0.9	1.0
$\tau_{\max}$	2.0645	1.8115	1.1530	0.7144	0.6096

and we can see that when $d(t) \leq \tau_{\max}$, and $\dot{d} \leq \mu$, system (14) is asymptoticlly stable.

5. CONCLUSION

This note consider the problem of delay-dependent stability criteria of the time-delay fuzzy system, we takes the relationship between $x(t-d(t))$, $x(t)$ and $\dot{x}(s)$ into account. Some free weighting matrices that express the influence of these three terms are determined based on linear matrix inequalities, which makes it easy to choose suitable ones. Finally, a numerical example suggest that the method presented here is very effective.

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