RELIABILITY CALCULUS USING MAX-PLUS ALGEBRA

Corneliu Caileanu

Technical University of Iasi, 67, D. Mangeron Street, Iasi, Romania

Abstract: Max-plus algebra has been used in various fields as optimal control, statistical physics, discrete event systems, decision theory. In this paper an attempt is made to use it for the reliability calculus in solving the following problems: find the structure function of a system, keep track of the lowest reliability the system can get and find all the possible combinations of the reliabilities of the system's elements (blocks) which ensure a desired lower limit for the system's reliability.

Keywords: Algebraic approaches, ELCP Algorithm, Max-plus algebra, Reliability analysis, Reliability evaluation, Reliability theory, Residuation theory.

1. INTRODUCTION

The exotic semirings have been used in various fields: performance evaluation of manufacturing systems and discrete event system theory, graph theory, Markov decision processes, Hamilton-Jacoby theory, asymptotic analysis, language theory. In this paper one of the exotic semirings, the idempotent semiring (dioid) named Max-plus algebra (Max algebra, (max,+) algebra), see (Baccelli, *et al.*, 1992; Cuninghame-Green, 1979), is used for reliability calculus for determining the structure function, see (Catuneanu, 1983; Ganciu and Tugurlan, 2002), of a system and all the possible combinations of the reliabilities of the system's elements (blocks) which ensure a desired lower limit for the reliability of the whole system, considering that the structure function method is very useful for reliability calculus for complex reliability diagrams which can not be reduced to simple series and parallel structures and obtaining a satisfactory reliability with reduced (minimum) cost is a constant, major objective.

The paper is structured as follows. In section 1 the algebraic structure 'Max-plus algebra' is introduced via it's axioms. In section 2 a method for determining the structure function of a system described by it's reliability diagram which cannot be reduced to simple combinations of series and parallel structure is presented and a lower limit for the reliability of the system is established using the algebraic structure introduced in section 1. In section 3 it is shown how the ELCP algorithm, see (De Schutter and De Moor, 1996), can be used to determine, when it's possible, all the combinations of the reliabilities of the elements (blocks) present in the reliability diagram which ensure an imposed lower limit for the reliability of the whole system. Section 4 deals with some considerations made on the determination of the smallest elements' reliabilities that ensure an imposed lower limit for the system's reliability. Section 5 is reserved to the conclusions.

2. MAX-PLUS ALGEBRA

A dioid is a set S endowed with two operations denoted ⊕ and ⊗ (called 'sum' or 'addition' and 'product' or 'multiplication') obeying the following axioms:

$$\forall a,b,c \in S, (a \oplus b) \oplus c = a \oplus (b \oplus c) \qquad (1)$$

$$\forall a,b \in S, a \oplus b = b \oplus a \qquad (2)$$

$$\forall a,b,c \in S, (a \otimes b) \otimes c = a \otimes (b \otimes c) \quad (3)$$

$$\forall a,b,c \in S, \\ (a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c), \\ c \otimes (a \oplus b) = c \otimes a \oplus c \otimes b \quad (4)$$

$$\exists \varepsilon \in S : \forall a \in S, a \oplus \varepsilon = a \quad (5)$$

$$\forall a \in S, a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon \quad (6)$$

$$\exists e \in S : \forall a \in S, a \otimes e = e \otimes a = a \quad (7)$$

$$\forall a \in S, a \oplus a = a \quad (8)$$

A dioid is commutative if multiplication is also commutative and it is complete if it is closed for infinite sums and axiom 4 extends to infinite sums. In a complete dioid the top element of the dioid, denoted T, exists and is equal to the sum of all elements in S. A subset S_1 of a dioid is called a subdioid of S if $\varepsilon \in S_1$, $e \in S_1$ and S_1 is closed for $\otimes$ and $\oplus$ that is,

$$\forall a,b \in S_1, a \otimes b \in S_1; a \oplus b \in S_1 \quad (9)$$

In a dioid S the following equivalence holds:

$$\forall a,b : a = a \oplus b \Leftrightarrow \exists c : a = b \oplus c \quad (10)$$

and these equivalent statements define a (partial) order relation denoted $\geq$ as follows:

$$a \geq b \Leftrightarrow a = a \oplus b \quad (11)$$

The set of real numbers R, together with $\{-\infty\}$ with max (the maximum value) as $\oplus$ and + (the usual addition) as $\otimes$ is the dioid called Max-plus algebra, R_{max}. It is not complete but can be 'completed' adding the element T=+∞.

Remark: The set of negative real numbers together with $\{-\infty\}$ is a subdioid of R_{max}. It is also complete, with T=e.

3. STRUCTURE FUNCTION DETERMINATION USING THE MAX-PLUS ALGEBRA

The structure function method is widely used for reliability calculus when the (logical) reliability diagram is complex and it is not composed only from combinations of simple series and/or parallel structures, see (Catuneanu, 1983; Ganciu and Tugurlan, 2002). Consider the simple reliability diagram presented in Fig. 1. For this reliability diagram corresponds the graph represented in Fig. 2. The associated matrix, D, of this graph is,

$$\mathbf{D} = \begin{bmatrix} \varepsilon & x_1 & x_4 & \varepsilon \\ x_1 & \varepsilon & x_3 & x_2 \\ x_4 & x_3 & \varepsilon & x_5 \\ \varepsilon & x_2 & x_5 & \varepsilon \end{bmatrix} \quad (12)$$

A connection is defined as the set of the elements, which guarantees the well functioning of the system, no matter the other elements' status. A minimal connection is defined as the connection that does not contain any other connection.

The maximum number of arcs of a minimal connection of a graph with 'n' nodes is 'n-1' .Taking into account that the element d_{ij} of the matrix $\mathbf{D}^r$ represents the greatest-weight path with 'r' arcs between the nod 'i' and the node 'j', the terms of the disjunction which gives the structure function of the system are the terms separated by $\oplus$ in the d_{ij} elements of all $\mathbf{D}^r$ matrices with r=1,...,n-1.

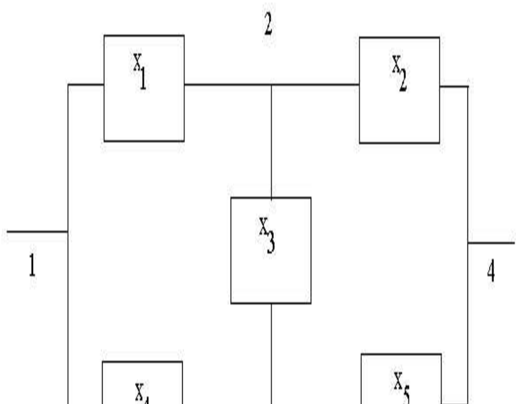

Fig.1. The reliability diagram of the system

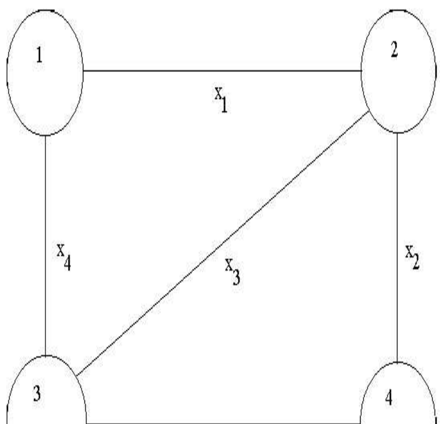

Fig. 2. The associated graph of the system

In the example above, n=4 so n-1=3. The matrices $\mathbf{D}^r$ with r=2,3 (filled only with the necessary elements considering the node 1 as the input node and the node 4 as the output node, omitting ⊗ and calculating the matrix product exactly as in the classical algebra but using the new operators for addition and multiplication) are,

$$\mathbf{D}^2 = \begin{bmatrix} . & . & . & x_1x_2 \oplus x_4x_5 \\ . & . & . & x_3x_5 \\ . & . & . & x_3x_2 \\ . & . & . & x_2x_2 \oplus x_5x_5 \end{bmatrix}$$
$$\mathbf{D}^3 = \begin{bmatrix} . & . & . & x_1x_3x_5 \oplus x_4x_3x_2 \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{bmatrix} \quad (13)$$

and the structure function of the system is

$$f(\mathbf{x}) = x_1x_2 \cup x_4x_5 \cup x_1x_3x_5 \cup x_4x_3x_2 \quad (14)$$

If the probabilistic reliability definition is considered, the new variables, LR_i, are introduced as in (15),

$$LR_i = \log(R_i), R_i \in [0,1], LR_i \in [-\infty,0] \quad (15)$$

and the variables x_i are replaced with LR_i , then the greatest-weight path from the node 'i' considered as the input node to the node 'j' considered as the output node is given by the element d_{ij} of the following matrix:

$$\mathbf{D}_{\max} = \mathbf{D} \oplus \mathbf{D}^2 \oplus ... \oplus \mathbf{D}^{n-1} \quad (16)$$

Moreover, considering the general form of the structure function, the mechanism of passing from the structure function to the reliability and using a basic result from the probability theory, the reliability corresponding to this element is the lowest reliability the system can get.

Remark: $\mathbf{D}_{\max}$ can be computed in $O(n^3)$ steps by the Floyd-Warshall algorithm.

4. THE ELCP ALGORITHM AND THE MAX-PLUS ALGEBRA

Consider the equation (17),

$$\mathbf{X} \oplus \mathbf{X}^2 \oplus ... \oplus \mathbf{X}^{n-1} = \mathbf{B} \quad (17)$$

where **B** is a known matrix. Equation (17) is obviously a system of multivariate polynomial equalities and inequalities in Max-plus algebra, see (De Schutter and De Moor, 1996), which has the general form (18),

$$\begin{aligned} &\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j^{\otimes c_{kij}} = b_k, k = 1,2,...,p_1 \\ &\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j^{\otimes c_{kij}} \le b_k, k = p_1+1,...,p_1+p_2 \end{aligned} \quad (18)$$

for i ∈ {1,2,...,m_k,}, j ∈ {1,2,...,n}, k∈{1,...,p1,p1+1,...,p1+p2} and where $x^{\otimes c}$ = x⊗x⊗...⊗x c times ($x^{\otimes c}$ is equal to cx in the classical linear algebra).

The system can be transformed for some ϕ_j sets, some matrices **A** and some vectors **c** in (19),

$$\sum_{j=1}^{p_1} \prod_{i \in \phi_j} (\mathbf{Ax} - \mathbf{c})_i = 0, \mathbf{Ax} \ge \mathbf{c} \quad (19)$$

where ϕ_j ={s_j + 1, s_j + 2, ... , s_j + m_j} for j=1, 2, ... , p_1 , s_1 = 0 and s_{j+1} = s_j + m_j for j=1,2, ... , p_1-1, and all the operations are in the linear algebra.

Moreover in the reliability calculus only two nodes are of interest, the input node 'i' and the output node 'j', and therefore only one element of the matrix **B**, b_{ij}. Therefore, after the Max-plus problem is transformed into an ELCP problem, see (De Schutter and De Moor, 1996), the solutions' set, if it exists, can be determined with the ELCP algorithm, see (De Schutter and De Moor, 1995).

Remark: The general ELCP problem is an NP-hard problem.

5. RESIDUATION AND THE MAX-PLUS ALGEBRA

Consider a complete dioid (e.g. the ones presented in Section 2). Starting from these scalar dioid, S, consider square nxn matrices with entries in S. The sum and the product of matrices are defined conventionally after the sum and the product of scalars in S. The set of nxn matrices endowed with these two operations is also a dioid which is denoted S^{nxn} .The zero element, **ε**, is the matrix with all the elements equal to ε and the identity element, **e**, is the matrix with all the elements equal to ε except the ones on the diagonal which are equal to e.

Remark: If the scalar dioid is complete, the matrix dioid defined as above is complete too.

Consider a mapping from a complete matrix dioid S, into itself defined as in (20),

$$f(\mathbf{X}) = \mathbf{X} \oplus \mathbf{X}^2 \oplus ... \oplus \mathbf{X}^{n-1} \quad (20)$$

and the equation

$$f(\mathbf{X}) = \mathbf{B} \quad (21)$$

The equation (21) does not have always solutions. In this case and in the reliability context it is interesting to find a set of 'supersolutions'

$$\mathbf{X} \mid f(\mathbf{X}) \geq \mathbf{B} \quad (22)$$

(inequality (22) holds componentwise) and to take the lower bound of this set (the minimum effort solution which ensures the imposed lower limit of the system's reliability).

The mapping f is isotone,

$$\forall \mathbf{X}_1, \mathbf{X}_2 \in S, \mathbf{X}_1 \geq \mathbf{X}_2 \Rightarrow f(\mathbf{X}_1) \geq f(\mathbf{X}_2) \quad (23)$$

the image of the top element through 'f' is the top element,

$$f(\mathbf{T}) = \mathbf{T} \quad (24)$$

where **T** are obvious extensions of the scalar case, and 'f' is upper- semicontinuous, that is for every (finite or infinite) subset S_1 of S,

$$f(\bigwedge_{\mathbf{X} \in S_1} \mathbf{X}) = \bigwedge_{\mathbf{X} \in S_1} f(\mathbf{X}) \quad (25)$$

where '$\wedge$' is the lower bound operator.

From (23), (24) and (25) results, see (Baccelli *et al*.. 1992), that the set of supersolutions is not empty and that it has a (unique) minimum element, **X**, which guarantees (22).

6. CONCLUSION

An attempt is made in this paper to use the Max-plus algebra to compute, analyze and enhance (optimize) the reliability of a system. First the structure function is determined and a lower bound for the reliability of the system is established. Then it is shown how the reliability of all the elements of the system can be computed in order to ensure an imposed lower bound of the reliability of the system. Finally, some considerations are made about using the residuation theory for calculating the lowest reliabilities of the elements of the system, which ensure for the system a greater reliability than an imposed value.

REFERENCES

Baccelli, F., G. Cohen, G.J. Olsder and J.P. Quadrat (1992). *Syncronization and Linearity*. John Wiley & Sons, New York.

Cuninghame-Green, R.A. (1979). *Minimax Algebra*. Springer-Verlag, Berlin.

Catuneanu, V. (1983). *Theoretical Basis of Reliability*. Editura Academiei, Bucuresti, Romania.

De Schutter, B. and B. De Moor (1995). The Extended Linear Complementarity Problem. *Mathemathical Programming*, **71**, 289-325.

De Schutter, B. and B. de Moor (1996). A method to find all solutions of a system of multivariate polynomial equalities and inequalities in the max algebra. *Discrete Event Dynamic Systems:Theory and Applications*, **6**, 115-138.

Ganciu, T. and C. Tugurlan (2002). *Statistics and Reliability*. Editura Gh. Asachi, Iasi, Romania.