LINEAR CONTINUOUS TIME SYSTEM RESPONSES

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Abstract: General closed–form expressions of linear continuous–time system responses, are derived. The system eigenvalues can be real and/or complex, and may be repeated. A recursive computationally attractive method is formulated by which the partial fraction expansion coefficients can be computed fast and accurately. The closed–form expressions include the numerator coefficients of the transfer function, a matrix containing the partial fraction expansion coefficients and the system's eigenvalues, and a vector containing the independent time–basis functions. Higher–order responses can easily be computed in closed form from the impulse response.

Keywords: Impulse response, partial fraction expansion, closed–form expression, higher–order response

1. INTRODUCTION

Closed–form transfer function responses for continuous–time systems are of considerable interest in the area of control systems and filter design, see, e.g., (Brogan, 1991), (Ogata, 1990) and (Oppenheim and Willsky, 1997). Closed–form transfer function responses in (Hauksdóttir, 1996) have opened up many new interesting applications, e.g., solving for optimal zero locations by minimizing transient responses (Hauksdóttir, 1996), minimizing the step response deviation from given reference step responses (Hauksdóttir, 1999*a*, 2001, 2002), and approaching the model reduction problem (Hauksdóttir, 2000) by minimizing the difference in the impulse response of the original and the reduced–order model, keeping a subset of the original eigenvalues and a desired relative degree. The closed-form expressions are further used in the direct computation of coefficients for PID controllers (Herjólfsson and Hauksdóttir, 2003) and (Herjólfsson, 2004).

The closed-form expressions are extended in this paper to include repeated eigenvalues as well as nonrepeated. The partial fraction expansion coefficients for a unitary numerator transfer function is treated in Section 2, including an example. The general partial fraction expansion coefficients are then treated together with the impulse response in Section 3 and the earlier example is revisited. Higher–order responses are finally presented in Section 4 and conclusions in Section 5.

2. COMPUTATION OF PARTIAL FRACTION EXPANSION COEFFICIENTS

We assume that the general nth–order differential equation of the form

$$\begin{aligned} & y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots + a_n y(t) \\ &= \ b_0 u^{(m)}(t) + b_1 u^{(m-1)}(t) + \cdots + b_m u(t) \quad (1) \end{aligned}$$

is given. The corresponding transfer function is

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$$\frac{Y(s)}{U(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n}$$
$$= \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{(s+\lambda_1)^{d_1}(s+\lambda_2)^{d_2}\cdots(s+\lambda_\nu)^{d_\nu}}$$
$$= \sum_{i=1}^{\nu}\sum_{j=1}^{d_i} \frac{\mu_{ij}}{(s+\lambda_i)^j} \qquad (2)$$

where μ_{ij} are the partial fraction expansion coefficients, given by

$$\mu_{ij} = \frac{1}{(d_i-j)!}\frac{d^{(d_i-j)}}{ds^{(d_i-j)}}\left[(s+\lambda_i)^{d_i}\frac{Y(s)}{U(s)}\right]\Bigg|_{s=-\lambda_i}. \qquad (3)$$

Thus, the impulse response is given by

$$y_i(t) = \sum_{i=1}^{\nu}\sum_{j=1}^{d_i} \mu_{ij}\frac{t^{(d_i-j)}}{(d_i-j)!}e^{-\lambda_i t}. \qquad (4)$$

It is assumed that the system has ν distinct eigenvalues, $-\lambda_1, -\lambda_2, \ldots, -\lambda_\nu$, the eigenvalue $-\lambda_i$ being repeated d_i times, $i = 1, 2, \ldots, \nu$, such that $\sum_{i=1}^{\nu} d_i = n$. Furthermore it is assumed that the system is causal, i.e., $m < n$.

The term *basic response* refers to the response of a transfer function containing only poles and a unity numerator, i.e., the basic impulse response $y_b(t)$ is the solution of

$$y_b^{(n)}(t) + a_1 y_b^{(n-1)}(t) + \cdots + a_n y_b(t) = \delta(t). \qquad (5)$$

Then, by linearity, the total impulse response $y_i(t)$ is given by

$$y_I(t) = b_o y_b^{(m)}(t) + b_1 y_b^{(m-1)}(t) + \cdots + b_m y_b(t). \qquad (6)$$

We can therefore write the basic impulse response in the Laplace domain as

$$Y_b(s) = \frac{1}{s^n + a_1 s^{n-1} + \cdots + a_n}$$
$$= \frac{1}{(s+\lambda_1)^{d_1}(s+\lambda_2)^{d_2}\cdots(s+\lambda_\nu)^{d_\nu}}$$
$$= \sum_{i=1}^{\nu}\sum_{j=1}^{d_i} \frac{\kappa_{ij}}{(s+\lambda_i)^j}, \qquad (7)$$

where κ_{ij} are the partial fraction expansion coefficients, given by

$$\kappa_{ij} = \frac{1}{(d_i-j)!}\frac{d^{(d_i-j)}}{ds^{(d_i-j)}}\left[(s+\lambda_i)^{d_i}Y_b(s)\right]\Big|_{s=-\lambda_i} \qquad (8)$$

Thus,

$$y_b(t) = \sum_{i=1}^{\nu}\sum_{j=1}^{d_i} \kappa_{ij}\frac{t^{(d_i-j)}}{(d_i-j)!}e^{-\lambda_i t}. \qquad (9)$$

Define

$$\hat{\kappa}_{ij}(s) = \frac{1}{(d_i-j)!}\frac{d^{(d_i-j)}}{ds^{(d_i-j)}}\left[(s+\lambda_i)^{d_i}Y_b(s)\right]. \qquad (10)$$

Now, it is easy to calculate $\hat{\kappa}_{id_i}$, i.e., for $j = d_i$,

$$\hat{\kappa}_{id_i}(s) = \prod_{q=1,q\neq i}^{\nu}(s+\lambda_q)^{-d_q}, \qquad (11)$$

and for $j = d_i - 1$,

$$\hat{\kappa}_{i(d_i-1)}(s) = \hat{\kappa}_{id_i}(s)(-1)\sum_{p=1,p\neq i}^{\nu}\frac{d_p}{(s+\lambda_p)}. \qquad (12)$$

Repeating this procedure for $j = d_i - 2, d_i - 3, \ldots, 1$, it is observed that generally $\hat{\kappa}_{ij}$ can be expressed by

$$\hat{\kappa}_{ij}(s) = \frac{1}{d_i-j}\sum_{q=1}^{d_i-j}\hat{\kappa}_{i(j+q)}(s)(-1)^q\sum_{p=1,p\neq i}^{\nu}\frac{d_p}{(s+\lambda_p)^q}, \qquad (13)$$

which can be proved by induction, see Appendix A. Finally, we can write

$$\kappa_{ij} = \hat{\kappa}_{ij}(s)\big|_{s=-\lambda_i}. \qquad (14)$$

2.1 Example 1

Consider the transfer function

$$G(s) = \frac{1}{(s+1)^2(s+2)^1(s+4)^4(s+7)^2},$$

applying (11) (13) and (14), gives

$$\begin{aligned}
\kappa_{11} &= -0.00091449, & \kappa_{12} &= 0.00034294,\\
\kappa_{21} &= 0.00250000, & & \\
\kappa_{31} &= -0.00145748, & \kappa_{32} &= -0.00291495,\\
\kappa_{33} &= -0.00308642, & \kappa_{34} &= -0.00617284,\\
\kappa_{41} &= -0.00012803, & \kappa_{42} &= -0.00006859.
\end{aligned}$$

This result agrees with the one obtained by the *int(diff(G(s)))* function in Matlab.

3. THE IMPULSE RESPONSE

It is possible to write the basic impulse response in (9), in vector form

$$y_b(t) = \sum_{i=1}^{\nu} h_{0i}\mathcal{E}_i(t), \qquad (15)$$

where

$$h_{0i} = \left[\,\kappa_{i1}\ \kappa_{i2}\ \cdots\ \kappa_{id_i}\,\right] \qquad (16)$$

and

$$\mathcal{E}_i(t) = \begin{bmatrix} e^{-\lambda_i t} \\ te^{-\lambda_i t} \\ \vdots \\ \frac{t^{(d_i-1)}}{(d_i-1)!}e^{-\lambda_i t} \end{bmatrix}. \qquad (17)$$

It can also be written as

$$y_b(t) = H_0\mathcal{E}(t), \tag{18}$$

where

$$H_0 = \left[\, h_{01} \;\; h_{02} \;\; \cdots \;\; h_{0\nu} \,\right] \tag{19}$$

and

$$\mathcal{E}(t) = \begin{bmatrix} \mathcal{E}_1(t) \\ \mathcal{E}_2(t) \\ \vdots \\ \mathcal{E}_\nu(t) \end{bmatrix} \tag{20}$$

In order to find the total impulse response in (6) $y_b(t)$ and its m differentials must be known. The first differential is easily obtained from (9)

$$\begin{aligned} y_b'(t) = \sum_{i=1}^{\nu} \Bigg(& (-\lambda_i\kappa_{i1} + \kappa_{i2})e^{-\lambda_i t} \\ & +(-\lambda_i\kappa_{i2} + \kappa_{i3})te^{-\lambda_i t} + \ldots \\ & +(-\lambda_i\kappa_{id_i})\frac{t^{(d_i-1)}}{(d_i-1)!}e^{-\lambda_i t}\Bigg) \end{aligned} \tag{21}$$

which can be expressed in matrix form as

$$y_b'(t) = \sum_{i=1}^{\nu} h_{1i}\mathcal{E}_i(t) \tag{22}$$

where

$$\begin{aligned} h_{1i} = \big[\, & (-\lambda_i\kappa_{i1} + \kappa_{i2}) \;\; (-\lambda_i\kappa_{i2} + \kappa_{i3}) \\ & \cdots \;\; (-\lambda_i\kappa_{i(d_i-1)} + \kappa_{id_i}) \;\; (-\lambda_i\kappa_{id_i}) \,\big]. \end{aligned} \tag{23}$$

But now it is possible to calculate the h_{1i} from h_{0i},

$$h_{1i} = h_{0i}W_i \tag{24}$$

where

$$W_i = \begin{bmatrix} -\lambda_i & 0 & \cdots & \cdots & 0 \\ 1 & -\lambda_i & \ddots & & \vdots \\ 0 & 1 & -\lambda_i & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & -\lambda_i \end{bmatrix}. \tag{25}$$

Thus, $y_b'(t)$ can be expressed in terms of the same functions as $y_b(t)$, only with different constants. To calulate $y_b''(t)$ the same procedure is repeated, i.e., by differentiating (21), we get that

$$h_{2i} = h_{1i}W_i, \tag{26}$$

and in general

$$h_{ki} = h_{(k-1)i}W_i, \qquad k = 1, 2, \ldots, m. \tag{27}$$

The total impulse response can thus be written as

$$y_I(t) = BH\mathcal{E}(t), \tag{28}$$

where

$$B = \left[\, b_m \;\; b_{m-1} \;\; \cdots \;\; b_0 \,\right], \tag{29}$$

$$H = \begin{bmatrix} h_{01} & h_{02} & \cdots & h_{0\nu} \\ h_{11} & h_{12} & \cdots & h_{1\nu} \\ \vdots & \vdots & & \vdots \\ h_{m1} & h_{m2} & \cdots & h_{m\nu} \end{bmatrix} \tag{30}$$

and $\mathcal{E}(t)$ as in (20).

It should be emphasized that (28) is the general closed–form impulse response for linear continuous–time systems corresponding to a general transfer function of the form (2). There are no restrictions, the eigenvalues can be real and/or complex, repeated and/or not and stable and/or unstable.

It should also be noted that

$$\mu = \left[\, \mu_{11} \;\; \cdots \;\; \mu_{1d_1} \;\; \cdots \;\; \mu_{\nu 1} \;\; \cdots \;\; \mu_{\nu d_\nu} \,\right] = BH \tag{31}$$

where μ_{ij} are the coefficients in (2). Thus we have a new easily computable recursive form of partial fraction expansion coefficients, given by the well known expression (3), for a general transfer function of the form (2).

Further, the new form (28) reduces to an earlier form in the case of nonrepeated eigenvalues as published in (Hauksdóttir, 1996, 1999*a*, 1999*b*, 2000, 2001, 2002), (Hauksdóttir and Hjaltadóttir, 2003) and (Herjólfsson, 2004) (Herjólfsson and Hauksdóttir, 2003). For non-repeated eigenvalues the elements of H can be expressed explicitly as

$$h_{ki} = \frac{(-\lambda_i)^{k-1}}{\prod_{q=1,q\neq i}^{\nu}(-\lambda_i + \lambda_q)} \tag{32}$$

and then the impulse response can be written as

$$\begin{aligned} y_I(t) &= BH \begin{bmatrix} e^{-\lambda_1 t} \\ e^{-\lambda_2 t} \\ \vdots \\ e^{-\lambda_\nu t} \end{bmatrix} \\ &= \mathcal{B}\Lambda \begin{bmatrix} \dfrac{e^{-\lambda_1 t}}{\prod\limits_{q=2}^{\nu}(-\lambda_1 + \lambda_q)} \\ \dfrac{e^{-\lambda_2 t}}{\prod\limits_{\substack{q=1 \\ q\neq 2}}^{\nu}(-\lambda_2 + \lambda_q)} \\ \vdots \\ \dfrac{e^{-\lambda_\nu t}}{\prod\limits_{q=1}^{\nu-1}(-\lambda_\nu + \lambda_q)} \end{bmatrix} \end{aligned} \tag{33}$$

where B is as in (29),

$$\mathcal{B} = \left[\, b_m \;\; -b_{m-1} \;\; \cdots \;\; (-1)^m b_0 \,\right] \tag{34}$$

and

$$\Lambda = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_\nu \\ \vdots & \vdots & & \vdots \\ \lambda_1^m & \lambda_2^m & \cdots & \lambda_\nu^m \end{bmatrix} \tag{35}$$

the Vandermonde matrix. The latter form can be extended to the case of repeated eigenvalues by replacing the Vandermonde matrix by an appropriate confluent Vandermonde matrix (Björck and Pereyra, 1970). However, the form (28) with the rows of the matrix (30) being calculated recursively according to (27), is computationally more efficient.

3.1 *Example 2*

Now, consider the transfer function

$$G(s) = \frac{3s^4 + 2s^3 + s^2 + 2s + 3}{(s+1)^2(s+2)^1(s+4)^4(s+7)^2}.$$

The κ–coefficients were calculated in Ex. 1. The μ–coefficients calculated from (16), (25), (27), (29), (30) and (31) become

$$\begin{aligned} \mu_{11} &= -0.004801, & \mu_{12} &= 0.001029, \\ \mu_{21} &= 0.087500, & & \\ \mu_{31} &= 0.493570, & \mu_{32} &= -1.440844, \\ \mu_{33} &= 2.175926, & \mu_{34} &= -4.018519, \\ \mu_{41} &= -0.576269, & \mu_{42} &= -0.449588. \end{aligned}$$

Which again agrees with *int(diff(G(s)))*.

3.2 *Example 3*

Having computed the μ–coefficients as in Ex. 2, it is trivial to calculate the impulse response from (28). The impulse response is shown in Figure 1, and compares well to standard results obtained by numerical integration. Here, however, no approximations or numerical methods are used, an accurate result is obtained, fast and efficiently.

4. HIGHER–ORDER RESPONSES

We can calculate the step response, $y_S(t)$ at time $t \geq 0$, by simply integrating the impulse response. Using the fact that (see any standard mathematics handbook)

$$\int_0^t \frac{t_1^r}{r!} e^{-\lambda_i t_1} dt_1 = -\frac{1}{\lambda_i^{r+1}} e^{-\lambda_i t_1} - \frac{1}{\lambda_i^r} t_1 e^{-\lambda_i t_1} - \cdots - \frac{1}{\lambda_i^2} \frac{t_1^{r-1}}{(r-1)!} e^{-\lambda_i t_1} - \frac{1}{\lambda_i} \frac{t_1^r}{r!} e^{-\lambda_i t_1} \Big|_0^t . \tag{36}$$

we get that,

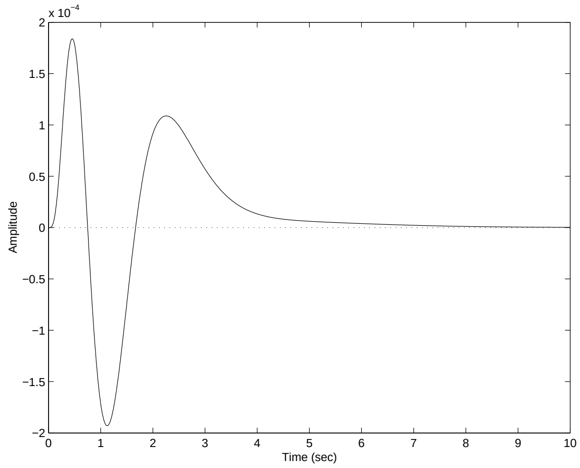

Fig. 1. Impulse response for transfer function in Example 2.

$$\int_0^t \mathcal{E}_i(t_1) dt_1 = V_i \mathcal{E}_i(t_1)\big|_0^t , \tag{37}$$

where

$$V_i = \begin{bmatrix} -\frac{1}{\lambda_i} & 0 & \cdots & 0 \\ -\frac{1}{\lambda_i^2} & -\frac{1}{\lambda_i} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ -\frac{1}{\lambda_i^{d_i}} & -\frac{1}{\lambda_i^{d_i-1}} & \cdots & -\frac{1}{\lambda_i} \end{bmatrix} . \tag{38}$$

Note that V_i is the inverse of the matrix W_i in (25). The step response is thus

$$\begin{aligned} y_S(t) &= \int_0^t BH\mathcal{E}(t_1) dt_1 \\ &= BHV\left(\mathcal{E}(t) - \mathcal{E}(0)\right) \\ &= BHV\mathcal{E}(t) + \frac{b_m}{a_n} \end{aligned} \tag{39}$$

where

$$V = \begin{bmatrix} V_1 & 0 & \cdots & 0 \\ 0 & V_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & V_\nu \end{bmatrix} . \tag{40}$$

The ramp response is found by integrating the step response

$$\begin{aligned} y_R(t) &= \int_0^t \left(BHV\mathcal{E}(t_1) + \frac{b_m}{a_n} \right) dt_1 \\ &= BHV^2\left(\mathcal{E}(t) - \mathcal{E}(0)\right) + \frac{b_m}{a_n} t. \end{aligned} \tag{41}$$

Higher–order responses can easily be obtained in a similar manner as

$$\begin{aligned} y_\gamma(t) &= BHV^\gamma \mathcal{E}(t)(\gamma - 1)! \\ &\quad - BH \sum_{p=1}^{\gamma} V^p \mathcal{E}(0) \frac{(\gamma-1)!}{(\gamma-p)!} t^{\gamma - p}. \end{aligned} \tag{42}$$

5. CONCLUSION

The closed–form expressions of transfer function responses derived in (Hauksdóttir, 1996) were extended in this paper to include the case of repeated eigenvalues, see also (Hauksdóttir and Hjaltadóttir, 2003) and (Herjólfsson, 2004) for earlier versions. Further, a recursive, computationally efficient equation for partial fraction expansion coefficients was developed. The closed–form expressions derived include the numerator coefficients of the transfer function, a matrix containing the partial fraction expansion coefficients and eigenvalues, and finally a vector containing the independent time–basis functions. Several examples were presented illustrating the proposed method. First, the partial fraction expansion coefficients for a unitary numerator transfer function were computed, and then that result was extended for a polynomial numerator. Finally, the corresponding impulse response was computed based on the partial fraction expansion coefficients and the linearly independent basis functions.

It should be emphasized, that the expressions derived are a new general closed–form solution for linear continuous–time system responses corresponding to a general transfer function. The system eigenvalues can be real and/or complex, and may be repeated. Further, the recursive form of partial fraction expansion coefficients for a general transfer function is also new and easily computable.

Software simulation tools for linear systems, in general rely on algorithms involving numerical integration, in order to simulate linear system responses. By using the new closed–form solution, linear system responses may be computed effectively, without iterations or approximations, as well as for computation of various system characteristics and controllers. Thus, the results of this paper could have an impact for software developement involving system simulation, various system analysis and computation of controllers. Being general solutions of ordinary differential equations, the results may also effect different areas of engineering and science.

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Appendix A. PROOF OF EQUATION (13)

First, we notice that for $j = d_i - 1$, (13) is true. Now, we suppose that (13) is satisfied for $j = d_i - k$ and calculate for $j = d_i - (k+1)$,

$$\begin{aligned}\hat{\kappa}_{i(d_i-(k+1))}(s) &= \frac{1}{(k+1)!}\frac{d^{k+1}}{ds^{k+1}}\left[(s+\lambda_i)^{d_i}\frac{Y_b(s)}{U(s)}\right]\\ &= \frac{1}{k+1}\frac{d}{ds}\left(\frac{1}{k!}\frac{d^k}{ds^k}\left[(s+\lambda_i)^{d_i}\frac{Y_b(s)}{U(s)}\right]\right)\\ &= \frac{1}{k+1}\frac{d}{ds}\hat{\kappa}_{i(d_i-k)}(s).\end{aligned}$$

Substituting (13) we get

$$\begin{aligned}&\hat{\kappa}_{i(d_i-(k+1))}(s)\\ &\quad= \frac{1}{k+1}\frac{d}{ds}\left(\frac{1}{k}\sum_{q=1}^{k}\hat{\kappa}_{i(d_i-k+q)}(s)(-1)^q\sum_{p=1,p\neq i}^{\nu}\frac{d_p}{(s+\lambda_p)^q}\right).\end{aligned}$$

Differentiating (13) with respect to s and setting $j = d_i - k$, we get

$$\begin{aligned}\hat{\kappa}_{i(d_i-(k+1))}(s) = &\frac{1}{k(k+1)}\sum_{q=1}^{k}(k+1-q)\hat{\kappa}_{i(d_i-k-1+q)}(s)\\ &\times(-1)^q\sum_{p=1,p\neq i}^{\nu}\frac{d_p}{(s+\lambda_p)^q}\\ &+\frac{1}{k(k+1)}\sum_{q=1}^{k}\hat{\kappa}_{i(d_i-k+q)}(s)\\ &\times(-1)^q\sum_{p=1,p\neq i}^{\nu}\frac{-qd_p}{(s+\lambda_p)^{q+1}}.\end{aligned}$$

Then expanding the first term and separating the first subterm from it, and expanding the second term and seperating the last subterm, gives

$$\begin{aligned}\hat{\kappa}_{i(d_i-(k+1))}(s) = &\frac{k}{k(k+1)}\hat{\kappa}_{i(d_i-k)}(s)\\ &\times(-1)\sum_{p=1,p\neq i}^{\nu}\frac{d_p}{(s+\lambda_p)}\\ &+\frac{1}{k(k+1)}\sum_{q=2}^{k}(k+1-q)\hat{\kappa}_{i(d_i-k-1+q)}(s)\\ &\times(-1)^q\sum_{p=1,p\neq i}^{\nu}\frac{d_p}{(s+\lambda_p)^q}\\ &+\frac{1}{k(k+1)}\sum_{q=1}^{k-1}\hat{\kappa}_{i(d_i-k+q)}(s)\\ &\times(-1)^{q+1}\sum_{p=1,p\neq i}^{\nu}\frac{qd_p}{(s+\lambda_p)^{q+1}}\\ &+\frac{k}{k(k+1)}\hat{\kappa}_{id_i}(s)\\ &\times(-1)^{k+1}\sum_{p=1,p\neq i}^{\nu}\frac{d_p}{(s+\lambda_p)^{k+1}}.\end{aligned}$$

Now increasing the index of the third term by one,

$$\hat{\kappa}_{i(d_i-(k+1))}(s) = \frac{1}{k+1}\hat{\kappa}_{i(d_i-k)}(s)$$
$$\times(-1)\sum_{p=1,p\neq i}^{\nu}\frac{d_p}{(s+\lambda_p)}$$
$$+\frac{1}{k(k+1)}\sum_{q=2}^{k}(k+1-q)\hat{\kappa}_{i(d_i-k-1+q)}(s)$$
$$\times(-1)^q\sum_{p=1,p\neq i}^{\nu}\frac{d_p}{(s+\lambda_p)^q}$$
$$+\frac{1}{k(k+1)}\sum_{q=2}^{k}(q-1)\hat{\kappa}_{i(d_i-k-1+q)}(s)$$
$$\times(-1)^q\sum_{p=1,p\neq i}^{\nu}\frac{d_p}{(s+\lambda_p)^q}$$
$$+\frac{1}{k+1}\hat{\kappa}_{id_i}(s)$$
$$\times(-1)^{k+1}\sum_{p=1,p\neq i}^{\nu}\frac{d_p}{(s+\lambda_p)^{k+1}}.$$

Combining the second term and the third term gives

$$\hat{\kappa}_{i(d_i-(k+1))}(s) = \frac{1}{k+1}\hat{\kappa}_{i(d_i-k)}(s)$$
$$\times(-1)\sum_{p=1,p\neq i}^{\nu}\frac{d_p}{(s+\lambda_p)}$$
$$+\frac{1}{k+1}\sum_{q=2}^{k}\hat{\kappa}_{i(d_i-k-1+q)}(s)$$
$$\times(-1)^q\sum_{p=1,p\neq i}^{\nu}\frac{d_p}{(s+\lambda_p)^q}$$
$$+\frac{1}{k+1}\hat{\kappa}_{id_i}(s)$$
$$\times(-1)^{k+1}\sum_{p=1,p\neq i}^{\nu}\frac{d_p}{(s+\lambda_p)^{k+1}}.$$

Now write all of this as one sum

$$\hat{\kappa}_{i(d_i-(k+1))}(s) = \frac{1}{k+1}\sum_{q=1}^{k+1}\hat{\kappa}_{i(d_i-(k+1)+q)}(s)$$
$$\times(-1)^q\sum_{p=1,p\neq i}^{\nu}\frac{d_p}{(s+\lambda_p)^q}.$$

Now we have shown that $\hat{\kappa}_{ij}(s)$ can be expressed with (11) for $j = d_i$ and (13) for $j = (d_i-1), (d_i-2), \ldots, 1$.

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