A NEW APPROACH FOR THE OBSERVER-BASED SYNCHRONIZATION OF CHAOTIC SYSTEMS

G. Iulia Bara * **Ali Zemouche** * **Mohamed Boutayeb** *

* *LSIIT-UMR CNRS-ULP 7005*
ENSPS, Bd. Sébastien Brandt, 67412 Illkirch Cedex France
bara@eavr.u-strasbg.fr, zemouche@eavr.u-strasbg.fr,
Mohamed.Boutayeb@ipst-ulp.u-strasbg.fr

Abstract: In this paper, a new framework for the synchronization of chaotic systems is presented. The synchronization problem of a large class of chaotic systems is formulated as an observer synthesis problem for an appropriate class of linear parameter-varying (LPV) systems. The result introduced in this paper shows that LPV techniques can successfully be used in the context of chaotic systems synchronization. Two examples are considered in order to show the applicability of the LPV approach.

Keywords: Chaotic systems, observer-based chaos synchronization, LPV approach.

1. INTRODUCTION

A new concept that evolved rapidly in the last few years in the modern communication technology is the secure communication using synchronization between chaotic systems. This has motivated a number of works in the area of chaos synchronization. A detailed survey of chaotic secure communication systems is given by Yang (2004). Since the pioneering works of Pecora and Carroll (1990),Carroll and Pecora (1991), many approaches, leading to a better understanding of the synchronization problem have been proposed by Ashwin et al. (1994); Haegy et al. (1994) and Ashwin et al. (1996).

As the chaos synchronization problem can be reformulated as an observer design one, the observer-based approach becomes one of the most attractive techniques. This kind of approach has extensively been investigated in the recent research works by Grassi and Mascolo (1997); Morgul and Solak (1996, 1997); Nijmeijer and Mareels (1997); Ushio (1999); H. (2001); Celikovsky and Chen (2002). We can mention in this context the approaches using nonlinear Lipschitz systems for describing the chaotic model as proposed by De Angeli et al. (1995); Grassi and Mascolo (1997); Liao and Huang (1999); Miller and Grassi (2001); Grassi and Miller (2002). The major drawback of the approach developed in these papers comes from the fact that the nonlinear term is measured in the output signal.

A parameter control method by a scalar signal is proposed for the implementation of hyper-chaos synchronization in Duan and Yang (1997). The synchronization of Rössler and Chen chaotic systems is obtained by using active control in Agiza and Yassen (2001). In addition to the above approaches, for the class of nonlinear discrete-time systems, an extended Kalman filtering was proposed by Cruz and Nijmeijer (1999); Boutayeb (2005) while extended observers are used by Huijberts et al. (2001); Lilge (1999).

In this paper, we propose a new approach for the synchronization of chaotic systems. This approach is based on the fact that many chaotic systems can be transformed into linear parameter-varying (LPV) systems when the output signal is chosen appropriately. The result introduced in this paper shows that LPV techniques can successfully be used in the context of chaotic systems synchronization. To our knowledge this is the first time LPV techniques are used in this context. Based on existing results, a sufficient condition is given in order to design the observer gain guaranteeing the stability of the synchronization error. This condition is expressed as a LMI solvability problem and hence easily tractable by convex optimization techniques. The proposed approach has several advantages over the existing methods. It proves to be simple and rigorous. It requires neither negative Lyapunov exponents nor initial conditions belonging to the same basin of attraction in order to guaranty the synchronization between the response-system and the drive-system. Moreover, global synchronization is achievable in a systematic way for many chaotic systems reported in the literature.

This paper is organized as follows. In section 2, we introduce the class of systems to be studied and we explain the chaos synchronization problem. The main contribution of our paper, which consists in a new LPV observer-based approach for this problem, is presented in section 3. An extension to continuous-time systems is given in section 4. In order to demonstrate the validity of our approach two numerical examples are presented in section 5. We end this paper by the conclusion.

Notations: The notation $(\star)$ is used for the blocks induced by symmetry.

2. PROBLEM FORMULATION

Consider the class of chaotic systems described by the following nonlinear state equations:

$$\begin{cases} x(k+1) = Ax(k) + f(x(k), y(k), k) \\ y(k) = Cx(k) \end{cases} \quad (1)$$

where $x(k) \in \mathbb{R}^n$ is the state vector and $y(k) \in \mathbb{R}$ is the scalar output signal. A and C are constant matrices of appropriate dimensions and f: $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{R}^n$ is a nonlinear function. The pair (A, C) is assumed to be detectable.

Given the chaotic drive-system (1), the chaos synchronization problem consists to find a response-system (also called a slave-system) whose state $\hat{x}(k)$ converges towards the drive-system state $x(k)$ using the transmitted signal $y(k)$. This principle was graphically represented in Figure 1.

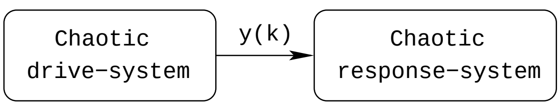


Fig. 1. Chaos synchronization.

In the following, we investigate the synchronization problem by using an observer-based approach.

3. MAIN RESULT

In this section, we present the main contribution of our paper which consists in a new framework for the chaos synchronization problem. This framework is based on reformulating the chaotic systems as an LPV one. This can be done under some nonrestrictive assumptions. In the rest of the paper, we make the following *assumptions*:

A1: For a particular choice of the output matrix C, the nonlinear part can be rewritten as:

$$f(x(k), y(k), k) = g_1(y(k), k)Hx(k) + g_2(y(k), k) \quad (2)$$

where $H \in \mathbb{R}^{n\times n}$, $g_1 : \mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{R}$ and $g_2 : \mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{R}^n$. Note that this particular choice of C is not restrictive. In fact, the class of systems satisfying the condition (2) includes an extensive variety of chaotic systems such as the discrete-time version of the Rössler's and Lorenz's systems presented in Liao and Huang (1999). In addition to those systems, we mention several other chaotic systems such as Chen's equation presented in Agiza and Yassen (2001) and Henon map given in De Angeli et al. (1995).

A2: We suppose that the function $g_1(y(k), k)$ is bounded when $y(k)$ is bounded. Note that this assumption is not restrictive because the state vector $x(k)$ and the transmitted signal $y(k)$ of a chaotic system are always bounded. This implies that the function $f(x(k), y(k), k)$ and implicitly the functions $g_1(y(k), k)$ and $g_2(y(k), k)$ are bounded when $y(k)$ is bounded.

Now, we introduce the following notations:

$$\rho(k) = g_1(y(k), k) \quad \text{and} \quad (3a)$$
$$\mathcal{A}(\rho(k)) = A + \rho(k)H. \quad (3b)$$

Using the relation (2) and the notations (3), the system (1) can be rewritten as:

$$\begin{cases} x(k+1) = \mathcal{A}(\rho(k))x(k) + g_2(y(k), k) \\ y(k) = Cx(k) \end{cases} \quad (4)$$

Using the output measurement, we can compute $\rho(k)$ at any instant k. Hence, $\rho(k)$ can be considered as a known time-varying parameter and the system (4) can be seen as a linear parameter-varying (LPV) system with a nonlinear term.

A state observer corresponding to (4) is given by:

$$\begin{cases} \hat{x}(k+1) = \mathcal{A}(\rho(k))\hat{x}(k) + g_2(y(k), k) + \\ \qquad + L(\rho(k))(y(k) - \hat{y}(k)) \\ \hat{y}(k) = C\hat{x}(k) \end{cases} \quad (5)$$

where $\hat{x}(k)$ denotes the estimate of the state $x(k)$. The synchronization problem is reduced to finding a gain $L(\rho(k))$ such that the synchronization error

$$e(k) = x(k) - \hat{x}(k) \quad (6)$$

converges asymptotically towards zero. The dynamic of this synchronization error is given by:

$$e(k+1) = (\mathcal{A}(\rho(k)) - L(\rho(k))C)e(k). \qquad (7)$$

Note that the estimation error equation (7) defines an LPV system. Hence, the LPV techniques can successfully be used in order to study the stability of this equation and to synthesize our observer. Next, we give the observer synthesis procedure by using the LPV approach. Note that these developments are standard for a well advised reader on LPV techniques. For clarity, however, we choose to present them in detail.

As the drive-system state matrix $\mathcal{A}(\rho(k))$ is affine in $\rho(k)$, we can choose a gain $L(\rho(k))$ which is also affine in $\rho(k)$ (see Bara (2001)). Then, we have

$$L(\rho(k)) = L_0 + \rho(k)L_1$$

where L_0, L_1 are constant matrices to be determined such that the synchronization error converges asymptotically and exponentially towards zero.

¿From the assumption **A2** and the definition of $\rho(k)$, we deduce that the parameter $\rho(k)$ is bounded. Then, we introduce the notations

$$\underline{\rho} = \min_k(\rho(k)), \quad \overline{\rho} = \max_k(\rho(k)), \qquad (8)$$

$$\underline{\mathcal{A}} = \mathcal{A}(\underline{\rho}) \quad \text{and} \quad \overline{\mathcal{A}} = \mathcal{A}(\overline{\rho}). \qquad (9)$$

We present in the following theorem a sufficient condition for the observer synthesis.

Theorem 1. The synchronization error (7) converges exponentially towards zero if there exist a symmetric matrix $P > 0$ and matrices R_0, R_1 of appropriate dimensions such that the following linear matrix inequalities (LMI) are feasible:

$$\begin{bmatrix} -P & \left(\underline{\mathcal{A}}^T P - C^T R_0 - \underline{\rho} C^T R_1\right) \\ (\star) & -P \end{bmatrix} < 0 \qquad (10)$$

$$\begin{bmatrix} -P & \left(\overline{\mathcal{A}}^T P - C^T R_0 - \overline{\rho} C^T R_1\right) \\ (\star) & -P \end{bmatrix} < 0. \qquad (11)$$

When these LMIs are feasible, the gain matrices L_0, L_1 are respectively given by $P^{-1}R_0^T$ and $P^{-1}R_1^T$.

PROOF. Consider the quadratic Lyapunov function

$$V(k) = V(e(k)) = e^T(k)Pe(k).$$

The variation of this Lyapunov function is:

$$\Delta V = V(k+1) - V(k) =$$
$$e^T(k)[\big(\mathcal{A}(\rho) - L(\rho)C\big)^T P\big(\mathcal{A}(\rho) - L(\rho)C\big) - P]e(k).$$

The time dependence of $\rho(k)$ will be omitted in the following for simplicity. According to the Lyapunov stability theory, the error (7) converges exponentially towards zero if:

- the function $V(k)$ is positive definite and
- ΔV is negative definite

for all $e(k) \neq 0$ and all possible trajectories $\rho(k)$. These conditions are satisfied if and only if $P > 0$ and

$$\big(\mathcal{A}(\rho) - L(\rho)C\big)^T P\big(\mathcal{A}(\rho) - L(\rho)C\big) - P < 0$$
$$\text{for all } \rho(k) \in [\underline{\rho}, \overline{\rho}]$$

which are equivalent, by Schur complement, to

$$\begin{bmatrix} -P & \big(\mathcal{A}(\rho) - L(\rho)C\big)^T P \\ P\big(\mathcal{A}(\rho) - L(\rho)C\big) & -P \end{bmatrix} < 0$$
$$\text{for all } \rho(k) \in [\underline{\rho}, \overline{\rho}]. \qquad (12)$$

The parameter dependence of the inequality (12) implies an infinite number of inequalities to satisfy. In order to reduce this infinite number to a finite one, we apply the convexity principle. Then, as (12) is affine according to the parameter $\rho(k)$, the inequality (12) is satisfied for all possible trajectories $\rho(k) \in [\underline{\rho}, \overline{\rho}]$ if it is satisfied on the vertices of $[\underline{\rho}, \overline{\rho}]$. Using the notations $PL_0 = R_0^T$ and $PL_1 = R_1^T$, this condition yields to the inequality conditions (10) and (11). As $P > 0$ then P is invertible and we can compute the gains L_0 and L_1 as, respectively, $P^{-1}R_0^T$ and $P^{-1}R_1^T$.

Remark 2. This approach can be also extended to a vector-valued output signal.

Remark 3. The LMI conditions of Theorem 1 are feasible for a large class of chaotic systems. However, if these LMIs are not feasible, we can use less restrictive synthesis conditions based on the existence of a parameter-dependent Lyapunov function $P(\rho)$. The interested reader can see the works presented by Bara et al. (2001); Bara (2001) for more details on the synthesis of parameter-dependent observers for LPV systems using the parameter-dependent quadratic stability concept.

In the next section, we give an extension of this result to the continuous-time chaos synchronization problem.

4. EXTENSION TO THE CONTINUOUS-TIME CASE

Consider the continuous-time chaotic system described by:

$$\begin{cases} \dot{x}(t) = Ax(t) + f(x(t), y(t), t) \\ y(t) = Cx(t) \end{cases} \qquad (13)$$

where $x(t) \in \mathbb{R}^n$ is the state vector and $y(t) \in \mathbb{R}$ is the scalar output signal. A and C are constant matrices of appropriate dimensions and $f : \mathbb{R}^n \times$

$\mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{R}^n$ is a nonlinear function. The pair (A, C) is assumed to be detectable.

As in the previous section, we assume that the nonlinear part of system (13) can be rewritten as:

$$f(x(t), y(t), t) = g_1(y(t), t)Hx(t) + g_2(y(t), t) \tag{14}$$

where $H \in \mathbb{R}^{n\times n}$, $g_2 : \mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{R}^n$ and $g_1 : \mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{R}$ are bounded when $y(t)$ is bounded. As explained in the previous section, this implies that the parameter $\rho(t)$ is bounded. Then, we introduce the notations:

$$\underline{\rho} = \min_t(\rho(t)), \quad \overline{\rho} = \max_t(\rho(t)),$$
$$\underline{\mathcal{A}} = \mathcal{A}(\underline{\rho}) \quad \text{and} \quad \overline{\mathcal{A}} = \mathcal{A}(\overline{\rho}).$$

By analogy with the discrete-time case, the chaotic response-system is described by the following equations:

$$\begin{cases} \dot{\hat{x}}(t) = \mathcal{A}(\rho(t))\hat{x}(t) + g_2(y(t), t) + \\ \qquad\qquad L(\rho(t))(y(t) - \hat{y}(t)) \\ \hat{y}(t) = C\hat{x}(t) \end{cases} \tag{15}$$

where $\hat{x}(t)$ denotes the estimate of the state $x(t)$, $\rho(t) = g_1(y(t), t)$ and $\mathcal{A}(\rho(t)) = A + \rho(t)H$.

The dynamic of the synchronization error is given by:

$$\dot{e}(t) = (\mathcal{A}(\rho(t)) - L(\rho(t))C)e(t). \tag{16}$$

When $L(\rho(t)) = L_0 + \rho(t)L_1$, the synthesis problem consists of finding the gains L_0 and L_1 such that the synchronization error (16) converges exponentially towards zero. The following theorem presents sufficient conditions which allow to compute these gains.

Theorem 4. The synchronization error (16) converges exponentially towards zero if there exist a symmetric matrix $P > 0$ and matrices R_0, R_1 of appropriate dimensions such that the following LMIs are feasible:

$$\underline{\mathcal{A}}^T P - C^T R_0 - \underline{\rho} C^T R_1 + P\underline{\mathcal{A}} - R_0^T C - \underline{\rho} R_1^T C < 0 \tag{17}$$

$$\overline{\mathcal{A}}^T P - C^T R_0 - \overline{\rho} C^T R_1 + P\overline{\mathcal{A}} - R_0^T C - \overline{\rho} R_1^T C < 0. \tag{18}$$

When these LMIs admit a solution, the gain matrices L_0 and L_1 are respectively given by $P^{-1}R_0^T$ and $P^{-1}R_1^T$.

PROOF. To study the stability of the synchronization error, we consider the quadratic Lyapunov function $V(e(t)) = e^T(t)Pe(t)$ where P is a symmetric matrix. The synchronization error (16) converges asymptotically and exponentially towards zero if:

$$V(e(t)) > 0 \text{ and} \tag{19a}$$
$$\dot{V}(t) < 0 \tag{19b}$$

for all $e(t) \neq 0$ and all trajectories $\rho(t)$. The time dependence of $\rho(t)$ will be omitted for simplicity. Note that

$$\dot{V}(t) = e^T(t)F(\rho)e(t) \quad \text{where}$$
$$F(\rho) = (\mathcal{A}(\rho) - L(\rho)C)^T P + P(\mathcal{A}(\rho) - L(\rho)C).$$

The condition (19a) implies that the matrix P must be positive-definite and from the condition (19b) we deduce that $F(\rho)$ must be negative-definite for all parameter trajectories. Note that the matrix function $F(\rho)$ is affine in ρ. Using the convexity principle, $\dot{V}$ is negative-definite for all possible trajectories $\rho \in [\underline{\rho}, \overline{\rho}]$ if $F(\rho)$ is negative definite on the vertices of $[\underline{\rho}, \overline{\rho}]$. Hence, using the notations $R_0 = L_0^T P$ and $R_1 = L_1^T P$, this yields to the inequality conditions (17) and (18). As $P > 0$ then P is invertible and we can compute the gains L_0 and L_1 as, respectively, $P^{-1}R_0^T$ and $P^{-1}R_1^T$.

Remark 5. The synchronization of Rössler systems has been studied by Peng et al. (1996); Grassi and Mascolo (1997). In these papers, no synchronization has been observed whenever the transmitted signal is a state variable of the drive-system. This drawback is eliminated by our approach. In fact, using our approach, the global synchronization can be obtained easily whenever the driving signal is the first component of the state vector $y(t) = x_1(t)$ or the third one $y(t) = x_3(t)$, as shown in Example 2.

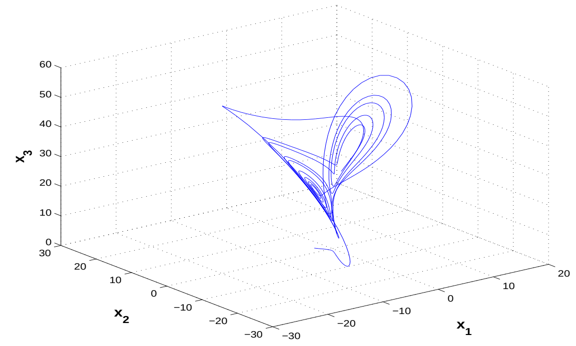
5. EXAMPLES

In this section, we apply our approach to the synchronization problem of two chaotic systems taken from the literature. The first one presents the Lorenz chaotic model in discrete time version as given by Boutayeb (2005). The second example is the Rössler chaotic model previously investigated by Wang and Wang (1998); Grassi and Mascolo (1997).

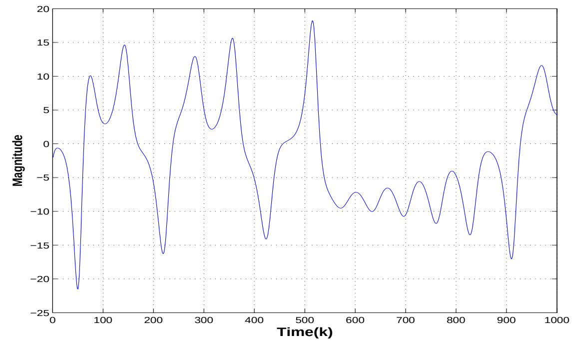
Example 1. Consider the discrete-time version of the Lorenz chaotic system. This discrete-time version is obtained by using the Euler discretization method with a sampling period of $T = 0.01$. The system is described by:

$$A = \begin{bmatrix} 1 - 10T & 10T & 0 \\ 28T & 1 - T & 0 \\ 0 & 0 & 1 - 8/3T \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

and $f(x(k), y(k), k) = T \begin{bmatrix} 0 \\ -x_3(k)x_1(k) \\ x_2(k)x_1(k) \end{bmatrix}$.



(a) Phase plot.



(b) The scalar signal $\rho(k)$.

Fig. 2. The Lorentz chaotic system.

The chaotic behavior of this system is presented in the phase plot of Figure 2(a), with initial conditions $x(0) = \begin{bmatrix}-2 \ 2 \ 0.1\end{bmatrix}^T$.

We can rewrite this system as in (4) with $\mathcal{A}(\rho(k))$ defined by (3b) where $H = T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$, $\rho(k) = g_1(y(k)) = y(k)$ and $g_2(y(k), k) = \begin{bmatrix}0 \ 0 \ 0\end{bmatrix}^T$. The Figure 2(b) shows the evolution of the parameter $\rho(k)$ which is identical to the system output. From this figure we can see that $\rho(k)$ is bounded and we have $\underline{\rho} = -25$ and $\overline{\rho} = 19$. Using Theorem 1, we can compute the following gain matrices for the slave-system:

$$L_0 = \begin{bmatrix}0.9609 \ 0.9077 \ 0.0106\end{bmatrix}^T \text{ and}$$
$$L_1 = \begin{bmatrix}0.0001 \ -0.0001 \ 0.0064\end{bmatrix}^T.$$

The dynamic of the synchronization error is shown on the Figure 3.

Example 2. Consider the continuous-time Rössler chaotic model described by the following set:

$$A = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ 0 & 0 & -c \end{bmatrix}, \quad C = \begin{bmatrix}0 \ 0 \ 1\end{bmatrix}$$

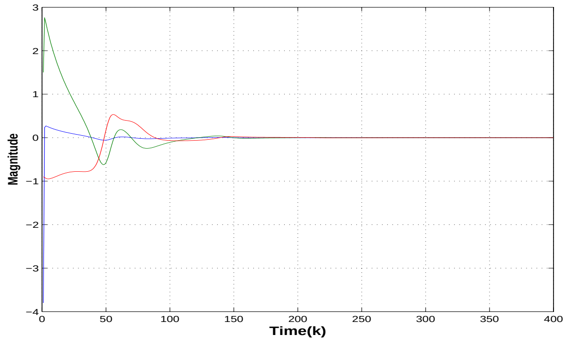


Fig. 3. The synchronization error $e(k)$.

and $f(x(t), y(t), t) = \begin{bmatrix}0 \ 0 \ x_1(t)x_3(t) + b\end{bmatrix}^T$ with $a = 0.398$, $b = 2$, $c = 4$ and $x = \begin{bmatrix}x_1 \ x_2 \ x_3\end{bmatrix}^T$.

The matrix C allows us to rewrite this system as in (4) with $H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $\rho(t) = g_1(y(t)) = y(t)$, and $g_2(y(t), t) = \begin{bmatrix}0 \ 0 \ b\end{bmatrix}^T$. The evolution of the parameter $\rho(k)$ is shown on Figure 4(b). We have $\underline{\rho} = 0.25$ and $\overline{\rho} = 5.8$.

The implementation of the LMI conditions of Theorem 4 gives the following gain matrices:

$$L_0 = \begin{bmatrix}-8.5867 \ 8.809 \ -5.3032\end{bmatrix}^T \text{ and}$$
$$L_1 = \begin{bmatrix}95.0233 \ -65.0105 \ 20.1112\end{bmatrix}^T.$$

The behavior of the synchronization error for this example is illustrated in Figure 5.

We can also measure the first state component *i.e.* choose the output matrix $C = \begin{bmatrix}1 \ 0 \ 0\end{bmatrix}$, and apply our approach in order to construct an observer-based synchronization scheme. The gain matrices computed using Theorem 4 are

$$L_0 = \begin{bmatrix}3.1482 \ -4.7058 \ -4.0962\end{bmatrix}^T \text{ and}$$
$$L_1 = \begin{bmatrix}0 \ 0 \ 1\end{bmatrix}^T.$$

6. CONCLUSION

In this paper, a new framework for the synchronization of chaotic systems was presented. We have shown that for a particular transmitted signal, the synchronization problem of a large class of chaotic systems can be reformulated as an observer synthesis problem for an appropriate LPV system. Our work shows that LPV techniques can be used successfully in the context of chaos synchronization.

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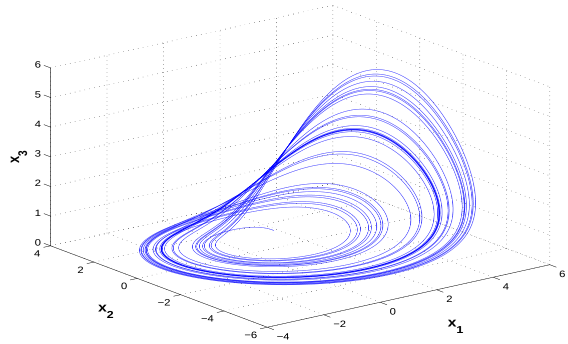
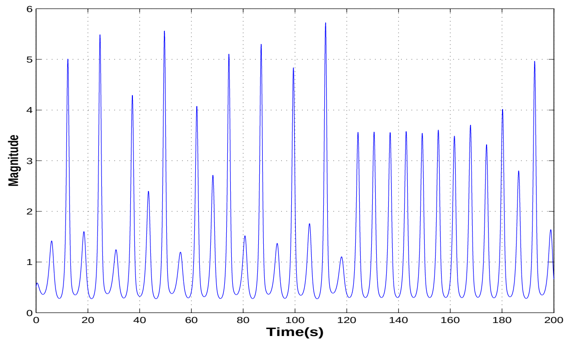


(a) The Rössler's attractor.



(b) The scalar signal $\rho(t)$.

Fig. 4. The synchronization error behavior.

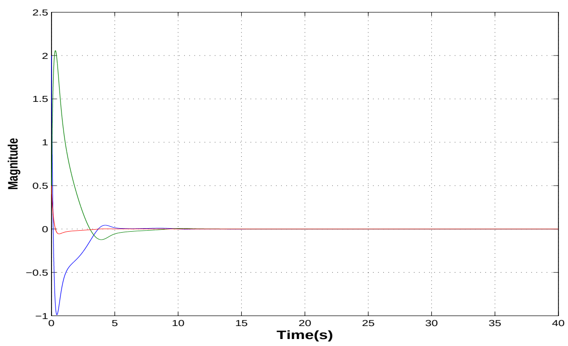


Fig. 5. The synchronization error $e(t)$.

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