PARAMETRIC APPROACH TO OPTIMAL NONLINEAR CONTROL PROBLEM USING ORTHOGONAL EXPANSIONS

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Abstract:
This article presents a parametric approach to the optimal nonlinear control problem. This contribution is based on reducing the nonlinear optimal control problem to a sequence of linear time-varying ones and approaching the desired optimal control by a sequence of linear algebraic equations. This approach provides a solution which can be obtained in a simpler way than the usual one. Alternative choices of orthonormal basis for the parametric approach are discussed.

Keywords: Nonlinear control systems, optimal control, linear time varying systems, orthonormal basis.

1. INTRODUCTION

In this paper the problem of obtaining an optimal control for a general nonlinear system is addressed. The classical method of using the Ricatti equation is replaced by an orthogonal expansion of both the state and the control law which lead to a linear algebraic equation for the unknown coefficients of the control. The extension of this approach to nonlinear systems is based on the following: first, the reduction of the original nonlinear optimal control problem to a sequence of linear time-varying ones whose solutions will converge to the solution of the original problem and secondly, the expansion of the state and the control of each of these LTV problems in terms of some orthonormal basis, such that the sequence of resulting coefficients will converge to the coefficients of the optimal control for the nonlinear system. This approach of parametrizing the control and/or the state, has been widely used by many authors (Sirisena and Chou, 1981), (Vlassenbroeck, 1988), (Yen and Nagurka, 1992), (Banks and Tomas-Rodriguez, 2001), and (Jaddu, 2002) providing satisfactory results.

Most of these parametrizations were done by using the Chebyshev polynomials as orthogonal basis to expand the control and/or state. In this paper, we also discuss the optimal choice of this basis. The iteration technique for nonlinear systems has been introduced previously in (Banks and Tomas-Rodriguez, 2002) were local and global convergence proofs were provided. This technique has been applied to different areas of control such as the design of nonlinear observers (Navarro Hernandez *et al.*, 2003), pole-placement for nonlinear systems (Tomas-Rodriguez and Banks, 2004), repetitive control (Owens *et al.*, 2003), identification of parameters (Tomas-Rodriguez and Banks, 2003) and closely related with the ideas presented here, in general optimal control (Salamci *et al.*, 2000), (Cymen and Banks, 2004), (Jianhua Zheng *et al.*, 2005), where the classical nonlinear non-

quadratic optimal control problem was addressed. The main purpose of this paper is to combine both ideas i.e; the parametrization of the control and/or state and the iteration method applied to a nonlinear system.
The paper structure is as follows: Section 2 recalls briefly the approximation technique for nonlinear systems. Section 3 presents the optimal control problem for the linear time-varying case and the solution is presented by using an orthonormal expansion in terms of some generic basis. Section 4 contains a numerical example for the LTV case. Section 5 addresses the general nonlinear optimal control problem by combining the iteration technique and the parametrization method from the previous sections. Section 6 contains a numerical example for the general nonlinear system. Section 7 gives the conclusions and suggestions for further research.

2. THE ITERATION TECHNIQUE

Consider a nonlinear system of the form

$$\dot{x}(t) = A[x(t)]x(t) + B[x(t)]u(t), \quad x(0) = x_0 \quad (1)$$

were $x(\cdot) \in \Re^n$, $u(\cdot) \in \Re^m$ and the matrices $A(x(\cdot))$ and $B(x(\cdot))$ are of appropriate dimension and satisfy a mild Lipschitz condition. Then if $x = 0$ is an equilibrium point, the system (1) can be replaced by a sequence of linear time-varying control systems whose solutions converge to the solution of the nonlinear system. This is,

$$\begin{aligned}
\dot{x}^{(1)}(t) &= A[x_0]x^{(1)}(t) + B[x_0]u^{(1)}(t) \quad (2)\\
\dot{x}^{(2)}(t) &= A[x^{(1)}(t)]x^{(2)}(t) + B[x^{(1)}]u^{(2)}(t)\\
&\vdots\\
\dot{x}^{(i)}(t) &= A[x^{(i-1)}(t)]x^{(i)}(t) + B[x^{(i-1)}]u^{(i)}(t)
\end{aligned}$$

with initial conditions $x^{(1)}(0) = x^{(2)}(0) = \cdots = x^{(i)}(0) = x_0$ at each iteration.

The sequence of solutions $x^{(i)}(t)$ converges uniformly on any compact time interval to the nonlinear solution $x(t)$ (Tomas-Rodriguez and Banks, 2003) and therefore linear control techniques are applicable to each of the equations defined in (2).

Remark 1. If $x = 0$ was not an equilibrium point of (1), it could always be obtained by an appropriate coordinate translation, provided the system had at least one such point.

Remark 2. The first iteration (only) is a LTI problem since the initial conditions are plugged into the original system matrix.

3. OPTIMAL CONTROL FOR LTV SYSTEMS USING ORTHOGONAL EXPANSIONS

Consider the optimal control problem

$$\begin{aligned}
minJ &= \frac{1}{2}x^T(T)Fx(T) \quad (3)\\
&+ \frac{1}{2}\int_0^T \frac{1}{2}x^T(t)Qx(t) + \frac{1}{2}u^T(t)Ru(t)dt
\end{aligned}$$

subject to the LTV dynamics,

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \ x(0) = x_0$$

where $x(\cdot) \in \Re^n$, $u(\cdot) \in \Re^m$ and the matrices $A(t)$,$B(t)$ are of appropiate dimensions.

Assuming stabilizability, there is a feasible control $u(t)$ which makes J finite, so that it must satisfy

$$\int_0^T x^T(t)Qx(t)dt < \infty, \quad \int_0^T u^T(t)Ru(t)dt < \infty$$

and so if Q (and R) are positive definite then $x(t)$ and $u(t)$ belong to $L^2[0,T]$ in which case we can expand them in terms of some orthonormal basis $[e_i]$ on the given time interval $[0,T]$:

$$x(t) = \sum_{i=1}^{\infty} \alpha_i e_i(t), \ u(t) = \sum_{i=1}^{\infty} \beta_i e_i(t)$$

where α_i and β_i are vector coefficients of the corresponding dimensions. The problem here is to find the unknown coefficients β_i such that the optimal control can be obtained. The solution of the time-varying system will be of the form

$$x(t) = \Phi(t;0)x_0 + \int_0^T \Phi(t;s)B(s)u(s)ds \quad (4)$$

where Φ is the evolution operator (transition matrix) for $A(t)$.

Hence, substituting the expansions of $x(t)$ and $u(t)$ in (4):

$$\begin{aligned}
\sum_i \alpha_i e_i(t) &= \Phi(t;0)x_0\\
&+ \sum_i \int_0^t \Phi(t;s)B(s)\beta_i e_i(s)ds \quad (5)
\end{aligned}$$

Expanding the initial term and the integral term gives

$$\Phi(t;0)x_0 = \sum_i \gamma_i e_i(t)$$

and

$$\int_0^t \Phi(t;s)B(s)e_i(s)ds = \sum_j \phi_{ij} e_j(t)$$

respectively for some coefficients γ_i depending on the initial state x_0, and for some coefficients ϕ_{ij}.

Hence,(5) becomes

$$\sum_i \alpha_i e_i(t) = \sum_i \gamma_i e_i(t) + \sum_i \sum_j \phi_{ij}\beta_i e_j(t) \quad (6)$$

and so, equating coefficients leads to an algebraic equation on the unknown parameters β_k ,

$$\alpha_i = \gamma_i + \sum_k \phi_{ki}\beta_k. \quad (7)$$

Next, in order to evaluate the cost functional J in this basis, it is necessary to evaluate the final term $\frac{1}{2}x^T(T)Fx(T)$. Since:

$$x(T) = \Phi(t;0)x_0 + \sum_i \int_0^T \Phi(T;s)B(s)\beta_i e_i(s)ds$$
$$= L(x_0) + \sum_i M_i\beta_i, \quad (8)$$

where

$$L(x_0) = \Phi(t;0)x_0 \quad (9)$$

and

$$M_i = \int_0^T \Phi(T;s)B(s)e_i(s)ds \quad (10)$$

The first term of the cost function takes the form

$$\frac{1}{2}x^T(T)Fx(T) =$$
$$= \frac{1}{2}\big(L^T(x_0) + \sum_i \beta_i^T M_i^T\big)F\big(L(x_0) + \sum_j M_j\beta_j\big)$$
$$= \frac{1}{2}L^T(x_0)FL(x_0) + \frac{1}{2}\big(\sum_i \beta_i^T M_i^T\big)FL(x_0)$$
$$+\frac{1}{2}L^T(x_0)F\big(\sum_j M_j\beta_j\big)$$
$$+\frac{1}{2}\sum_i\sum_j \beta_i^T M_i^T F M_j\beta_j$$

and the remaining terms of J are just quadratic forms due to the orthonormality of the basis:

$$J = \frac{1}{2}L^T(x_0)FL(x_0) + \frac{1}{2}\big(\sum_i \beta_i^T M_i^T\big)FL(x_0)$$
$$+ \frac{1}{2}L^T(x_0)F\big(\sum_j M_j\beta_j\big)$$
$$+ \frac{1}{2}\sum_i\sum_j \beta_i^T M_i^T F M_j\beta_j + \frac{1}{2}\sum_i \alpha_i^T Q\alpha_i$$
$$+ \frac{1}{2}\sum_i \beta_i^T R\beta_i$$

Now,

$$\frac{1}{2}\sum_i \alpha_i Q\alpha_i =$$
$$= \frac{1}{2}\sum_i (\gamma_i^T + \sum_k \beta_k^T\phi_{ki}^T)Q(\gamma_i + \sum_k \beta_k\phi_{ki})$$
$$= \frac{1}{2}\sum_i \gamma_i^T Q\gamma_i + \frac{1}{2}\sum_i\sum_k \beta_k^T\phi_{ki}^T Q\gamma_i$$
$$+ \frac{1}{2}\sum_i\sum_l \gamma_i^T Q\phi_{li}\beta_l + \frac{1}{2}\sum_i\sum_k\sum_l \beta_k^T\phi_{ki}^T Q\phi_{li}\beta_l$$

Therefore, J can be written as a second order infinite polynomial in the β_i's

$$J = \frac{1}{2}\sum_i\sum_j \beta_i^T K_{ij}\beta_j + \frac{1}{2}\sum_i \beta_i^T\Pi_i$$
$$+ \frac{1}{2}\sum_i^T \Pi_i^T\beta_i + \Gamma \quad (11)$$

where

$$K_{ij} = M_i^T F M_j + \sum_k \phi_{ik}^T Q\phi_{jk} + R\delta_{ij}, \quad (12)$$
$$\Pi_i = M_i^T FL(x_0) + \sum_k \phi_{ik}^T Q\gamma_k, \quad (13)$$
$$\Gamma = \frac{1}{2}L^T(x_0)FL(x_0) + \frac{1}{2}\sum_i \gamma_i^T Q\gamma_i. \quad (14)$$

From the necessary condition for an optimum,

$$\frac{\partial J}{\partial \beta_i} = 0$$

is required , so that gives

$$\sum_j K_{ij}\beta_j + \Pi_i = 0. \quad (15)$$

For simplicity, assume a scalar control, so that it can be solved for the infinite vector $\beta = (\beta_1, \beta_2, \cdots)^T$:

$$K\beta = -\Pi \quad \rightarrow \quad \beta = -K^{-1}\Pi$$

where

$$K = (K_{ij}),\ \Pi = (\Pi_1, \Pi_2, \cdots)^T$$

The infinite vector $\beta = (\beta_1, \beta_2, \cdots)^T$ is the set of parameters which generates the optimal control

$$u(t) = \sum_{i=1}^{\infty} \beta_i e_i(t).$$

The approximation to this optimal control $u(t)$ can be obtained by truncating the expansions to finite- dimensional ones. This approach is not difficult to apply, as the optimal control can be solved directly by solving a linear algebraic equation rather than a nonlinear Riccati equation backwards in time.
In the next section it will be shown how this method can be applied and computed for a finite

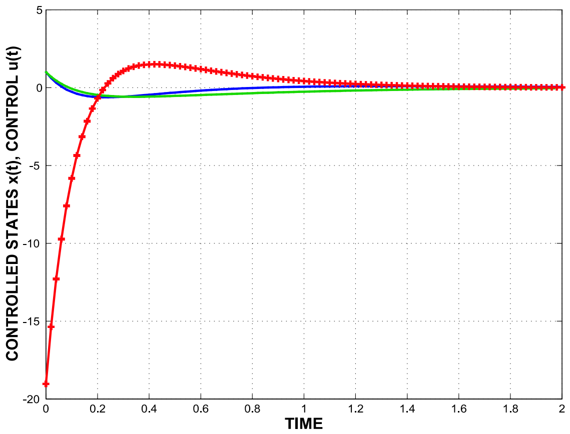

Fig. 1. Optimal Control and controlled states using Ricatti method

expansion. The optimal control will be obtained in this way for different choices of orthonormal basis.

4. NUMERICAL EXAMPLE LTV CASE

Consider the optimal control problem,

$$J = \frac{1}{2}x(t_f)^T F x(t_f) + \int_{t_0}^{t_f} (x^T Q x + u^T R u) ds$$

with $F = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, $Q = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and $R = 1$, subject to the linear time-varying constrain

$$\dot{x} = \begin{pmatrix} 0 & t & t \\ 3 & t & 1 \\ 5 & 6 & 7 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} u(t)$$

for some initial conditions $x(0) = (1, 1, 1)^T$ and a final time $t_f = 2$.

method

The task is to find the optimal control $u(t)$ that makes the state $x(t)$ minimize the given cost functional J. Solving in the classical way with the Ricatti equation, the optimal control law can be observed in figure (1) ($'+'$ line) as well as the stabilized states $x_1(t)$ and $x_2(t)$, (solid lines).

This particular example was solved using two different orthonormal basis: first by using the well known fourier orthonormal basis

$$(1, sint, cost, sin2t, cos2t, ..)$$

and secondly using Chebyshev polynomials

$$(1, t, 2t^2 - 1, \cdots, 2te_{n-1} - e_{n-2})$$

In both cases the number of terms in the expansions were chosen to be 30.

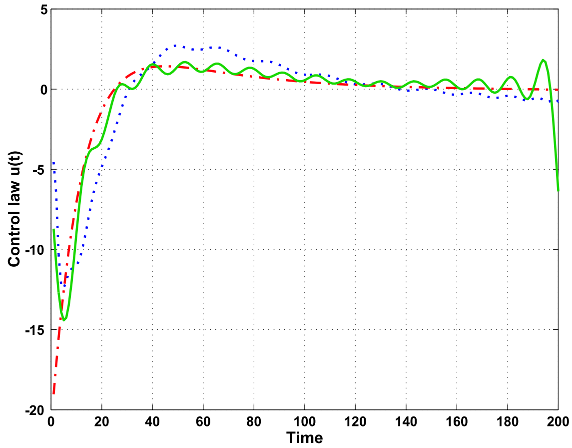

Fig. 2. Approximated Optimal Control for different choices of basis

basis

By choosing the fourier basis $(1, sint, cost, ..)$ to expand the control and state, it can be seen in figure (2), that the approach to the optimal control is only good at those intermediate points of the time interval (solid line), converging at both extremes to the middle value. In the other hand, using the Chebyshev polynomials as orthonormal basis, it is easy to see in the same figure (dotted line) how in this case the convergence towards the Ricatti optimal control (dash-dotted line) improves in the sense that the behaviour of this signal at the extremes of the interval follow the behaviour of the Ricatti control. This support the idea that the Chebyshev basis is generally a better choice.

This leads to the interesting question about which orthonormal basis are better or if there is an optimal procedure to find such a basis. In future reports, the authors would like to develop a best choice of basis "criteria" and are planing to show comparative results between the expansions in terms of different basis.

The next section consists of a generalization of the contents here presented to the nonlinear optimal control problem: In this case, bearing in mind the iteration technique in Section 2, a sequence of "i" LTV optimal control problems is generated such that the orthogonal expansion of the control $u^{(i)}(t)$ and state $x^{(i)}(t)$ for each of them can be applied. This will lead to a sequence of linear algebraic equations whose solutions will be a sequence of control coefficients $\beta^{(i)}$s converging to the control coefficients of the original nonlinear optimal control problem.

5. NONLINEAR CASE

Considering the nonlinear optimal control problem

$$
\begin{aligned}
minJ = & \frac{1}{2}x^T(T)F(x(T))x(T) \\
& + \frac{1}{2}\int_0^T \big(\frac{1}{2}x^T(t)Q(x(t))x(t) \\
& + \frac{1}{2}u^T(t)R(x(t))u(t)\big)dt
\end{aligned}
$$

subject to the dynamics

$$\dot{x}(t) = A(x(t))x(t) + B(x(t))u(t), \;\; x(0) = x_0.$$

Introducing the approximating problems

$$\dot{x}^{(i)}(t) = A(x^{(i-1)}(t))x^{(i)}(t) + B(x^{(i-1)}(t))u^{(i)}(t), \tag{16}$$

being $x^{(i)}(0) = x_0$ the initial conditions and with the cost functionals

$$
\begin{aligned}
J^{(i)}(u) = & \frac{1}{2}\int_0^T \big(x^{(i)T}(s)Q(x^{(i-1)}(s))x^{(i)}(s) \\
& + \frac{1}{2}u^{(i)T}(s)R(x^{(i-1)}(s))u(s)\big)ds \\
& + \frac{1}{2}x^{(i)T}(T)F(x^{(i-1)}(T))x^{(i)}(T)
\end{aligned} \tag{17}
$$

By the theory presented in Section 3, each of these problems can be expanded in a basis:

$$x^{(i)}(t) = \sum_{k=1}^{\infty} \alpha_k^{(i)} e_k(t), \;\; u^{(i)}(t) = \sum_{k=1}^{\infty} \beta_k^{(i)} e_k(t)$$

where $\alpha^{(i)}$, $\beta^{(i)}$ satisfy the equations:

$$\alpha_k^{(i)} = \gamma_k^{(i)} + \sum_l \phi_{lk}^{(i)} \beta_l$$

where

$$\gamma_k^{(i)} = \Big\langle \Phi^{(i)}(t,0)x_0, e_k \Big\rangle \tag{18}$$

and $\Phi^{(i)}$ is the transition matrix of $A(x^{(i-1)}(t))$. (Note that in here $< \cdot, \cdot >$ denotes the usual inner product in a Hilbet space H.)

The solution is then given, as in the previous section by:

$$\beta = -Lim_{i\to\infty}(K^{(i)})^{-1}\Pi^{(i)} \tag{19}$$

where $K^{(i)}$, $\Pi^{(i)}$ are given by

$$K^{(i)} = \Big(K_{ij}^{(i)}\Big), \;\; \Pi^{(i)} = \Big(\Pi_1^{(i)}, \Pi_2^{(i)}, \cdots\Big)^T.$$

and

$$
\begin{aligned}
K_{jk}^{(i)} &= M_j^{(i)T} F M_k^{(i)} + \sum_l \phi_{jl}^{(i)T} Q \phi_{jl}^{(i)} + R\delta_{ij} \\
\Pi_j^{(i)} &= M_j^{(i)T} F L^{(i)}(x_0) + \sum_l \phi_{jl}^{(i)T} Q \gamma_l^{(i)}.
\end{aligned}
$$

So it has been shown how a nonlinear optimal control problem can be solved by applying an approximation procedure which replaces the nonlinear problem by a sequence of linear, time varying quadratic optimization problems. Each of these problems can be solved efficiently by expanding the state and control functions in terms of a basis of L^2, the coefficients of which lead directly to a simple discrete quadratic optimization problem.

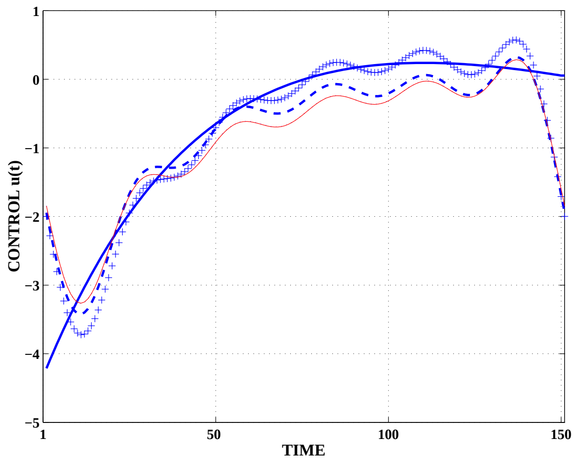

Fig. 3. Optimal Control at different iterations

6. NUMERICAL EXAMPLE FOR NONLINEAR CASE

Consider the nonlinear optimal control problem,

$$
\begin{aligned}
minJ = & \frac{1}{2}x^T(t_f)Fx(t_f) \\
& + \frac{1}{2}\int_0^{t_f} \big(\frac{1}{2}x^T(t)Q(x(t))x(t) \\
& + \frac{1}{2}u^T(t)R(x(t))u(t)\big)dt
\end{aligned}
$$

with $F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $R = 1$, subject to the nonlinear constrain

$$\dot{x} = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & -1 & -1 \\ x_2 & 1 & 1 \end{pmatrix} x + \begin{pmatrix} x_1 \\ 1 \\ 1 \end{pmatrix} u(t)$$

for some initial conditions $x(0) = (1,1,1)^T$ and a final time $t_f = 1.5$.

The first step is to iterate the nonlinear system and generate a sequence of LTV optimal control problems of the form (16). By applying the orthonormal expansion to each iteration in the states $x^{(i)}(t)$ and control $u^{(i)}(t)$ as in section 4, a sequence of optimal controls coefficients is obtained for each iteration $\beta^{(i)}(t)$. Figure (6) shows the successive iterated control obtained for each of the iterations: (thin solid line) for the first iteration,$(--)$for the 2^th iteration ..up to $(++)$ for the 10^th iteration. The convergence of the controls to the optimal control obtained using the

Ricatti equation (thick solid line) is clear. In this example only 10 iterations were required in order to obtain a satisfactory approach to the nonlinear control obtained with the Ricatti algorithm. The basis used in this case, was the fourier basis. The authors expect to continue this work by investigating other possible choices of basis.

7. CONCLUSIONS AND FURTHER WORK

In this paper the authors have presented a method to approach optimal control for a general nonlinear system. The method is based in replacing the nonlinear optimal control problem by a sequence of LTV problems. The optimal control for each of these LTV problems is obtained by expanding the control and state as a series of terms in some orthonormal basis and determining the unknown coefficients of the control. By the convergence of the sequence of LTV problems to the nonlinear problem, the convergence of the sequence of coefficients towards the coefficients corresponding to the optimal control for the nonlinear case can be inferred.

This method has its main advantage in the fact that for the general optimal nonlinear control problem, the classical method of Ricatti equation does not need to be applied; instead a sequence of linear algebraic equations is generated, which are easier to solve.

In this paper the authors have mentioned how the different choices of orthonormal basis for the state and control expansions may influence the performance of the method. It has been shown how Chebyshev polynomials perform in a better way than Fourier orthonormal basis: This best choice of orthonormal basis is currently under study by the authors and some new results will be given in a separate paper.

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