SUM OF SQUARES RELAXATIONS FOR ROBUST POLYNOMIAL SEMI-DEFINITE PROGRAMS

Camile Hol [*,1] **Carsten Scherer** [*,2]

[*] *Delft University of Technology, Delft Center for Systems and Control, Mekelweg 2, 2628 CD Delft, The Netherlands. Phone +31-15-2781660, Fax +31-15-2786679.* c.w.j.hol@dcsc.tudelft.nl *and* c.w.scherer@dcsc.tudelft.nl

Abstract: A whole variety of robust analysis and synthesis problems can be formulated as robust Semi-Definite Programs (SDPs), i.e. SDPs with data matrices that are functions of an uncertain parameter which is only known to be contained in some set. We consider uncertainty sets described by general polynomial semi-definite constraints, which allows to represent norm-bounded and structured uncertainties as encountered in μ-analysis, polytopes and various other possibly non-convex compact uncertainty sets. As the main novel result we present a family of Linear Matrix Inequalities (LMI) relaxations based on sum-of-squares (sos) decompositions of polynomial matrices whose optimal values converge to the optimal value of the robust SDP. The number of variables and constraints in the LMI relaxations grow only quadratically in the dimension of the underlying data matrices. We demonstrate the benefit of this a priori complexity bound by an example and apply the method in order to asses the stability of a fourth order LPV model of the longitudinal dynamics of a helicopter.

Keywords: Sum-of-squares (sos), Polynomials, robustness, Linear Matrix Inequalities (LMI), convex programming

1. INTRODUCTION

Many robust analysis and synthesis problems can be translated into so-called robust Semi-Definite Programs (SDPs). The modelling power of this framework, in particular for robust optimization and robust linear algebra is known for long (Ben-Tal and Nemirovski, 2001; El Ghaoui *et al.*, 1999). It captures a large class of robust performance analysis and synthesis problems, such as in standard singular value theory and considerable generalizations thereof (Packard and Doyle, 1993), and stability and performance analysis of Linear Parameter Varying (LPV) systems with quadratic-in-state Lyapunov functions (Trofino and de Souza, 1999; Iwasaki and Shibata, 2001).

[1] Sponsored by Philips CFT

[2] Supported by the Technology Foundation STW, applied science division of NWO and the technology programme of the Ministry of Economic Affairs.

Robust SDP problems are variants of standard SDPs where the data matrices $F_i(x)$, $i = 0, 1, \ldots, n$, are functions of some parameter x that is only known to be contained in some set $\mathbf{\Delta}$ and the goal is to infimize $c^T y$ over all $y \in \mathbb{R}^n$ such that $F(x, y) \succ 0$ for all $x \in \mathbf{\Delta}$, where

$$F(x, y) := F_0(x) + y_1 F_1(x) + \ldots + y_n F_n(x). \quad (1)$$

For a large number of problems it can be assumed that the data matrices $F_i(x)$, $i = 1, \ldots, n$ are symmetric $p \times p$ matrix-valued polynomial functions (i.e. polynomials with matrix-coefficients) and that the set $\mathbf{\Delta}$ is described by a polynomial semi-definite constraint $\mathbf{\Delta} := \{x \in \mathbb{R}^N \mid G(x) \preccurlyeq 0\}$, where G is a $q \times q$ symmetric matrix-valued polynomial. This brings us to the problem considered in this paper:

$$\begin{array}{ll} \text{infimize} & c^T y \\ \text{subject to} & F(x, y) \succ 0 \\ & \text{for all } \ x \in \mathbb{R}^m \text{ with } \ G(x) \preccurlyeq 0 \end{array} \quad (2)$$

where $F(x, y)$ is as in (1). Let us denote its optimal value by p_{opt}. If $F(x, y)$ is rationally dependent on x, we can, under a well-posedness condition, multiply by the smallest common denominator of $F(x, y)$ that is positive on $\mathbf{\Delta}$ to render the SDP polynomial in x. In (Jibetean and De Klerk, 2003) this technique is applied to problems with scalar rational constraints. Complex-valued uncertainty are reduced to real values in a standard fashion.

As an example of a problem that can be molded into (2), consider the uncertain system $\dot{z}(t) = A(\delta(t))z(t)$, where $A(\cdot)$ is a matrix-valued polynomial and $\delta(\cdot)$ varies in the set of continuously differentiable parameter curves $\delta : [0, \infty) \mapsto \mathbb{R}^m$ with $\delta(t) \in \mathbf{\Delta}$ for all $t \in [0, \infty)$. Consider for instance the following combination of polytopic and norm-bounded uncertainties $\mathbf{\Delta} := \{p = \begin{pmatrix} p_1 & p_2 \end{pmatrix}^T : p_1 \in \mathbb{R}^{m_1},\ p_2 \in \mathbb{C}^{m_2},\ H(p_1) \leq 0,\ \|p_2\| \leq 1\}$ where H is affine. Then the system is uniformly exponentially stable if there exists a $Y \succ 0$ such that $A(p)^T Y + YA(p) \prec 0$ for all $p \in \mathbf{\Delta}$. With $x_1 = p_1$, $x_2 = \Re(p_2)$, $x_3 = \Im(p_2)$, $y = \text{vec}(Y)$, $c = 0$, $G(x) = \text{diag}(H(x_1), x_2^T x_2 + x_3^T x_3 - 1)$ and $F(x, y) = \text{diag}(Y, -A(x)^T Y - YA(x))$ this problem is of type (2), where $\Re$ and $\Im$ denote real and imaginary part respectively and diag denotes block diagonal augmentation.

In addition to robustness problems, the framework of (2) includes polynomial Semi-definite Programming (Hol and Scherer, 2004; Kojima, 2003) as a special case. Since (2) is not a standard polynomial SDP, the class of problems considered in this paper is significantly larger.

Solving (2) is very difficult and various relaxations have been proposed in the literature. The full block S-procedure (Iwasaki and Shibata, 2001; Scherer, 2001) allows to construct various relaxations, but in general it cannot be expected that these relaxations are exact. We focus here on a sequence of Sum-Of-Squares (sos) relaxations that are asymptotically exact. Such schemes have recently been applied to robust stability analysis problems as described above (Chesi *et al.*, 2003*a*; Chesi *et al.*, 2003*b*; Henrion *et al.*, 2003; Scherer, 2003). These papers only consider polytopic uncertainty sets. The uncertainty description in (2) allows to describe many other sets, such as norm-bounded and structured uncertainty as encountered in μ-analysis and various other possibly non-convex compact uncertainty sets.

This paper presents asymptotically exact sos relaxations for (2), and is as such a natural extension of (Scherer and Hol, 2004) to robustness problems with matrix-valued constraints. More precisely the main novel result is the construction of a sequence of sos polynomial relaxations of (2)

- that require the solution of standard Linear Matrix Inequalities (LMI) problems whose size grows only quadratically in the dimension p and q (i.e. the number of rows/columns) of F and G respectively and
- whose optimal values converge from below to p_{opt} if a certain constraint qualification is satisfied.

A crucial concept in this construction is the 'sum of squares of polynomial matrices', as will be explained in Section 2. Based on this concept and on scalar sos results (Jacobi and Prestel, 2001; Putinar, 1993; Lasserre, 2001; Parrilo and Sturmfels, 2001), we will present a sequence of sos relaxations for (2) with the desired properties in Section 3. In Section 4 we demonstrate the benefit of the a priori bound on the LMI size by an example and discuss why straightforward scalarisation of the matrix valued problem fails in general to admit such bounds. Finally in Section 5 we apply the approach to stability analysis for a fourth order LPV model of the longitudinal dynamics of a helicopter.

2. SUM OF SQUARES OF POLYNOMIAL MATRICES

A symmetric matrix-valued $p \times p$-polynomial matrix $S(x)$ in $x \in \mathbb{R}^m$ is said to be a sum-of-squares (sos) if there exists a (not necessarily square and typically tall) polynomial matrix $T(x)$ such that

$$S(x) = T(x)^T T(x). \tag{3}$$

If $u_j(x)$ $j = 1, \ldots, n_u$ are pairwise different monomials, then $S(x)$ is said to be sos *with respect to monomial basis* $u(x) = \text{col}(u_1(x), \ldots, u_{n_u}(x))$, if T in (3) can be chosen as $T(x) = \sum_{j=1}^{n_u} T_j u_j(x)$, where $T_j = T_j^T \in \mathbb{R}^{p \times p}$, $j = 1, \ldots, n_u$. To compactly represent the sos decompositions we define for $M \in \mathbb{R}^{pq \times pq}$, partitioned in blocks $M_{i,j} \in \mathbb{R}^{q \times q}$, $i, j = 1, \ldots, p$, the operator

$$\text{Trace}_p(M) := \begin{pmatrix} \text{Trace}(M_{11}) & \cdots & \text{Trace}(M_{1p}) \\ \vdots & \ddots & \vdots \\ \text{Trace}(M_{p1}) & \cdots & \text{Trace}(M_{pp}) \end{pmatrix},$$

and for $A, B \in \mathbb{R}^{pq \times pq}$ the bilinear mapping

$$\langle A, B \rangle_p = \text{Trace}_p(A^T B).$$

If $w_j(x)$, $j = 1, \ldots, s$, denote the pairwise different monomials that appear in $u(x)u(x)^T$ one can determine the unique symmetric matrices Z_j with

$$u(x)u(x)^T = \sum_{j=1}^{s} Z_j w_j(x).$$

Using these definitions the following result reduces the question of whether $S(x)$ is sos with respect to $u(x)$ to an LMI feasibility problem.

Lemma 1. The polynomial matrix $S(x)$ of dimension p is sos with respect to the monomial basis $u(x)$ iff there exist symmetric S_j with $S(x) = \sum_{j=1}^{s} S_j w_j(x)$ and the following linear system has a solution $W \succeq 0$:

$$\langle W, I_p \otimes Z_j \rangle_p = S_j, \quad j = 1, \ldots, s. \tag{4}$$

If W solves (4) then $S(x) = \langle W, I_p \otimes u(x)u(x)^T \rangle_p = (I_p \otimes u(x))^T W (I_p \otimes u(x))$.

$\mathrm{Trace}_p(\cdot)$ satisfies the following easily verified properties: for all A and B of appropriate size

$$\mathrm{Trace}_p((I_p \otimes B)A) = \mathrm{Trace}_p(A(I_p \otimes B)) \tag{5}$$

and (Choi, 1975)

$$\mathrm{Trace}_p(A) \succeq 0 \ \text{ for every } \ A \succeq 0, A \in \mathcal{S}^{pq}, \tag{6}$$

for arbitrary $p, q \in \mathbb{N}$.

3. MATRIX VALUED SOS DECOMPOSITION

Consider the optimization problem

$$\begin{array}{ll} \text{infimize} & c^T y \\ \text{subject to} & \\ & F(x,y) - \epsilon I_p + \langle S(x), I_p \otimes G(x) \rangle_p = S_0(x) \\ & S(x) \text{ and } S_0(x) \text{ are sos}, \ \epsilon > 0 \end{array} \tag{7}$$

with optimal value d_{opt}. Note that the sizes of $S_0(x)$ and $S(x)$ are $p \times p$ and $pq \times pq$ respectively. In this section we present our main result, which shows that the optimal values of (2) and (7) are equal if G satisfies a constraint qualification. This result allows to construct an asymptotically exact family of LMI relaxations of (2). Indeed, by choosing fixed monomial bases $u_0(x)$ and $u(x)$ of the sos matrices $S_0(x)$ and $S(x)$ respectively, upper bounds on the optimal value of (7) can be computed by solving an LMI problem, as explained in the previous section. These upper bounds converge to the optimal value of (7) if the monomial basis vectors are infinitely extended with new monomials.

Before discussing the main result let us first assume that G is diagonal, i.e.

$$G = \mathrm{diag}(g_1(x), g_2(x), \ldots, g_r(x)). \tag{8}$$

Feasibility of y_* for (2) comes down to computationally verifying whether

$$F(x, y_*) \succ 0 \ \text{ for all } \ G(x) \preccurlyeq 0. \tag{9}$$

The following theorem shows that this is possible using a representation with matrix-valued sos polynomials.

Theorem 2. Suppose G is as in (8) for some g_i, $i = 1, \ldots, r$, and suppose the following constraint qualification holds true: There exists some $M > 0$, an sos polynomial $\psi(x)$ and an sos matrix $\Psi(x)$ such that

$$M - \|x\|^2 + \langle \Psi(x), G(x) \rangle = \psi(x). \tag{10}$$

Then (9) implies there exist $\epsilon > 0$ and matrix sos $S_0(x), S_1(x), \ldots, S_r(x)$ such that

$$F(x, y_*) - \epsilon I_p + \sum_{i=1}^{r} S_i(x) g_i(x) = S_0(x). \tag{11}$$

Proof. The proof is a straightforward extension of Theorem 2 in (Scherer and Hol, 2004) and therefore omitted.

■

Now let us drop the assumption on $G(x)$ being diagonal. This brings us to the central contribution of this paper.

Theorem 3. Suppose there exist $M > 0$, an sos polynomial $\psi(x)$ and an sos matrix $\Psi(x)$ such that (10) holds true. If p_{opt} and d_{opt} are the optimal values of (2) and (7) respectively then $p_{\mathrm{opt}} = d_{\mathrm{opt}}$.

Proof. We first prove $p_{\mathrm{opt}} \le d_{\mathrm{opt}}$, by showing that the constraint in (2) is implied by the constraint in (7). Consider arbitrary $y_* \in \mathbb{R}^n$ and x_* with $G(x_*) \preccurlyeq 0$. Let us now suppose that $S(x)$ and $S_0(x) = F(x, y_*) - \epsilon I_p + \langle S(x), I_p \otimes G(x) \rangle_p$ are sos. Due to (6) one infers

$$F(x_*, y_*) \succeq F(x_*, y_*) - \epsilon I_p + \langle S(x_*), I_p \otimes G(x_*) \rangle_p \succeq 0.$$

Since x_* with $G(x_*) \preccurlyeq 0$ was arbitrary the implication is shown which implies $p_{\mathrm{opt}} \le d_{\mathrm{opt}}$.

To prove $p_{\mathrm{opt}} \ge d_{\mathrm{opt}}$ note that, as a consequence of the constraint qualification, if $G(x) \preccurlyeq 0$ is replaced by

$$\tilde{G}(x) := \mathrm{diag}(G(x), \|x\|^2 - M) \preccurlyeq 0$$

then the value of (2) is not modified. In a first step of the proof of $p_{\mathrm{opt}} \ge d_{\mathrm{opt}}$, let us show that the same is true for the sos reformulation (7).

Indeed suppose $F(x,y) - \epsilon I_p + \langle S(x), I_p \otimes G(x) \rangle_p = S_0(x)$ with sos matrices $S_0(x)$ and $S(x)$. If we partition $S(x) = (S_{jk}(x))_{jk}$ into $q \times q$-blocks then $\tilde{S}(x) := (\mathrm{diag}(S_{jk}(x), 0))_{jk}$ satisfies $\langle \tilde{S}(x), I_p \otimes \tilde{G}(x) \rangle_p = \langle S(x), I_p \otimes G(x) \rangle_p$ and therefore $F(x,y) + \langle \tilde{S}(x), I_p \otimes \tilde{G}(x) \rangle_p - \epsilon I_p = S_0(x)$.

Conversely suppose $F(x,y) - \epsilon I_p + \langle \tilde{S}(x), I_p \otimes \tilde{G}(x) \rangle_p = \tilde{S}_0(x)$ with sos matrices $\tilde{S}_0(x)$, $\tilde{S}(x)$. Now we make explicit use of (10) with sos matrices $\psi(x)$, $\Psi(x)$. Let us partition

$$\tilde{S}(x) = \left(\begin{pmatrix} S_{jk}(x) & * \\ * & s_{jk}(x) \end{pmatrix} \right)_{jk}$$

into blocks of size $(q+1) \times (q+1)$ and define

$$S(x) := (S_{jk}(x) + s_{jk}(x)\Psi(x))_{jk}$$

and $s(x) = (s_{jk}(x))_{jk}$ of dimension pq and p respectively. It is easy to verify that both matrices are sos and satisfy

$$\langle \tilde{S}(x), I_p \otimes \tilde{G}(x) \rangle_p = \langle S(x), I_p \otimes G(x) \rangle_p - s(x)\psi(x).$$

This implies $F(x,y) - \epsilon I_p + \langle S(x), I_p \otimes G(x)\rangle_p = \tilde{S}_0(x) + s(x)\psi(x)$ and it remains to observe that $\tilde{S}_0(x) + s(x)\psi(x)$ is sos.

Therefore, from now on we can assume w.l.o.g. that

$$v_1^T G(x) v_1 = \|x\|^2 - M \tag{12}$$

where v_1 is the last standard unit vector. It remains to show $p_{\text{opt}} \geq d_{\text{opt}}$, and for this purpose it suffices to choose an arbitrary y_* which is feasible for (2) and to prove that y_* is as well feasible for (7).

Let us hence assume $F(x, y_*) \succ 0$ for all $x \in \mathbf{\Delta}$. Choose a sequence of unit vectors $v_2, v_3, \ldots$ such that v_i, $i = 1, 2, \ldots$ is dense in the unit sphere $\{v \in \mathbb{R}^q : \ \|v\| = 1\}$. Define

$$\mathbf{\Delta}_N := \{x \in \mathbb{R}^m : v_i^T G(x) v_i \leq 0, \quad i = 1, \ldots, N\}$$

to infer that $\mathbf{\Delta}_N$ is compact (by (12)) and that $\mathbf{\Delta}_N \supset \mathbf{\Delta}_{N+1} \supset \mathbf{\Delta}$ for $N = 1, 2, \ldots$. Therefore $p_N := \min\{\lambda_{\min}(F(x, y_*)) : \ x \in \mathbf{\Delta}_N\}$ is attained by some x_N and $p_N \leq p_{N+1}$ for all $N = 1, 2, \ldots$. Let us prove that there exists some N_0 for which $p_{N_0} > 0$ which implies

$$F(x, y_*) \succ 0 \ \text{ for all } \ x \in \mathbf{\Delta}_{N_0}. \tag{13}$$

Indeed otherwise $p_N \leq 0$ for all $N = 1, 2, \ldots$ and hence $\lim_{N\to\infty} p_N \leq 0$. Choose a subsequence N_ν with $x_{N_\nu} \to x_0$ to infer $0 \geq \lim_{\nu\to\infty} \lambda_{\min}(F(x_{N_\nu}, y_*)) = \lambda_{\min}(F(x_0, y_*))$. This contradicts the choice of y_* if we can show that $G(x_0) \preccurlyeq 0$. In fact, otherwise there exists a unit vector v with $\delta := v^T G(x_0) v > 0$. By convergence there exists some K with $\|G(x_{N_\nu})\| \leq K$ for all ν. By density there exists a sufficiently large ν such that $K\|v_i - v\|^2 + 2K\|v_i - v\| < \delta/2$ for some $i \in \{1, \ldots, N_\nu\}$. Since $v^T G(x_{N_\nu}) v \to v^T G(x_0) v$ we can increase ν to even guarantee $v^T G(x_{N_\nu}) v \geq \delta/2$ and arrive at the following contradiction:

$$\begin{aligned} 0 &\geq v_i^T G(x_{N_\nu}) v_i = \\ &= (v_i - v)^T G(x_{N_\nu})(v_i - v) + 2v^T G(x_{N_\nu})(v_i - v) + \\ &\quad + v^T G(x_{N_\nu}) v \geq \\ &\geq -K\|v_i - v\|^2 - 2K\|v_i - v\| + \delta/2 > 0. \end{aligned}$$

We are now in the position to apply Theorem 2 to (13) since, due to (12), the constraint qualification is trivially satisfied. Hence there exist $\epsilon > 0$ and polynomial matrices $U_i(x)$ with p columns, $i = 1, \ldots, N_0$, such that

$$F(x, y_*) - \epsilon I + \sum_{i=1}^{N_0} [U_i(x)^T U_i(x)](v_i^T G(x) v_i) \tag{14}$$

is sos in x. With elementary Kronecker product manipulations and (5) we conclude

$$\begin{aligned} &[U_i(x)^T U_i(x)](v_i^T G(x) v_i) = \\ &= \mathrm{Trace}_p \left([U_i(x)^T U_i(x)] \otimes (v_i^T G(x) v_i)\right) \\ &= \mathrm{Trace}_p \left(([U_i(x)^T U_i(x)] \otimes v_i^T)(I_p \otimes G(x))(I_p \otimes v_i)\right) \\ &= \mathrm{Trace}_p \left(([U_i(x)^T U_i(x)] \otimes v_i v_i^T)(I_p \otimes G(x))\right) \\ &= \langle (U_i(x) \otimes v_i^T)^T (U_i(x) \otimes v_i^T), I_p \otimes G(x)\rangle_p. \end{aligned}$$

With the sos polynomial matrix

$$S(x) := \sum_{i=1}^{N_0} (U_i(x) \otimes v_i^T)^T (U_i(x) \otimes v_i^T)$$

we infer that $F(x, y_*) - \epsilon I + \langle S(x), I_p \otimes G(x)\rangle$ equals the left-hand side in (14) and is hence sos in x. Therefore y_* is feasible for (7).

$\blacksquare$

Remark. The constraint qualification can be equivalently formulated as follows: There exist sos polynomials $s_0(x), S(x)$ such that

$$\{x \in \mathbb{R}^m : \ \mathrm{Trace}(S(x)G(x)) - s_0(x) \geq 0\}$$

is compact.

Theorem 3 is a natural extension of a theorem of Putinar (Putinar, 1993) for scalar polynomial problems in two directions:

- the set $G(x) \preccurlyeq 0$ is described by **matrix-valued** instead of a scalar polynomials;
- a sos representation of the **matrix-valued** (instead of scalar) polynomial $F(x)$ is obtained.

If the variable y is absent, F is scalar and G is diagonal as in (8), Lasserre's approach (Lasserre, 2001) for minimizing $f(x)$ over scalar polynomial constraints $g_i(x) \geq 0$, $i = 1, \ldots, r$ is recovered. Moreover the constraint qualification in Theorem 3 is a natural generalization of that used by Schweighofer (Schweighofer, 2003) for scalar polynomial optimization problems.

4. COMPARISON WITH SCALARIZATION

In this section we shed some light on the benefits of exploiting the matrix structure in the sos relaxations compared to straightforward scalarisation. In particular we explain why scalarisation fails to lead to the desired properties (of quadratic growth in the matrix sizes) of the corresponding LMI relaxations. Observe that $G(x) \preccurlyeq 0$ is equivalent to $M_i(G(x)) \leq 0, i = 1, \ldots, r$ where $M_i(A), i = 1, \ldots, r$ are all the principal minors of a matrix $A \in \mathbb{R}^{q\times q}$ (Horn and Johnson, 1985). Hence if we define $f(v, x, y) := v^T F(x, y) v$ and

$$\begin{aligned} &h_i(v, x) = M_i(G(x)) \ \ i = 1, \ldots, r, \\ &h_{r+1}(v, x) = 1 - v^T v, \quad h_{r+2}(v, x) = v^T v - 2, \end{aligned}$$

then (2) is equivalent to infimizing $c^T y$ subject to

$$f(v, x, y) > 0 \ \text{ for all } \ (x, v) \tag{15}$$

with $h_i(v, x) \leq 0 \ i = 1, \ldots, r + 2$. If $h_i, i = 1, \ldots, r + 2$ satisfy a constraint qualification then the scalar results of Putinar (Putinar, 1993) (15) imply that there exist sos polynomials $s_i(v,x)$, $i=1,\ldots,r+2$, such that

$$f(v,x,y)+\sum_{i=1}^{r+2} s_i(v,x)h_i(v,x) \quad \text{is sos.} \qquad (16)$$

However, although $f(v,x,y)$ and $h_i(v,x)$ are quadratic in v, no available result allows to guarantee that the sos polynomials $s_i(v,x)$, $i=1,\ldots,r+2$, can be chosen quadratic in v without loosing the relaxation's exactness. Without such a priori degree information, the corresponding LMI relaxation size needs to grows fast in the length of v which equals the dimension of $F(x,y)$. Theorem 3 implies that one can indeed confine the search to $s_{q+1}(v,x)=0$, $s_{q+2}(v,x)=0$ and to $s_j(v,x)=v^TS_j(x)v$, $j=0,1,\ldots,q$, which are homogenously quadratic in v, without violating $p_{\text{opt}}=d_{\text{opt}}$.

Regarding the matrix-valued constraint polynomial $G(x)$, the maximum of the total degrees of the minors $M_i(G(x)), i=1,\ldots,r$ is at least as high as the total degree of $G(x)$ and will in practice often be higher. A larger polynomial degree often requires to use a larger monomial basis and hence more variables and constraints in the LMI relaxation to obtain good approximations of p_{opt}. This is illustrated by the following example (inspired by a personal communication with Didier Henrion): Computate lower bounds on $p_{\text{opt}}=\inf_{G(x)\preccurlyeq 0} F(x)$ where $x=\begin{pmatrix} x_1 & x_2\end{pmatrix}^T\in\mathbb{R}^2$, $F(x)=x_1+x_2$ and

$$G(x)=\begin{pmatrix} 1 & x_1^2 & 0\\ x_1^2 & 9-x_2^2 & 0\\ 0 & 0 & 1-\dfrac{x_2^2+x_1^2}{100}\end{pmatrix}.$$

Table 1 shows lower bounds on the optimal value and the sizes of the LMI problems for sos relaxations based on (7) and based on two ways of scalarisation, where we used for "Scalar 1" $g_1(x)=\det(G(1:2,1:2))$, $g_2(x)=\text{Trace}(G(1:2,1:2))$ and $g_3(x)=G(3,3)$ and for "Scalar 2" the minors $g_i(x):=M_i(G(x))\le 0, i=1,\ldots,7$. We choose monomial bases $u_0(x)$ as shown in the table and $u_i(x)=1$, $i=1,2,\ldots$ respectively to represent $S_i(x)$, $i=0,1,2,\ldots$ as in Lemma 1. An upper bound on the optimal value p_{opt} is $F(-1.148,-2.695)=-3.843$, which was obtained by gridding. As is clear from the table, the matrix-valued relaxation is (almost) exact for $u_0=(1,x_1,x_2)^T$, obtained by LMI optimization with 18 constraints and 13 variables. The scalarised relaxations are exact if $u_0=(1,x_1,x_2,x_1^2,x_1x_2,x_2^2)^T$, which required the solution of an LMI problem with 27 constraints and 16 variables. The table shows that "Scalar 2" requires even more LMI variables to obtain close to exact results.

Relax-ation	optim. value	monomial in $u_0(x)^T$	LMI constr	LMI vars
Matrix	-3.85	$(1,x_1,x_2)$	18	13
Scalar 1	-12.65	$(1,x_1,x_2)$	16	10
Scalar 1	-3.85	$(1,x^T,x_1x^T,x_2^2)$	27	16
Scalar 2	-2.6e4	$(1,x_1,x_2)$	29	14
Scalar 2	-3.85	$(1,x^T,x_1x^T,x_2^2)$	48	29

Table 1. Optimal values and LMI size for matrix and scalar relaxations

5. APPLICATION

We consider the stability analysis of an LPV model of a closed-loop Vertical TakeOff and Landing (VTOL) helicopter (Gahinet *et al.*, 1994; Iwasaki and Shibata, 2001). The linearized longitudinal dynamic equations of the helicopter, after applying a static feedback law as in (Iwasaki and Shibata, 2001), are $\dot z=A(p)z$ where

$$A(p)=\begin{pmatrix} -0.0366 & -0.096 & 0.018 & -0.45\\ 0.0482 & a_3(p) & 0.0024 & -4.02\\ 0.10 & a_1(p) & -0.707 & a_2(p)\\ 0 & 0 & 1 & 0\end{pmatrix}$$

and $a_1(p)=14.0+0.05p_1$, $a_2(p)=1.42+0.01p_2$ and $a_3=-18.2-0.0399p_3$. We analyze its stability for all uncertainties satisfying $\|p\|\le\gamma$ and $|\dot p_k|\le\rho$, $k=1,2,3$ for fixed values of γ and ρ. We consider affine Lyapunov functions $P(p)=\sum_{i=1}^4 P_im_i(p)$ where $m(p)=(1,p_1,p_2,p_3)$. Then the system is robustly stable if there exist $P_i=P_i^T\in\mathbb{R}^{4\times4}$, $i=1,\ldots,4$ such that

$$A(p)^TP(p)+P(p)A(p)+\sum_{k=1}^{3}\frac{\partial A(p)}{\partial p_k}q_k\prec 0$$

for all $\|p\|\le\gamma$ and all $|q_k|\le\rho$, $k=1,2,3$. Hence with the definitions $x:=\begin{pmatrix}p^T & q^T\end{pmatrix}^T$, $y:=\begin{pmatrix}\text{vec}(P_1),\ldots,\text{vec}(P_N)\end{pmatrix}^T$,

$$F(x,y):=A(p)^TP(p)+P(p)A(p)+\sum_{k=1}^{3}\frac{\partial A(p)}{\partial p_k}q_k,$$

$G=\text{diag}\,(g_1,\ldots,g_4)$, $g_1(x):=\|p\|^2-\gamma^2$ and $g_{1+i}(x):=|q_i|-\rho$ $i=1,2,3$, feasibility of y in (2) implies robust stability.

We compute sos relaxations of (2) with sos bases $u_0(x)=\begin{pmatrix}1 & x\end{pmatrix}$ and $u_i(x)=1$, $i=1,\ldots,4$. Figure 1 shows the results. Note that the results can not directly be compared with those in (Iwasaki and Shibata, 2001), (Gahinet *et al.*, 1994) and (Montagner and Peres, 2003), since we consider a norm-bounded instead of a polytopic set and the relaxations in those reference can only be applied to polytopic sets. This illustrates the additional flexibility of our framework, since it can be applied to any uncertainty set that admits a polynomial SDP description. For comparison, we also computed bounds for the polytope $|p_k|\le\gamma$, $k=1,2,3$ together with $|\dot p_k|\le\rho$, $k=1,2,3$ and compared them to the results of (Gahinet *et al.*, 1994). The figure shows that the resulting γ values are similar.

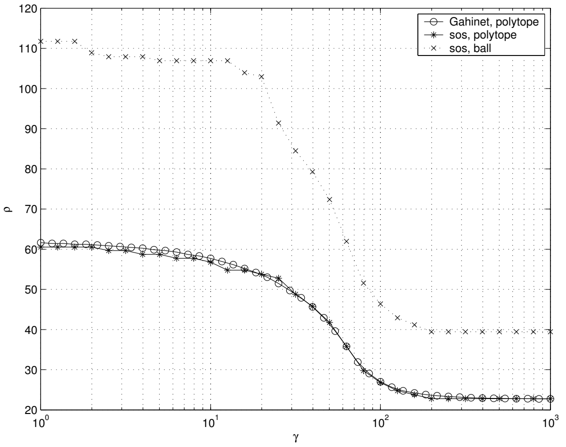

Fig. 1. Sos lower bounds for the polytopic ($*-$) and norm-bounded set ($\times\cdot\cdot$) and Gahinet's lower bounds for the polytopic set ($\circ-$).

ACKNOWLEDGEMENTS.

The authors would like to thank Didier Henrion, Pablo Parrilo, Andrea Garulli, Michael Overton and Pedro Peres for stimulating discussions and Tetsuya Iwasaki for making data available for the example.

6. REFERENCES

Ben-Tal, A. and A. Nemirovski (2001). *Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications.* SIAM-MPS Series in Optimizaton. SIAM Publications. Philadelphia.

Chesi, G., A. Garulli, A. Tesi and A. Vicino (2003*a*). Homogeneous Lyapunov functions for systems with structured uncertainties. *Automatica* **39**(6), 1027–1035.

Chesi, G., A. Garulli, A. Tesi and A. Vicino (2003*b*). Robust stability of polytopic systems via polynomially parameter-dependent lyapunov functions. In: *42nd IEEE Conference on Decision and Control.* Maui, Hawaii. pp. 4670–4675.

Choi, M.D. (1975). Completely positive linear maps on complex matrices. *Linear Algebra Appl.* **10**, 285–290.

El Ghaoui, L., F. Oustry and H. Lebret (1999). Robust solutions to uncertain semidefinite programs. *SIAM J. Optim.* **9**(1), 33–52.

Gahinet, P., A. Nemirovskii, A.J. Laub and M. Chilali (1994). The LMI control toolbox. In: *Proc. 33rd IEEE Conf. Decision and Control.* Orlando, FL. pp. 2038–2041.

Henrion, D., M. Sebek and V. Kucera (2003). Positive polynomials and robust stabilization with fixed-order controllers. *IEEE Transactions on Automatic Control* **48**, 1178–1186.

Hol, C.W.J. and C.W. Scherer (2004). Sum of squares relaxations for polynomial semidefinite programming. In: *Proceedings MTNS 2004.* Leuven, Belgium.

Horn, R.A. and C.R. Johnson (1985). *Matrix Analysis.* Cambrigde University Press. New York.

Iwasaki, T. and G. Shibata (2001). LPV system analysis via quadratic separator for uncertain implicit systems. *IEEE Trans. Aut. Contr.* **46**(8), 1195–1208.

Jacobi, T. and A. Prestel (2001). Distinguished representations of strictly positive polynomials. *J. Reine Angew. Math.* **532**, 223–235.

Jibetean, D. and E. De Klerk (2003). Global optimization of rational functions: a semidefinite programming approach. Technical report. CWI (Centrum voor Wiskunde en Informatica).

Kojima, M. (2003). Sums of squares relaxations of polynomial semidefinite programs. Technical report. Tokyo Institute of Technology.

Lasserre, J. B. (2001). Global optimization with polynomials and the problem of moments. *SIAM J. Optim.* **11**(3), 796–817.

Montagner, V.F. and P.L.D. Peres (2003). A new LMI condition for the robust stability of linear time-varying systems. In: *Proceedings of the* 42^{nd} *Conference on Decision and Control, Maui, Hawaii USA.* pp. 6133–6138.

Packard, A. and J. Doyle (1993). The complex structured singular value. *Automatica* **29**, 71–109.

Parrilo, P. and B. Sturmfels (2001). Minimizing polynomial functions. *DIMACS Series in Discrete Mathematics and Theoretical Computer Science.*

Putinar, M. (1993). Positive polynomials on compact semi-algebraic sets. *Indiana Univ. Math. J.* **42**, 969–984.

Scherer, C.W. (2001). LPV control and full block multipliers. *Automatica* **37**, 361–375.

Scherer, C.W. (2003). Higher-order relaxations for robust LMI problems with verifications for exactness. In: *Proc. 42nd IEEE Conf. Decision and Control.* Maui, Hawaii. pp. 4652–4657.

Scherer, C.W. and C.W.J. Hol (2004). Asymptotically exact relaxations for robust LMI problems based on matrix-valued sum-of-squares. In: *MTNS 2004.* Leuven, Belgium.

Schweighofer, M. (2003). Optimization of polynomials on compact semialgebraic sets. In: *Preprint.*

Trofino, A. and C.E. de Souza (1999). Biquadratic stability of uncertain linear systems. In: *Proc. 38th IEEE Conf. Decision and Control.* Phoenix, Arizona. pp. 5016–5021.