APPLICATION OF POLYNOMIAL SYSTEMS THEORY TO NONLINEAR SYSTEMS

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Abstract: A global linearization of nonlinear input-output descriptions by relaxing the appropriate appearances of inputs, outputs and their derivatives is applied. The resulting time-varying linear systems are described by skew polynomial systems accepting most of the methods of ordinary polynomial systems. The properties like stability and controllability of nonlinear systems can be analyzed using the linear theory. The feedback designs can also be carried out by linear methods. The methodology is illustrated by examples.

Keywords: Nonlinear systems, time-varying systems, skew polynomials, feedback control.

1. INTRODUCTION

The polynomial systems theory for time-invariant linear differential and difference systems is a well-established and efficient tool for control system analysis and design (Blomberg and Ylinen, 1983). The methodology utilises the algebraic properties of polynomials with real or complex coefficients . The theory is computational in nature, i.e. the ring of polynomials over a field in an operator normally satisfies a division algorithm which can be used to find common factors and to manipulate polynomial matrices into suitable canonical forms in an algorithmic way. Thus all computations can be implemented with a computer.

The ordinary polynomial systems theory has been generalized to time-varying (Ylinen, 1980; Ylinen and Zenger, 1991) and distributed parameter systems (Hätönen and Ylinen, 2002) and the resulting theory is, in principle, similar to the time-invariant one. The main differences are that symbolic parameters and in time-varying case the algebraic methodology related to noncommutative skew polynomials are needed. Thus all computations must be done symbolically which in practice limits the complexity of the problems to be handled.

For nonlinear systems there does not exist any similar machinery. Therefore their analysis and design are most commonly based on local linearization near to some nominal input-output pair. This leads to linear but often time-varying systems which can be used for analysis and design by applying standard methods of linear systems theory.

However, in many nonlinear problems there is difficult or even impossible to find such a nominal pair that the local linearization is accurate enough. The nominal solution is not needed, if the model is linearized in the neigborhood of the actual solution and the differentials of the variables are used as local variables (Zheng *et al.*, 2001). The resulting time-varying models can be used for analysis but for control design the differentials are not suitable.

One way to obtain global linear models is to use non-linear coordinate (or variable) transformations and the tools and methods of differential geometry. This leads to relatively sophisticated non-linear algebra and complicated calculations (Isidori, 1995).

In this paper the idea to embed a non-linear system globally into a more general, linear, time-varying system is utilized. This is done by replacing the appropriate appearances of inputs, outputs and their derivatives by free time-varying parameters (Ylinen, 2002). The properties like stability, controllability, etc. of the linear system can be analysed using the skew polynomial methods and the *structural* results are applicable also to the original non-linear system.

Feedback design can also be carried out by applying linear skew polynomial methods to the linear model. The final non-linear feedback controller is obtained by putting the input-output signals and their derivatives back into the parameters. The methodology is illustrated by examples.

2. EMBEDDING NONLINEAR SYSTEMS INTO LINEAR SYSTEMS

Consider a nonlinear *input-output (IO-) relation* S consisting of *input-output pairs* (u, y) satisfying the equation

$$f(y(t), y^{(1)}(t), ..., y^{(n)}(t), u(t), u^{(1)}(t), ..., u^{(m)}(t)) = 0 \qquad (1)$$

where $t \in T \stackrel{\triangle}{=}$ *time set* $=$ open interval $\subset \mathbf{R}$, u,$y \in \mathcal{X} \stackrel{\triangle}{=}$ *signal space* $\subset \mathbf{C}^T$, $u^{(i)}, y^{(i)}$ are ith derivates of u and y, respectively, and f is a function $\mathbf{C}^{n+m+2} \to \mathbf{C}$. $\mathbf{R}$ and $\mathbf{C}$ above denote the real and complex numbers, respectively. In order to present a real system, the model should be *realizable*, which means that the knowledge of the past of the system and the future input uniquely determine the future output. This usually means that in (1) the highest derivative $y^{(n)}(t)$ should be uniquely solvable.

In what follows it will be shown that under some smoothness properties the nonlinear model (1) can be written to the form

$$\sum_{i=0}^{n} a_i(x(t))y^{(i)}(t) = \sum_{i=0}^{m} b_i(x(t))u^{(i)}(t) \qquad (2)$$

with $x(t) = (y(t), y^{(1)}(t), ..., y^{(n)}(t), u(t), u^{(1)}(t), ..., u^{(m)}(t))$ or an appropriate sublist of it. Obviously at least some of the coefficients a_i, b_i will be rational functions in time, so that the whole model must also be extended to rational signals.

At first sight, the linearization above seems to be only a trick because the model is still nonlinear.

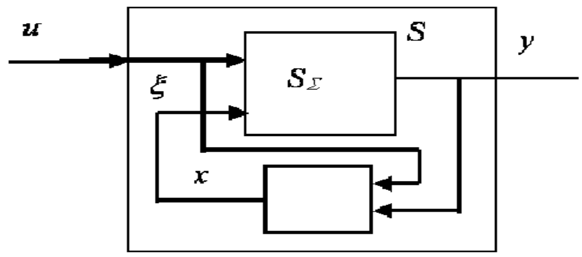

Fig. 1. Embedding

However, in many analysis and design problems the dependence of the parameters a_i, b_i on the variables u, y can be *relaxed*, so that x in (2) is replaced by a list ξ of *independent* time-varying parameters resulting in

$$\sum_{i=0}^{n} a_i(\xi(t))y^{(i)}(t) = \sum_{i=0}^{m} b_i(\xi(t))u^{(i)}(t) \qquad (3)$$

Let S_ξ be the IO-relation determined by (3). Varying ξ in a set Σ of possible parameter values the original $S(= S_x)$ can be *embedded* into

$$S_\Sigma \stackrel{\triangle}{=} \{((u, \xi), y) | (u, y) \in S_\xi, \xi \in \Sigma\} \qquad (4)$$

and S can be considered as its subset determined by $\xi = x$. The situation is depicted by Fig.1. Now the properties like stability, controllability, etc. of each S_ξ can be analyzed using the linear theory, and if the properties are *structural*, i.e. they are valid for all parameter values ξ, they are directly applicable to S. Furthermore, feedback and observer designs can be carried out by linear methods.

3. TIME-VARYING POLYNOMIAL SYSTEMS

Time-varying linear differential input-output systems are usually described by differential equations of the form

$$\sum_{i=0}^{n} a_i(t)y^{(i)}(t) = \sum_{i=0}^{m} b_i(t)u^{(i)}(t) \qquad (5)$$

where a_i,$b_i \in K \stackrel{\triangle}{=}$ *coefficient space* $\subset \mathbf{C}^T$.

If the signal space is closed with respect to differentiation, these equations can be presented in operator equation form

$$(\sum a_i p^i)y = (\sum b_i p^i)u \qquad (6)$$

where $p \stackrel{\triangle}{=} \frac{d}{dt} \stackrel{\triangle}{=}$ the *differentiation operator* on $\mathcal{X}$.

Under appropriate differentiability and closedness properties of the coefficient space the operators $a(p) = \sum a_i p^i$ constitute the (noncommutative) ring $K[p; 1_K, p_K]$ of *skew polynomials* (or *skew polynomial forms*) with respect to addition $\sum a_i p^i + \sum b_i p^i = \sum (a_i + b_i) p^i$ and multiplication $(\sum a_i p^i)(\sum b_i p^i) = \sum c_i p^i$ which can be constructed using the equation

$$pb = 1_K(b)p + p_K(b) = bp + b^{(1)} \tag{7}$$

repeatedly. Here $1_K \triangleq$ the identity operator on K and $p_K \triangleq$ the differentiation operator on K. Note that K can be considered as a subring of $K[p; 1_K, p_K]$.

Let $\mathcal{X}$ be 'sufficiently rich' in the sense that the powers $p^0, p^1, p^2, \ldots$ are *linearly independent* over K, i.e. the representation of each $a(p) \in K[p; 1_K, p_K]$ is unique and the *degree* $\deg a(p) \triangleq \max\{i | a_i \neq 0\}$ with $\deg 0 \triangleq -\infty$ is well-defined.

In order to obtain stronger algebraic structures, the coefficient space K should have some additional properties. For instance, if K is a field, i.e. each nonzero coefficient is pointwise invertible, $a(p) \in K[p; 1_K, p_K]$ satisfies the *left division algorithm (LDA)*

$$a(p) = q(p)b(p) + r(p), \ \deg r(p) < \deg b(p) \tag{8}$$

$K[p; 1_K, p_K]$ satisfies also the *right division algorithm (RDA)* defined correspondingly. Note that there are very few suitable coefficient spaces which are fields.

Suppose that K contains a unit element 1 (= the constant parameter equal to 1). Let D be a subset of $K[p; 1_K, p_K]$ closed under multiplication and let $1 \in D$. Suppose further that all elements of D are nonzerodivisors (an element a is a zerodivisor if $ab =$ for some $b \neq 0$) and for all $c(p) \in K[p; 1_K, p_K]$, $d(p) \in D$ there exist $b(p) \in K[p; 1_K, p_K]$, $a(p) \in D$ such that

$$a(p)c(p) = b(p)d(p) \tag{9}$$

Then D is a *denominator set* and $K[p; 1_K, p_K]$ can be extended via *embedding* $a(p) \mapsto a(p)/1$ to the *ring* $K[p; 1_K, p_K]_D$ of *(left) fractions* (or *quotients, rationals, rational forms*)

$$b(p)/a(p) \triangleq (a(p)/1)^{-1}(b(p)/1)$$
$$b(p) \in K[p; 1_K, p_K] \ , \ a(p) \in D \tag{10}$$

Eq. (9) is needed for addition and multiplication of fractions. If K is a field, the construction of $a(p), b(p)$ in (9) can be accomplished by the repeated use of the RDA.

If K is not a field but an integral domain, i.e it lacks zerodivisors (others than 0), then $K^* \triangleq K - \{0\}$ is a denominator set in K and K can be *extended* to its *field of fractions* $K_{K^*} \triangleq F$. K^* is a denominator set also in $K[p; 1_K, p_K]$, and $K[p; 1_K, p_K]_{K^*} = F[p; 1_F, p_F]$, where 1_F and p_F are natural extensions of 1_K and p_K, respectively. Now $F[p; 1_F, p_F]$ satisfies the LDA and RDA. Note that the elements of F can usually be identified with corresponding rational functions.

The signal space $\mathcal{X}$ is a *left module* both over K and $K[p; 1_K, p_K]$. When K is extended to $K_{K^*} = F$, $\mathcal{X}$ has also to be extended via embedding $x \mapsto x/1$ to the *module* $\mathcal{X}_{K^*}$ of *(left) fractions*

$$x/a \triangleq (a/1)^{-1}(x/1) \tag{11}$$

over F or over $F[p; 1_F, p_F]$. The original $\mathcal{X}$ can be considered as a module over F or over $F[p; 1_F, p_F]$ only if for each $d \in K^*$ the mapping $x \mapsto dx$ is an automorphism of $\mathcal{X}$.

In order to maintain the possibility for varying the initial conditions of systems, the signal space is supposed to be so 'rich' that it contains all complex-valued solutions y to all equations $a(p)y = 0$, with monic $a(p)$ (the leading coefficient equal to 1). A suitable signal space is for instance the space of all complex-valued infinitely continuously differentiable functions on an open interval T of $\mathbf{R}$ denoted usually by C^∞.

The space of complex-valued *analytic functions* denoted here by $\mathcal{A}$ is an integral domain and satisfies thus the assumptions for coefficients above. In particular, in extension of $\mathcal{A}$ to $\mathcal{A}_{\mathcal{A}^*}$ the embeddings $a \mapsto a/1$ and $x \mapsto x/1$ are injections (monomorphisms) so that there are one-one correspondences between the original signals and coefficients and the extended ones. C^∞ itself cannot be considered as a module over $\mathcal{A}_{\mathcal{A}^*}$ but it has first to be extended to the space $C^\infty_{\mathcal{A}^*}$ of 'rational signals' x/a which then is a module (in fact a *vector space*) over the field $\mathcal{A}_{\mathcal{A}^*}$. Note that $\mathcal{A}_{\mathcal{A}^*}$ can be identified with the space of *meromorphic functions* denoted here by $\mathcal{M}$(Dieudonne, 1969). Each $b/a \in \mathcal{A}_{\mathcal{A}^*}$ defines a meromorphic function $m : T - \{\text{zeros of } a\} \to C, t \mapsto b(t)/a(t) = (b/a)(t)$ and for each meromorphic function m there exists a fraction $b/a \in \mathcal{A}_{\mathcal{A}^*}$ with which it can be defined.

Consider now an arbitrary IO-relation $S \subset \mathcal{X} \times \mathcal{X}$ and suppose that $\mathcal{X}$ is extended to a module of fractions $\mathcal{X}_D$. If $\mathcal{X}$ cannot be identified with $\mathcal{X}_D$, the problem is, whether the model S is still applicable, and if not, whether there exists an extended IO-relation S_D in $\mathcal{X}_D \times \mathcal{X}_D$ such that the composite relations satisfy $j \circ S = S_D \circ j$, where j is the embedding $j : x \mapsto x/1$. In (Hätönen and Ylinen, 2002) the following proposition is given:
If every $d(p) \in D$ is a monomorphism then $j \circ S = S_D \circ j$ if for every $(u, y) \in \mathcal{X} \times \mathcal{X}$ there holds: $(u, y) \in S \iff (u/1, y/1) \in S_D$.

4. NONLINEAR POLYNOMIAL SYSTEMS

Suppose now that the signal space $\mathcal{X}$ is restricted to the space $\mathcal{A}$ of complex valued analytic functions on T and that the function $f : \mathbf{C}^{n+m+2} \to \mathbf{C}$ in (1) is analytic. This implies that for an arbitrary $x \in \mathcal{A}^{n+m+2}$ the composite function $t \mapsto f(x(t))$, is also analytic (Dieudonne, 1969). This further means that f determines pointwise a function $\mathcal{A}^{n+m+2} \to \mathcal{A}$, $x \mapsto f(x(.))$, which can notationally be identified with f.

Consider the signal space $\mathcal{A}$ as a module over $\mathcal{A}[p; 1_{\mathcal{A}}, p_{\mathcal{A}}]$ and extend it to the module of fractions $\mathcal{A}_{\mathcal{A}^*} = \mathcal{M}$ over $\mathcal{A}[p; 1_{\mathcal{A}}, p_{\mathcal{A}}]_{\mathcal{A}^*} = \mathcal{M}[p; 1_{\mathcal{M}}, p_{\mathcal{M}}]$. The function f can be extended to a function $\mathcal{M}^{n+m+2} \to \mathcal{M}$ pointwise by assignment $t \mapsto f(x_1(t)/d_1(t), ..., x_{n+m+2}(t)/d_{n+m+2}(t))$, $t \in T - \{$ zeros of $d_1, ..., d_{n+m+2}\}$.

Each point in the range of f can be presented as a linear combination over $\mathcal{M}$ of some arbitrary set of elements. Thus

$$\begin{aligned} f(z/d) &= f(x_1/d_1, ..., x_{n+m+2}/d_{n+m+2}) \\ &= \sum_{i=1}^{n+m+2} c_i(z/d)(x_i/d_i) \end{aligned} \quad (12)$$

where $z/d = (x_1/d_1, ..., x_{n+m+2}/d_{n+m+2}$. According to the proposition above, the extension $S_{\mathcal{A}^*}$ of S has to satisfy $(u, y) \in S \iff (u/1, y/1) \in S_{\mathcal{A}^*}$. This condition is satisfied if $S_{\mathcal{A}^*}$ is chosen as

$$\begin{aligned} S_{\mathcal{A}^*} = \{(u/c, y/d) |\ & f(y/d, ..., p^n(y/d), \\ & u/c, ..., p^m(u/c)) = 0\} \end{aligned} \quad (13)$$

Now, for pairs $(u, y) = (u/1, y/1)$ Eq. (1) is equivalent to a linear combination over $\mathcal{M}$ of $p^i y, i = 0, ..., n,\ p^i u, i = 0, ..., m$

$$\sum_{i=0}^{n} a_i(x)p^i y - \sum_{i=0}^{m} b_i(x)p^i u = 0 \quad (14)$$

with $x = (y, py, ..., p^n y, u, pu, ..., p^m u)$. Finally by relaxing the coefficients a time-varying linear model

$$\sum_{i=0}^{n} a_i(\xi)p^i y = \sum_{i=0}^{m} b_i(\xi)p^i u \quad (15)$$

is obtained. Note that Eqs. (14,15) can always be brought to equations over $\mathcal{A}$ by multiplying them by a common denominator of the coefficients. The problem is that there are infinitely many ways to write the original equation (1) to the form (15). The structure depends on the problem to be studied using the model, but some general rules can be given:
(i) The original structure i.e. the inputs and outputs as well as the order (the order of the highest derivative of y) of the system should be maintained.
(ii) The behavior of the coefficients should be as "constant" as possible, i.e. they should be constants or analytic functions dependent only on low order derivatives of u and y.
(iii) In control design, in order to avoid complicated nonlinear differential equations, the coefficients should preferably be dependent only on outputs.

5. MULTIVARIABLE SYSTEMS

The approach can be generalized to multivariable systems. Then the linear IO-relations are described by matrix equations over skew polynomials. Most of the definitions and methods of time-invariant systems are applicable (Blomberg and Ylinen, 1983; Ylinen, 1980).

A multivariable, linear, time-varying IO-relation $S \subset\in \mathcal{X}^r \times \mathcal{X}^s$ can be defined by the matrix equation

$$[A(p) \vdots - B(p)] \begin{bmatrix} y \\ u \end{bmatrix} = 0 \quad (16)$$

where and $A(p)$, $B(p)$ are matrices with skew polynomial entries. The matrix $[A(p) \vdots - B(p)]$ is called a *generator* for S. The generators can be brought to equivalent forms by multiplying them by *unimodular* matrices, i.e. invertible skew polynomial matrices (Blomberg and Ylinen, 1983; Ylinen, 1980). In particular, *elementary row and column operations* can be used. Applying elementary row operations and the LDA, skew polynomial matrices can be brought to *canonical forms*, for instance to *Canonical Upper Triangular (CUT-) form* or to *Canonical Row Proper (CRP-) form*(Blomberg and Ylinen, 1983; Ylinen, 1980). In nonlinear case, all operations should be unimodular irrespective of the parameter values.

Example 1. Consider a nonlinear IO-relation described by the model

$$\begin{aligned} y_1^{(1)}(t) + y_1^2(t) + y_1(t)y_2^{(1)}(t) &= y_1(t)u(t) \\ y_1(t) + y_2^{(1)}(t) + y_1(t)y_2(t) &= u(t) \end{aligned} \quad (17)$$

Choosing $\xi_1 = y_1$ leads to the model

$$\begin{bmatrix} p + \xi_1 & \xi_1 p \\ 1 & p + \xi_1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \xi_1 \\ 1 \end{bmatrix} u \quad (18)$$

Using the elementary row operations a model of CRP-form

$$\begin{bmatrix} p & -\xi_1^2 \\ 1 & p + \xi_1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad (19)$$

and of CUT-form

$$\begin{bmatrix} 1 & p+\xi_1 \\ 0 & p^2+\xi_1 p+\xi_1^2+\xi_1^{(1)} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ p \end{bmatrix} u \quad (20)$$

are obtained. These correspond to the nonlinear models

$$\begin{aligned} y_1^{(1)}(t) - y_1^2(t)y_2(t) &= 0 \\ y_1(t) + y_2^{(1)}(t) + y_1^2(t)y_2(t) &= u^{(1)}(t) \end{aligned} \quad (21)$$

$$\begin{aligned} y_1(t) + y_2^{(1)}(t) + y_1(t)y_2(t) &= u(t) \\ y_2^{(2)}(t) + y_1(t)y_2^{(1)}(t) + y_1^2(t)y_2(t) & \\ +y_1^{(1)}(t)y_2(t) &= u^{(1)}(t) \end{aligned} \quad (22)$$

Note that all operations were unimodular irrespective of the parameter ξ_1.

6. STABILITY AND CONTROLLABILITY

If an IO-relation S is *globally asymptotically stable* then every pair $(0, y) \in S$ is such that $y(t)$ approaches 0 when the time t approaches infinity.

The stability of S generated by $[A(p)\vdots - B(p)]$ usually *cannot* be tested from the *'pointwise'* roots of $\det A(t)(p)$, where $A(t)(p)$ denotes the ordinary polynomial matrix obtained from $A(p)$ by replacing the coefficients by their values at time t. The concepts of time-varying poles and zeros are much more complicated than the time-invariant ones (Zenger and Ylinen, 2002) .

In nonlinear case, the original nonlinear IO-relation is stable if the relaxed linear IO-relation is structurally stable. This is obvious because the the feedback in Fig.1 only restricts the number of pairs in the relaxed IO-relation. However, the converse is not necessarily true.

Example 2. Let S be a bilinear IO-relation determined by

$$y^{(1)}(t) = u(t)y(t) \quad (23)$$

and linearize it by $\xi_3 = u$. This gives a linear model

$$py = \xi_3 y \quad (24)$$

which obviously means unstable behavior for some parameters ξ_3. However, the original model gives $py = 0$, i.e. the model is stable but not asymptotically stable. If the linearization is done by $\xi_1 = y$

$$py = \xi_1 u \quad (25)$$

the linear IO-relation is structurally stable implying correctly the stability of the IO-relation S.

In general, an IO-relation said to be *controllable*, if all its modes can be affected by inputs. If S is generated by $[A(p)\vdots - B(p)] = L(p)[A_1(p)\vdots - B_1(p)]$ with $A_1(p), B_1(p)$, then S can be decomposed to a parallel composition consisting of two IO-relations S_1 generated by $[A_1(p)\vdots - B_1(p)]$ and S_2 generated by $[L(p)A_1(p)\vdots\ 0]$ (Blomberg and Ylinen, 1983; Ylinen, 1980). If $L(p)$ is not unimodular, S contains modes in S_2 related to $L(p)$, which cannot be affected by the input. This means that S is not controllable. Thus the IO-relation S generated by $[A(p)\vdots - B(p)]$ is controllable if $A(p), B(p)$ are *left coprime*, i.e. have no common left divisors apart from unimodular ones.

In nonlinear case, the structural controllability of the relaxed linear IO-relation implies the controllability of the original IO-relation but not *vice versa*.

Example 3. Consider the model (23) in the previous example. The linearized model (24) is generated by $[p - \xi_3 \vdots 0]$ and is obviously structurally uncontrollable. However, the model (25) is generated by $[p \vdots -\xi_1]$ which implies controllability for all parameter values except the trivial case $\xi_1 = 0$.

Example 4. Consider the example presented in (Zheng *et al.*, 2001). Linearize the IO-relation described by

$$y^{(2)} - y^{(1)} - \frac{(y^{(1)})^2 - y^{(1)}u}{y} - u^{(1)} + u = 0 \quad (26)$$

by $\xi_1 = y$ and $\xi_2 = y^{(1)}$. This leads to the model

$$(\xi_1 p^2 - (\xi_1 + \xi_2)p)y = (\xi_1 p - (\xi_1 + \xi_2))u \quad (27)$$

Obviously, the corresponding IO-relation is generated by $(\xi_1 p - (\xi_1 + \xi_2))[p\vdots - 1]$ so that it is not controllable except in the trivial case $\xi_1 = 0$.

7. FEEDBACK COMPENSATOR DESIGN

Consider the feedback composition depicted by Fig.2 consisting of an IO-relation S_1 to be compensated and a feedback compensator S_2 to be designed so that the resulting composition is stable, robust, realizable etc. Let S_1 be generated by

$$[A(p)\vdots - B(p)] = L(p)[A_1(p)\vdots - B_1(p)] \quad (28)$$

where $A_1(p), B_1(p)$ are left coprime. Then $[A(p)\vdots - B(p)]$ and $[L(p)\vdots 0]$ are *column equivalent*, i.e.

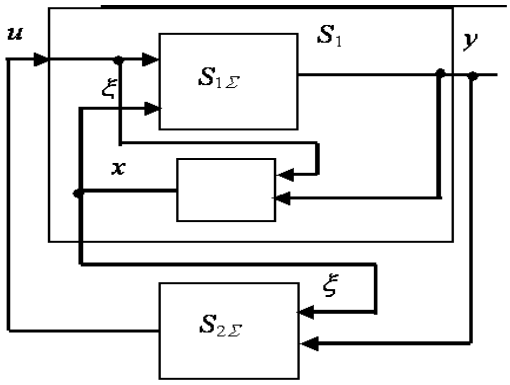

Fig. 2. Feedback design

$$[A(p)\vdots - B(p)] = [L(p)\vdots\ 0\ ] \underbrace{\begin{bmatrix} A_1(p) & B_1(p) \\ Q_3(p) & Q_4(p) \end{bmatrix}}_{Q(p)} \quad (29)$$

where $Q(p)$ is unimodular and can be constructed by elementary column operations.

Let the feedback IO-relation S_2 be generated by $[C(p)\vdots - D(p)]$. Then the feedback composition is generated by

$$\begin{bmatrix} A(p) & B(p) \\ -D(p) & C(p) \end{bmatrix} = \begin{bmatrix} L(p) & 0 \\ T_3(p) & T_4(p) \end{bmatrix} \times \begin{bmatrix} A_1(p) & B_1(p) \\ Q_3(p) & Q_4(p) \end{bmatrix} \quad (30)$$

where $T_3(p), T_4(p)$ are appropriate matrices (Blomberg and Ylinen, 1983; Ylinen, 1980). The dynamic behaviour of the system depends on $T_4(p)$ and the uncontrollable part corresponding to $L(p)$. Thus the feedback compensator can be designed starting from the first candidate $Q(p)$ by constructing first a suitable $T_4(p)$ and then a $T_3(p)$ so that the resulting feedback compensator is realizable and the whole composition is robust against the parameter variations. The construction can be carried out step by step using elementary row operations.

In nonlinear case the feedback composition should be structurally stable in order to guarantee the stability of the original nonlinear IO-relation.

Example 5. Consider again the system of Example 1. Starting from the CRP-form (19) the first candidate

$$Q(p) = \begin{bmatrix} p & -\xi_1^2 & 0 \\ 1 & p+\xi_1 & -1 \\ 1 & 0 & 0 \end{bmatrix} \quad (31)$$

can be constructed using elementary column operations. If $p+\alpha$ and $p+\beta$ are chosen as suitable dynamics for the closed loop system then elementary row operations result in the generator

$$\begin{bmatrix} p & -\xi_1^2 & 0 \\ 1 & p+\xi_1 & -1 \\ \alpha\beta & 2\xi_1\xi_1^{(1)} + \beta\xi_1^2 - \xi_1^3 & \xi_1^2 \end{bmatrix} \quad (32)$$

Thus the stabilizing controller is

$$u = -\alpha\beta/y_1 - (2y_1^{(1)}/y_1 + \beta - y_1)y_2 \quad (33)$$

The singularity at $y_1 = 0$ is cancelled in the closed loop. The robustness of this cancellation depends on the system and controller realization.

8. CONCLUDING REMARKS

Linearization by relaxation of variables offers new tools for nonlinear system theory. The main problems are the complicated calculations and the choice of variables to be relaxed.

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