EFFICIENT OFF-LINE SOLUTIONS TO ROBUST MODEL PREDICTIVE CONTROL USING ORTHOGONAL PARTITIONING

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Abstract: The main limitation of many robust model predictive control (MPC) schemes is the formidable real-time computational complexity. In this paper, a new algorithm for computing efficient approximate solutions to the min-max MPC problem for discrete-time polytopic systems is proposed. It is shown that the resulting control profile is, in fact, piecewise affine (PWA) defined on an orthogonal partition of the state space. This explicit structure is exploited for efficient real-time implementation via binary search trees. Conditions for robust exponential stability of the closed-loop system can be derived in terms of linear matrix inequality (LMI) constraints.

Keywords: Model predictive control, Robust stability, Binary search trees, Linear matrix inequalities.

1. INTRODUCTION

Practical difficulties associated with the implementation of stabilizing robust model predictive control (MPC) laws are well known. The algorithms typically rely on the solution of a min-max optimization problem (Mayne, *et al.*, 2000) in which the worst-case performance cost is minimized over the control input while satisfying input and state constraints. The requirement to solve the min-max problem on-line greatly restricts the MPC application range to systems with relatively slow dynamics or high-performance computers.

Despite the complex nature of the problem, several different approaches to reduce the computational complexity of robust MPC have been proposed. For a linear cost function and parametric uncertainty Bemporad, *et al.*, (2003) show that solutions to min-max control problems can be pre-computed off-line in an explicit piecewise affine (PWA) state feedback form defined on a polyhedral partition of the state space. The advantage of the formulation is that the real-time computation simply reduces to a function evaluation problem. For quadratic cost functions and parametric uncertainty, explicit feedback solutions are, in general, not available (Lee and Yu, 1996). On the other hand, (Wan and Kothare, 2003) develop an approximate algorithm, that is based on the earlier work (Kothare, *et al.*, 1996), in which a sequence of explicit state feedback laws associated with invariant ellipsoidal regions of attraction is obtained. Although the approach appears attractive, it often suffers excessive conservativeness. To resolve the problem Ding, *et al.*, (2004) include a sequence of N free control moves separately from the feedback law, therefore enabling a balance between computational burden and reduction of conservativeness to be achieved. However, there is no technique to compute an efficient off-line solution to this problem.

The contribution of this paper is to propose a novel approach for computing an approximate explicit solution to the problem studied in (Ding, *et al.*, 2004). The presented technique can be viewed as a direct extension of approximate explicit solutions recently developed for linear constrained systems (Johansen and Grancharova, 2003) and nonlinear systems (Johansen, 2004). The proposed algorithm consists of two independent parts. First, an orthogonal state space partition is computed along with the associated approximate PWA feedback law.

Second, the resulting partition is tested for robust stability. The analysis is based on the computation of globally quadratic Lyapunov functions, the existence of which, guarantee robust exponential closed-loop stability. It will be shown that the real-time computational effort required for the implementation of the approximate controller can be reduced to a simple search in a finite dimensional tree.

The following notation is used: for a vector $x \in \mathbb{R}^n$ and positive definite matrix Q the weighted norm $\| x\|_Q^2$ is denoted by $x^T Q x$. $x_{k+i|k}$ is the value of a vector x at a future time $k+i$ predicted at time k. The symbol * induces a symmetric structure, e.g., when H and R are symmetric matrices, then

$$\begin{bmatrix} H+S+* & * \\ T & R \end{bmatrix} \triangleq \begin{bmatrix} H+S+S^T & T^T \\ T & R \end{bmatrix}.$$

2. PROBLEM FORMULATION

Consider the following time varying and/or uncertain model

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k u_k \\ [A_k, B_k] &\in \Omega \end{aligned} \quad (1)$$

where $k \geqslant 0$; $u \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$ are the input and the measurable state respectively. Also $A_k \in \mathbb{R}^{n\times n}$, $B_k \in \mathbb{R}^{n\times m}$ and (A_k, B_k) is a controllable pair. For a polytopic uncertainty description, Ω is a polytope

$$\Omega \triangleq Co\{[A^1, B^1], [A^2, B^2], ..., [A^L, B^L]\}, \quad (2)$$

where $Co\{\}$ denotes convex hull and $[A^l, B^l]$ $\forall l \in \mathcal{L} \triangleq \{1,...,L\}$ are vertices of the convex hull. For the current state $x_k = x_{k|k}$, a typical robust constrained MPC algorithm (Mayne, *et al.*, 2000) solves the following min-max optimization problem:

$$\min_{\mathcal{U}} \max_{[A_{k+i}, B_{k+i}]\in\Omega} \sum_{i=0}^{N-1} [\|x_{k+i|k}\|_{\mathcal{Q}}^2 + \|u_{k+i|k}\|_{\mathcal{R}}^2] + \|x_{k+N|k}\|_{\mathcal{P}_k}^2 \quad (3)$$

subject to

$$u_{\min} \leqslant u_{k+i|k} \leqslant u_{\max}, \quad \forall i \geqslant 0, \quad (4)$$

$$x_{\min} \leqslant \Psi x_{k+i|k} \leqslant x_{\max}, \quad \forall i \geqslant 1, \quad (5)$$

$$x_{k+i|k} \in \mathcal{T}_k, \quad \forall i \geqslant N, \quad (6)$$

$$u_{k+i|k} = \mathcal{F}_k x_{k+i|k}, \quad \forall i \geqslant N, \quad (7)$$

$$x_{k+i+1|k} = A_{k+i} x_{k+i|k} + B_{k+i} u_{k+i|k}, \quad \forall i \geqslant 0. \quad (8)$$

Here, $\mathcal{U} \triangleq [u_{k|k}^T, u_{k+1|k}^T, ..., u_{k+N-1|k}^T]^T$ is the vector of control moves, $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{P}_k$ are symmetric positive definite weighting matrices and $N > 0$ denotes the control horizon. The set $\mathcal{T}_k$ in (6) is typically chosen to be control invariant (Blanchini, 1999) with respect to $\mathcal{F}_k$ in the specified polytopic family (2). The following assumption is in order:

(A1) $u_{\min} < 0 < u_{\max}$ and $x_{\min} < 0 < x_{\max}$.

As shown in (Ding *et al.*, 2004), the robust MPC problem (3)-(8) can be formulated as a semi-definite programming (SDP) problem involving LMI constraints. To see this, consider the following parameterization of the cost index:

$$\gamma_1 \geqslant \|\mathcal{A}x_k + \mathcal{B}\mathcal{U}\|_{\tilde{\mathcal{Q}}}^2 + \|\mathcal{U}\|_{\tilde{\mathcal{R}}}^2, \quad \forall[\mathcal{A}, \mathcal{B}] \in \bar{\Omega}, \quad (9)$$

$$\gamma_2 \geqslant \|\mathcal{A}_N x_k + \mathcal{B}_N \mathcal{U}\|_{\mathcal{P}_k}^2, \quad \forall[\mathcal{A}_N, \mathcal{B}_N] \in \bar{\Omega}_N, \quad (10)$$

where $\gamma_1 > 0$, $\gamma_2 > 0$, and the matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{A}_N$, $\mathcal{B}_N$, $\tilde{\mathcal{Q}}$ and $\tilde{\mathcal{R}}$ are easily obtained from (1)-(2), $\mathcal{Q}$ and $\mathcal{R}$. The sets $\bar{\Omega}$ and $\bar{\Omega}_N$ can be constructed as in (Ding *et al.*, 2004) and shall satisfy the following assumption:

(A2) $\bar{\Omega}$ and $\bar{\Omega}_N$ are convex and compact.

Suppose now that the assumptions (A1)-(A2) hold and the cost function is parameterized according to (9), (10). The following lemma is stated without a proof (see Ding *et al.*, 2004 for a complete description).

Lemma 1. (On-line robust MPC) *Consider the uncertain system* (1) *with input and state constraints* (4)-(5). *The min-max optimization problem* (3) *can be formulated as the following SDP problem:*

$$\min_{\gamma_1, \gamma_2, \mathcal{U}, \mathcal{Q}, Y, G} \|x_k\|_{\mathcal{Q}}^2 + \gamma_1 + \gamma_2, \text{ subject to} \quad (11)$$

$$\mathcal{G}\mathcal{U} \leqslant \mathcal{W} + \mathcal{E}x_k, \quad (12)$$

$$\begin{bmatrix} Z & * \\ Y^T & G+G^T-Q_l \end{bmatrix} \geqslant 0, \; Z_{jj} \leqslant z_j^2, \quad \forall l \in \mathcal{L}, \quad (13)$$

$$\begin{bmatrix} G+G^T-Q_l & * \\ \Psi(A^l G + B^l Y) & \Gamma \end{bmatrix} \geqslant 0, \; \Gamma_{ss} \leqslant x_s^2, \; \forall l \in \mathcal{L}, \quad (14)$$

$$\begin{bmatrix} 1 & * \\ \mathcal{A}_N x_k + \mathcal{B}_N \mathcal{U} & Q_l \end{bmatrix} \geqslant 0, \quad \forall[\mathcal{A}_N, \mathcal{B}_N] \in \bar{\Omega}_N, \quad (15)$$

$$\begin{bmatrix} \gamma_1 & * & * \\ \tilde{\mathcal{Q}}^{1/2}(\mathcal{A}x_k + \mathcal{B}\mathcal{U}) & I & * \\ \tilde{\mathcal{R}}^{1/2}\mathcal{U} & 0 & I \end{bmatrix} \geqslant 0, \; \forall[\mathcal{A}, \mathcal{B}] \in \bar{\Omega}, \quad (16)$$

$$\begin{bmatrix} G+G^T-Q_l & * & * & * \\ A^l G + B^l Y & Q_t & * & * \\ \mathcal{Q}^{1/2} G & 0 & \gamma_2 I & * \\ \mathcal{R}^{1/2} Y & 0 & 0 & \gamma_2 I \end{bmatrix} \geqslant 0, \; \forall l, t \in \mathcal{L}, \quad (17)$$

where

$Q_l \triangleq \gamma_2 \mathcal{P}_l^{-1}$, $Q_t \triangleq \gamma_2 \mathcal{P}_t^{-1}$, $\forall l, t \in \mathcal{L}$, $\mathcal{F}_k \triangleq YG^{-1}$, $z_j = \min\{|u_{\min,j}|, u_{\max,j}\}$, $x_s = \min\{|x_{\min,s}|, x_{\max,s}\}$ *and* Z_{jj} (Γ_{ss}) *is the jth (sth) diagonal element of* Z

(Γ). *The matrices* $\mathcal{G}$, $\mathcal{W}$ *and* $\mathcal{E}$ *can be easily obtained from* (4)-(5).

Certain properties follow immediately from the SDP formulation in Lemma 1 (Ding, *et al.*, 2004):

Property 1. *Any feasible solution of the optimization problem* (11) *at time k is also feasible for all times* $t > k$.

Property 2. *The set* $\mathcal{T}_k \triangleq \{x \in \mathbb{R}^n | x^T \mathcal{P}_k x \leqslant \gamma_2\}$ *is a robust positively invariant ellipsoid for the system* (1) *in feedback with the controller* (7).

Property 3. *The on-line implementation of the min-max MPC algorithm guarantees exponential closed-loop stability, once a feasible solution is found.*

Despite the fact that SDP problems can be solved in polynomial time using interior point algorithms, the computation effort required to solve the robust MPC problem (11) on-line can be quite prohibitive for many real-time applications. In the following, a new technique to obtain controllers of significantly lower computational complexity will be presented. The proposed algorithm *off-line* determines a sequence of approximate feedback laws defined on the state space partition of hyper-rectangles. Subsequently, the constructed partition as well as the associated local controllers will be analyzed for robust stability.

3. OFF-LINE APPROXIMATE ROBUST MPC

3.1 Local Feasible Controller

Suppose the set $X \subset \mathbb{R}^n$ of feasible initial states can be decomposed into n-dimensional boxes or hyper-rectangles $\Theta(R) \triangleq \{R_r\}_{r\in\mathcal{I}} \subseteq X$, given by polyhedra of the form

$$R_r \triangleq \{x \in \mathbb{R}^n | H_r x \leqslant d_r\},$$
$$H_r = \begin{bmatrix} I \\ -I \end{bmatrix},\ d_r = \begin{bmatrix} h^u \\ -h^l \end{bmatrix},\ \forall r \in \mathcal{I}, \tag{18}$$

where $I \in \mathbb{R}^{n\times n}$ denotes the identity matrix and the lower and upper limits h^l and h^u are real n-vectors satisfying $h^l < h^u$ in element wise. The index set of boxes is denoted $\mathcal{I}$. Moreover, let $\mathcal{V} \triangleq \{\upsilon_1,...,\upsilon_M\}$ represent a set of M vertices of R_r. The set of all explored regions $\Theta(R)$ will henceforth be referred to as a partition. In each region of the partition $\Theta(R)$ a local affine feedback controller

$$\tilde{\mathcal{U}}(x) \triangleq \tilde{F}x + \tilde{g},\ \ \tilde{F} \in \mathbb{R}^{n\times mN},\ \ \tilde{g} \in \mathbb{R}^{mN}, \tag{19}$$

is defined. Notice that (19) implies that $\tilde{\mathcal{U}} : X' \to \mathbb{R}^{mN}$ is a PWA function restricted to the set $X' \triangleq \bigcup_{r\in\mathcal{I}} R_r$ which is the union of all regions in $\Theta(R)$. The feedback parameters $\tilde{F}$ and $\tilde{g}$ can be computed as suggested in (Bemporad and Filippi, 2001), by considering the optimal solutions to the min-max optimization problem in Lemma 1 at vertices of a hyper-rectangle only. Consider any given region $R \subseteq X$ with vertices $\mathcal{V} = \{\upsilon_1,...,\upsilon_M\}$, and let $\mathcal{U}_h^{o}$ denote the optimal control sequence computed at υ_h. Suppose the feedback parameters $\tilde{F}$ and $\tilde{g}$ satisfy the optimization problem:

$$\min_{\tilde{F},\tilde{g}} \sum_{h=1}^{M} \| \mathcal{U}_h^{o} - \tilde{F}\upsilon_h - \tilde{g} \|_{\mathcal{H}_0}^2,\ \mathcal{H}_0 > 0, \tag{20}$$

subject to $\forall \upsilon_h \in \mathcal{V}$

$$\mathcal{G}(\tilde{F}\upsilon_h + \tilde{g}) \leqslant \mathcal{W} + \mathcal{E}\upsilon_h. \tag{21}$$

Lemma 2. (Feasible approximate controller). *The least squares approximate solution* $\tilde{\mathcal{U}}(x) = \tilde{F}x + \tilde{g}$, *is robustly feasible for all* $x \in R$ *and all uncertainty realizations* $[A_k, B_k] \in \Omega$.

Proof. Follows directly from convexity. □

The accuracy of the approximation will be measured by the difference between the optimal and approximate solutions restricted to a region R, i.e.

$$\varepsilon = \max_h \| \mathcal{U}_h^{o} - \tilde{F}\upsilon_h - \tilde{g} \|_{\mathcal{H}}^2,\ \ \forall \upsilon_h \in \mathcal{V}, \tag{22}$$

where $\mathcal{H} \geqslant 0$ is a weighting matrix typically having non-zero elements only on the first m components of the solution (Bemporad and Filippi, 2001). Notice that satisfying the error bound (22) at the vertices does not necessarily imply that the bound will be satisfied for all $x \in R$ (Grancharova and Johansen, 2002). One heuristic approach is to include some interior points of R and consider the following estimate:

$$\hat{\varepsilon} = \max_{\upsilon_h \in \bar{\mathcal{V}}} \varepsilon, \tag{23}$$

where the set $\bar{\mathcal{V}} \triangleq \{\upsilon_1, \upsilon_2, ..., \upsilon_M, ..., \upsilon_{\bar{M}}\}$ $\forall \bar{M} \geqslant M$ contains, in addition to $\mathcal{V}$, a finite number of arbitrary points in R. Moreover, it is assumed that for all regions in the partition, the error bound (23) should respect the following tolerance:

$$\bar{\varepsilon} = \max\{\bar{\varepsilon}_a, \bar{\varepsilon}_r \min_h \| \mathcal{U}_h^{o} \|_{\mathcal{H}}^2\}, \tag{24}$$

where $\bar{\varepsilon}_a > 0$ and $\bar{\varepsilon}_r > 0$ can be interpreted as absolute and relative tolerances respectively.

3.2 Exploration Algorithm

An immediate consequence of enforcing an orthogonal structure (18) on the state space partition is that the partition can be organized as a multidimensional binary search tree, or *quad-tree* (de Berg, *et al*., 2000), yielding a search complexity that is logarithmic in the number of regions. With d levels of search in the tree, the computation required to determine an active region (the one that contains a given state) reduces, in the worst-case, to a total of $n \times d$ scalar comparisons leading to an extremely fast real-time implementation.

The objective of the algorithm presented in this section is to compute, off-line, a partition of boxes defined on a feasible set $X \subset \mathbb{R}^n$ along with the associated approximate local controllers $\tilde{\mathcal{U}}_r(x)$ such that the approximation accuracy satisfies the tolerance bound (24).

Let the initial region $R_0 \supseteq X$ be a minimal bounding box containing the set *X*. Following Johansen and Grancharova (2003), the off-line algorithm for computing the approximate solution to the min-max optimization problem (11) is presented:

Algorithm 1. (Off-line approximate robust MPC).

1. *Let the set of all unexplored boxes be denoted as P. Initialize the partition to the region* $R_0 \supseteq X$ *i.e.* $P = \{R_0\}$.

2. *Select any unexplored region* $R_1 \in P$. *If* $P \in \varnothing$ *then the algorithm terminates successfully.*

3. *Substitute* x_k *in* (11), (12), (15) *and* (16) *by* v_h $\forall h \in \{1,2,\ldots,\bar{M}\}$ *and solve* (11) *to obtain the set of optimizers* $\{\mathcal{U}_1^{\text{o}},\ldots,\mathcal{U}_{\bar{M}}^{\text{o}}\}$. *If all solutions are feasible, go to step* 4. *Otherwise, compute the largest Euclidean distance between any pair of vertices of* R_1. *If it is smaller than some tolerance, mark* R_1 *explored and infeasible and go to step* 2. *Otherwise, go to step* 6.

4. *Compute an approximation* $\tilde{\mathcal{U}}(x)$ *using* (20). *If a feasible solution is not found, go to step* 6.

5. *Compute the error in the solution* $\hat{\varepsilon}$ *using* (22), (23). *If* $\hat{\varepsilon} \leqslant \bar{\varepsilon}$, *add the region* R_1 *to* $\Theta(R)$ *and go to step* 2.

6. *Partition* R_1 *into* 2^n *equal hyper-rectangles* $R_2,\ldots,R_{2^n+1}$. *Remove* R_1 *from P and add* $R_2,\ldots,R_{2^n+1}$. *Go to step* 2. □

The algorithm terminates after a finite number of steps with the piecewise approximation $\tilde{\mathcal{U}}(x)$ and a partition $\Theta(R)$ inside of which this approximation is valid. The finite-time termination of the algorithm is, in general, not guaranteed, though in extensive simulations the termination was always attained.

4. STABILITY

4.1 Computing a partition around the origin

Since the approximate feedback controller proposed in this paper does not directly inherit the robust stability properties of the on-line LMI-based controller (Lemma 1), *a posteriori* stability analysis is required to ensure that the feedback is also robustly stabilizing. In addition, any nonzero tolerance $\bar{\varepsilon}$ imposed on the approximation error renders the asymptotic convergence to the origin impossible. This is to be expected as no additional conditions have been imposed on the approximate solution of Algorithm 1 to enforce asymptotic stability in a close neighborhood of the origin.
In order to make the origin robustly stable, it is required that in all boxes containing the origin a local stabilizing feedback gain matrix, denoted F_0, is utilized. The matrix F_0 may be determined from the infinite-time solution of the robust MPC problem (11) computed for $N = 0$ where x_k is properly chosen. It can be shown (Kothare, *et al.*, 1996) that $u_k = F_0 x_k$, $\forall k \geqslant 0$ is a stabilizing local feedback controller for the uncertain system (1) with the region of attraction being an invariant ellipsoid.
Given a partition $\Theta(R)$ generated by Algorithm 1, the goal of the procedure presented here is to extract from regions containing the origin a subset $\Theta_0 \triangleq \{R_r\}_{r\in\mathcal{I}_0} \subseteq \Theta(R)$ such that $u_k = F_0 x_k$, $\forall k \geqslant 0$ is optimal everywhere in Θ_0 (Johansen and Grancharova, 2003). Henceforth, the index set of boxes belonging to Θ_0 will be denoted $\mathcal{I}_0$. An immediate shortcoming of this approach is that the regions for which the feedback matrix F_0 is optimal tend to be excessively small. Consequently, this may significantly limit the set of states steerable to the origin for which the off-line approximate robust MPC is feasible and closed-loop stable. A heuristic solution to this problem may involve enlarging the regions contained in Θ_0 until stability is recovered.

4.2 Stability of off-line approximate robust MPC

In order to establish stability of the approximate robust MPC algorithm, various classes of Lyapunov functions may be considered (see Ferrari-Trecate, *et al.*, 2002 for an excellent review). In this paper, the attention is restricted to the following common quadratic Lyapunov functions (Grieder, *et al.*, 2003):

$$V(x) \triangleq x^T P_c x, \ \ \forall x \in \Theta(R), \tag{25}$$

with $P_c \in \mathbb{R}^{n\times n} > 0$. Suppose the following robust stability condition is satisfied for all boxes in $\Theta(R)$ and all uncertainty realizations $l \in \mathcal{L}$

$$V(x_{k+1}) - V(x_k) \leqslant -\rho\|x_k\|^2, \qquad \forall k \geqslant 0, \tag{26}$$

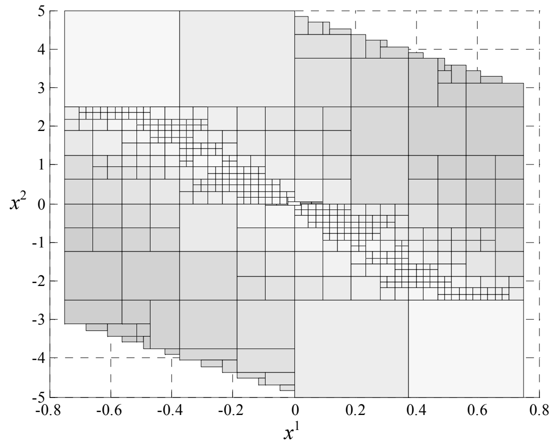

Fig. 1. Quad-tree partition of the approximate robust MPC.

where the scalar $\rho > 0$ is introduced to enforce exponential stability. This condition leads to the so-called quadratic Lyapunov stability widely studied in the past (Ferrari-Trecate, *et al.*, 2002). The requirement (26) can be reformulated along the lines of (Grieder, *et al.*, 2003) to obtain a description suitable for LMI solvers. To see this, define the variation of the Lyapunov function associated with region r as follows

$$\Delta V_r^l(x) = x^T \Delta Q_r^l x + 2x^T \Delta l_r^l + \Delta c_r^l, \tag{27}$$

where

$$\begin{aligned}\Delta Q_r^l &= (A^l + B^l \tilde{F}_r^0)^T P_c (A^l + B^l \tilde{F}_r^0) - P_c,\\ \Delta l_r^l &= (A^l + B^l \tilde{F}_r^0)^T P_c B^l \tilde{g}_r^0,\\ \Delta c_r^l &= (B^l \tilde{g}_r^0)^T P_c B^l \tilde{g}_r^0.\end{aligned} \tag{28}$$

In (28), $\tilde{F}_r^0$ and $\tilde{g}_r^0$ denote the first m components of the approximate affine feedback law associated with a region r. The following lemma is stated without a proof (see Grieder, *et al.*, 2003 for a complete description).

Lemma 3. *There exists a quadratic function* (25) *meeting the requirement* (26) *if for all* $r \in \mathcal{I}$ *and* $l \in \mathcal{L}$ *there exist symmetric matrices* $N_r^l \geqslant 0$ *with a proper dimension and a nonnegative scalar* ρ *satisfying the following LMI:*

$$\begin{bmatrix}(H_r)^T N_r^l H_r + \Delta Q_r^l + \rho I & * \\ -d_r N_r^l H_r + (\Delta l_r^l)^T & \Delta c_r^l\end{bmatrix} \leqslant 0. \tag{29}$$

Theorem 1. *If there exists a quadratic function* (25) *such that* (29) *is satisfied, then the off-line approximate robust MPC guarantees exponential closed-loop stability.*
Proof. See (Grieder, *et al.*, 2003). □

It should be noted that due the conservative nature of quadratic stability analysis, the numerical procedure of Lemma 3 might not lead to a Lyapunov function in all cases, even if the closed-loop system is stable.

Table 1 Characteristics of the approximate solution

$\bar{\varepsilon}_r$	Number of regions	Average error	Maximum error
0.5	271	0.0179	0.1963
0.2	358	0.0074	0.1071
0.1	466	0.0035	0.0531
0.05	586	0.0016	0.0306
0.02	865	0.0009	0.0175
0.01	976	0.0005	0.0117

Alternatively, a class of smooth non-quadratic Lyapunov functions (Johansen, 2000) may be exploited to increase the likelihood of successful Lyapunov function identification, at the expense of greater computational complexity.

5. NUMERICAL EXAMPLE

Consider the following uncertain system (Ding, et al., 2004):

$$x_{k+1} \triangleq \begin{bmatrix} x_{k+1}^1 \\ x_{k+1}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \beta_k & 1 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_k,$$

where $\beta_k \in [0.5, 2.5]$ is a time varying parameter. The system is required to satisfy the input and state constraints

$$-1 \leqslant u_k \leqslant 1, \qquad \forall k \geqslant 0,$$
$$\begin{bmatrix} -0.5 \\ -5 \end{bmatrix} \leqslant \begin{bmatrix} x_k^1 \\ x_k^2 \end{bmatrix} \leqslant \begin{bmatrix} 0.5 \\ 5 \end{bmatrix}, \qquad \forall k \geqslant 1.$$

The weighting matrices are selected as $Q = I$ and $\mathcal{R} = 1$ and the control horizon is $N = 4$. The initial box is defined by $R_0 = [-0.75, 0.75] \times [-5, 5]$ and the region size is restricted to be larger than $\Delta x = 0.02$. The tolerance on the approximation error is chosen according to (24) with $\bar{\varepsilon}_a = 0.0001$ and $\bar{\varepsilon}_r = 0.1$.

The off-line solution computed with Algorithm 1 is depicted in Fig. 1 and consists of 466 hyper-rectangles and 7 levels of search. The computational complexity of the approximate approach consists, in the worst-case, of a total of 18 arithmetic operations per sample (14 comparisons, 2 multiplications and 2 additions). For comparison, on a Pentium IV machine (1.8 GHz and total memory 500 MB) the average time for the on-line robust MPC algorithm (11) to compute a solution is 0.9 s, which indicates that millions of arithmetic operations are required in real-time to solve the LMI optimization problem.

Given an initially disturbed state $x_0 = [0.75, 1.5]^T$ and assuming $\beta_k = 1.5 + \sin(k)$, the state and input trajectories for the on-line (Lemma 1) and off-line algorithms are shown in Fig. 2 and Fig. 3 respectively. Notice that the approximate min-max MPC controller achieves nearly the same performance as its exact (on-line) counterpart.

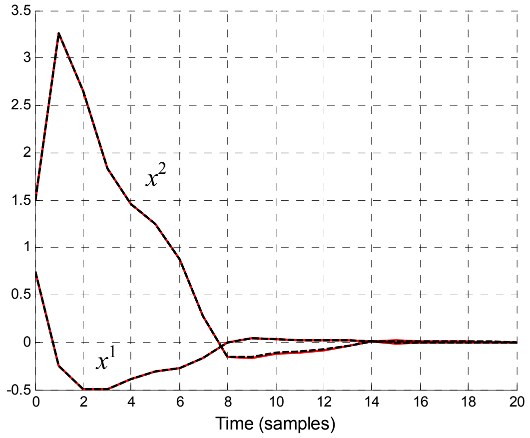

Fig. 2. The state trajectories for on-line (dashed) and off-line (solid) solutions.

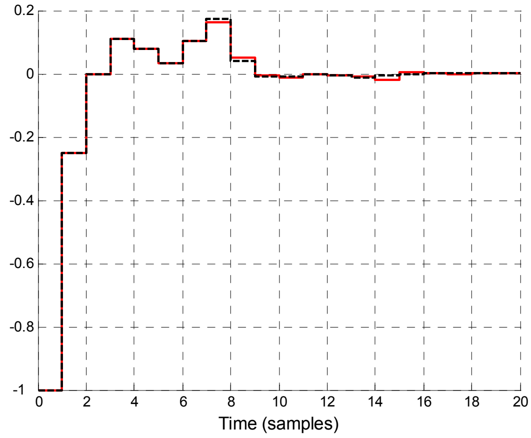

Fig. 3. The input trajectories for on-line (dashed) and off-line (solid) solutions.

It can also be observed that the proposed controller keeps the input and state evolutions within the constraints despite the time varying uncertainty. The accuracy of the approximation is validated by computing the absolute error between the first components of the off-line and on-line solutions, based on simulations for 2525 initial states. Table I reports how the average and maximum values of this error depend on the relative tolerance $\bar{\varepsilon}_r$.

The approximate control law is robustly stabilizing as the LMI problem in Lemma 3 provides a common Lyapunov function $P_c = \begin{bmatrix} 0.8974 & 0.2923 \\ 0.2923 & 0.1673 \end{bmatrix}$ and a decay rate of $\rho = 0.0057$.

6. CONCLUSIONS

A new off-line algorithm to address min-max model predictive control of systems with parametric uncertainty is proposed. It is shown that the approximate solution to this problem can be pre-computed off-line in an explicit form as a PWA state feedback law defined on an orthogonal partition of the state space. The proposed algorithm allows computationally demanding constrained min-max optimization to be avoided by a simpler search in a finite dimensional tree. This makes the presented method an attractive alternative to the existing robust MPC schemes.

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