MPC FOR TRACKING OF PIECE-WISE CONSTANT REFERENCES FOR CONSTRAINED LINEAR SYSTEMS

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Abstract: Model predictive control (MPC) is one of the few techniques which is able to handle with constraints on both state and input of the plant. The admissible evolution and asymptotically convergence of the closed loop system is ensured by means of a suitable choice of the terminal cost and terminal contraint. However, most of the existing results on MPC are designed for a regulation problem. If the desired steady state changes, the MPC controller must be redesigned to guarantee the feasibility of the optimization problem, the admissible evolution as well as the asymptotic stability. In this paper a novel formulation of the MPC is proposed to track varying references. This controller ensures the feasibility of the optimization problem, constraint satisfaction and asymptotic evolution of the system to any admissible steady-state. Hence, the proposed MPC controller ensures the offset free tracking of any sequence of piece-wise constant admissible set points. Moreover this controller requires the solution of a single QP at each sample time, it is not a switching controller and improves the performance of the closed loop system.

Keywords: Predictive control, Tracking, Constrained linear systems, Invariants, Asymptotic stability.

1. INTRODUCTION

Model predictive control (MPC) is one of the few control techniques which is able to consider constraints (on both state and inputs of the system) in the design of the control law. This is achieved by predicting the evolution of the system and computing the admissible sequence of control inputs which makes the system evolve satisfying the constraints. This problem can be posed as an optimization problem. To obtain a feedback policy, the obtained sequence of control inputs is applied in a receding horizon manner, solving the optimization problem at each sample time.

Nowadays the theoretical foundation of MPC is well-known and under some assumptions, asymptotic stability is guaranteed. This is achieved by means of a suitable penalization of the terminal state and an additional terminal constraint (Mayne, Rawlings, Rao & Scokaert 2000).

Most of the results on MPC consider the regulation problem, i.e. steering the system to the desired steady state which is assumed to be the origin. The purpose of this paper is to design an MPC algorithm to track a sequence of admissible set points. It is clear that for

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a given non-zero set point, a suitable choice of the steady state can be chosen and the problem can be posed as a regulation problem translating the state and input of the system (Muske & Rawlings 1993). However, since the stabilizing choice of the terminal cost and constraints depends on the desired steady state, a changing reference requires to re-design the MPC at each change of the reference. The computational amount that the design of a stabilizing MPC requires may make this approach not viable.

Some results exist to solve the tracking problem for constrained linear system. The first remarkable approach is the so-called command governor (Gilbert, Kolmanovsky & Tan 1994), where a nonlinear low-pass filter of the reference is added to guarantee the admissible evolution of the system to the reference. This can be seen as adding an artificial reference (the output of the filter). This is computed at each sampling time to ensure the admissible evolution of the system, converging to the desired reference. In (Bemporad, Casavola & Mosca 1997) a command governor is designed to minimize a performance index of the predicted evolution of the system. In (Blanchini & Miani 2000) it is proved that any control invariant set for the constrained system is a tracking domain of attraction and an interpolation-based control law is proposed.

In (Bemporad et al. 1997, Rossiter & Kouvaritakis 1998) is shown that there exists similarities between predictive controllers and command governors: both compute the *control action* to guarantee the constraint satisfaction and the convergence to the reference. The main difference is how this *control action* is considered in the associated optimization problem. In (Chisci & Zappa 2003) a dual-mode strategy for tracking based on MPC is presented: if the MPC is not feasible, the controller switches to a feasibility recovery mode, which steers the system to the feasibility region of the MPC.

In this paper a novel MPC strategy for tracking is presented. In a similar way to the command governors, an artificial reference is considered as decision variable of the MPC. The feasibility is ensured by considering an invariant set for tracking as terminal constraint and the offset free control is ensured by penalizing the deviation of the artificial reference from the desired reference. The obtained optimization problem is a standard QP and the MPC control law is able to track any admissible reference, and hence, any piece-wise constant sequence of references.

Notation: vector (x,t,r) denotes $[x^T,t^T,r^T]^T$; for a given λ, $\lambda X = \{\lambda \cdot x : x \in X\}$; $int(X)$ denotes the interior of set X; a matrix T definite positive is denoted as $T > 0$ and $T > P$ denotes that $T - P > 0$. Consider $a \in \mathbb{R}^{n_a}$, $b \in \mathbb{R}^{n_b}$, and set $\Gamma \subset \mathbb{R}^{n_a+n_b}$, then projection operation is defined as $Proj_a(\Gamma) = \{a \in \mathbb{R}^{n_a} : \exists b \in \mathbb{R}^{n_b}, (a,b) \in \Gamma\}$.

2. PROBLEM DESCRIPTION

Let a discrete-time linear system be described by:

$$
\begin{aligned}
x^+ &= Ax + Bu \qquad (1)\\
y &= Cx + Du
\end{aligned}
$$

where $x \in \mathbb{R}^n$ is the current state of the system, $u \in \mathbb{R}^m$ is the current input, $y \in \mathbb{R}^p$ is the current otput and x^+ is the successor state. The state of the system and the control input applied at sampling time k are denoted as $x(k)$ and $u(k)$ respectively. The system is subject to hard constraints on state and control:

$$x(k) \in X, \quad u(k) \in U$$

for all $k \geq 0$. The sets X and U are compact convex polyhedra containing the origin in their interior. They are given by

$$X = \{x \in \mathbb{R}^n : A_x \cdot x \leq b_x\} \qquad (2)$$

$$U = \{u \in \mathbb{R}^m : A_u \cdot u \leq b_u\} \qquad (3)$$

The problem we consider is to track a piece-wise constant sequence of set points or references $s(k)$ in such a way that the constraints are satisfied for all the time. For this purpose we propose an MPC formulation which allows one to reach any admissible set point s with offset-free.

3. CALCULATION OF THE ADMISSIBLE STEADY STATES

Consider a set-point t and a steady state of the system $(\mathrm{x}_s, \mathrm{u}_s)$ associated to this set-point satisfying:

$$\begin{bmatrix} A-I & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} \mathrm{x}_s \\ \mathrm{u}_s \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix} \qquad (4)$$

Denote

$$E = \begin{bmatrix} A-I & B \\ C & D \end{bmatrix} \quad F = \begin{bmatrix} A-I & B & 0 \\ C & D & I_p \end{bmatrix}$$

where I_p is the identity matrix of order p. It is assumed that the rank of E is equal to the rank of F. This assures that the steady state equation (4) has a solution for any set-point t. Thus, any solution of (4) can be posed as

$$z_s = \begin{bmatrix} \mathrm{x}_s \\ \mathrm{u}_s \end{bmatrix} = M \cdot t + N \cdot r = \begin{bmatrix} M_x \\ M_u \end{bmatrix} \cdot t + \begin{bmatrix} N_x \\ N_u \end{bmatrix} \cdot r \qquad (5)$$

where r is an auxiliary variable and its size depends on the rank of E; notice that if E is full column rank, then the solution is given by $z_s = M \cdot t$. If rank of E is less than $n+m$, then vector r can be thought as free variables in the selection of the steady state and input for a desired set point t.

The set of all admissible steady states of the system, is denoted as X_s, i.e. $X_s = \{\mathrm{x}_s \in X : \exists \mathrm{u}_s \in U$ such that $(A-I)\cdot \mathrm{x}_s + B\cdot \mathrm{u}_s = 0\}$. Analogously, the set of admissible steady inputs is denoted as U_s . The set of all admissible set-points is denoted as S i.e. $S = \{s = C\cdot \mathrm{x}_s + D\cdot \mathrm{u}_s : \mathrm{x}_s \in X, \mathrm{u}_s \in U$ and $(A-I)\cdot \mathrm{x}_s + B\cdot \mathrm{u}_s = 0\}$.

4. CALCULATION OF AN INVARIANT SET FOR TRACKING

Assume that the following controller given by

$$u = K\cdot(x - \mathrm{x}_s) + \mathrm{u}_s, \tag{6}$$

where x_s and u_s is the steady state we want to reach, asymptotically stabilizes the closed loop system. It is well known that if the controller gain K is such that $A+BK$ has all its eigenvalues inside the unit circle, then the system is steered to the desired steady state. Since the system is constrained, this controller leads to an admissible evolution of the system only in a neighborhood of the origin.

Substituting (5) in (6), matrices L_t and L_r can be found such that

$$u = K\cdot x + L_t\cdot t + L_r\cdot r \tag{7}$$

Consider the extended state $w = (x,t,r)$, then the closed loop system can be posed as

$$\begin{bmatrix} x \\ t \\ r \end{bmatrix}^+ = \begin{bmatrix} A+BK & BL_t & BL_r \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \cdot \begin{bmatrix} x \\ t \\ r \end{bmatrix} \tag{8}$$

that is, $w^+ = A_w\cdot w$.

Because of reasons that will be clearer later, define set $W_\lambda = \{w = (x,t,r) : u = K\cdot x + L_t\cdot t + L_r\cdot r \in U, x \in X, x_s = M_x\cdot t + N_x\cdot r \in \lambda X$ and $u_s = M_u\cdot t + N_u\cdot r \in \lambda U\}$; this set is a polyhedron given by

$$\begin{bmatrix} A_x & 0 & 0 \\ A_uK & A_u\cdot L_t & A_u\cdot L_r \\ 0 & A_x\cdot M_x & A_x\cdot N_x \\ 0 & A_u\cdot M_u & A_u\cdot N_u \end{bmatrix} \cdot \begin{bmatrix} x \\ t \\ r \end{bmatrix} \leq \begin{bmatrix} b_x \\ b_u \\ \lambda\cdot b_x \\ \lambda\cdot b_u \end{bmatrix} \tag{9}$$

It is clear that the set of constraints for system (8) is $W = W_1$. We say that a set Ω^w is an admissible invariant set for tracking, for system (8) constrained to W, if for all $w \in \Omega^w$, then $A_w\cdot w \in \Omega^w$ and $\Omega^w \subseteq W$. The maximal admissible invariant set for tracking is given by (Gilbert & Tan 1991):

$$O_\infty^w = \{w : A_w^i w \in W, \forall i \geq 0\}$$

This set might be not finitely determined by a finite set of constraints.

Consider the maximal admissible invariant set for tracking considering W_λ as constraint set, which is given by

$$O_{\infty,\lambda}^w = \{w : A_w^i w \in W_\lambda, \forall i \geq 0\}$$

Following similar arguments to (Gilbert et al. 1994), it can be shown that for any $\lambda \in (0,1)$, $O_{\infty,\lambda}^w$ is finitely determined and $\lambda O_\infty^w \subset O_{\infty,\lambda}^w \subset O_\infty^w$. Notice that since λ can be chosen arbitrarily close to 1, the obtained invariant set is arbitrarily close to the real maximal invariant set O_∞^w.

In what follows, superscript w denotes that set Ω^w is defined in the extended state, while no superscript denotes that set Ω is defined in the state vector space x, i.e. $\Omega = Proj_x(\Omega^w)$.

5. MPC FOR TRACKING

Assume that a stabilizing gain controller K for the system (1) is computed and an admissible invariant set for tracking X_f^w is obtained. Based on these, we present an MPC formulation which guarantees offset-free tracking to any admissible steady state $\hat{\mathrm{x}}_s$ contained in X_f.

In a similar way to the command governors, the proposed MPC considers an artificial reference given by $(\mathrm{x}_s, \mathrm{u}_s)$ as decision variable of the associated cost. Moreover, the deviation between the artificial steady state x_s and the desired steady state $\hat{\mathrm{x}}_s$ is penalized. If $\hat{\mathrm{x}}_s$ is an admissible steady state, then this penalization guarantees offset-free tracking; however if $\hat{\mathrm{x}}_s$ is not admissible, this penalization makes the system evolve to an admissible steady state such that its deviation with the desired (although unreachable) steady state is minimized.

The proposed cost is

$$V_N(x, s, \mathbf{u}, t, r) = \sum_{i=0}^{N-1} \left(\|x(i) - \mathrm{x}_s\|_Q^2 + \|u(i) - \mathrm{u}_s\|_R^2 \right) + \|x(N) - \mathrm{x}_s\|_P^2 + \|\mathrm{x}_s - \hat{\mathrm{x}}_s\|_T^2$$

where x is the current state, s is desired set point to be tracked, $\mathbf{u}$ is a sequence of N future control inputs, $x(i)$ is the predicted state of the system at time i given by $x(i+1) = A\cdot x(i) + B\cdot u(i)$, with $x(0) = x$, $x_s = M_x\cdot t + N_x\cdot r$ and $u_s = M_u\cdot t + N_u\cdot r$. For a given desired set point s, an associated steady state $\hat{\mathrm{x}}_s$ is obtained by means of a linear mapping $\hat{\mathrm{x}}_s = H_x\cdot s$ in such a way that some performance index is minimized. Matrices Q, R, P and T are assumed to be definite positive.

Therefore, $\mathbf{u}$,t, and r are the decision variables and x and s are the parameters of the proposed cost function. Note that this cost can be posed as a quadratic function of the decision variables. The MPC optimization problem $P_N(x,s)$ is given by

$$
\begin{aligned}
V_N^*(x,s) \;=\; & \min_{\mathbf{u},t,r} V_N(x,s,\mathbf{u},t,r) \\
& s.t.\; x(0) = x. \\
& \quad x(j+1) = A{\cdot}x(j) + B{\cdot}u(j), \\
& \quad u(j) \in U, \\
& \quad x(j) \in X, \qquad j = 0,\cdots,N-1. \\
& \quad (x(N),t,r) \in \mathcal{X}_f^w.
\end{aligned}
$$

where $\mathcal{X}_f^w$ is an invariant set for tracking in the extended space (x,t,r). Since this region is a polyhedron, the optimization problem is a standard quadratic problem, that can be efficiently computed.

6. STABILITY ANALYSIS

In this section we provide sufficient conditions to guarantee asymptotic stability of the proposed controller, in such a way that the closed loop system asymptotically reaches any desired admissible steady state $s \in S$.

Before stating the main result of the paper, some technical lemmas must be proved. All the proofs can be found in the appendix.

We denote hereafter $O_\infty(\mathrm{x}_s)$ as the maximal invariant set of states that can be steered to x_s in an admissible way by the control law (6).

Lemma 1. Let $\hat{\mathrm{x}}_s$ be an admissible steady state and let x_s and u_s be a steady state and input for system (1) such that $\mathrm{x}_s \in int(X)$ and $u \in int(U)$. Let K be a stabilizing linear controller with a Lyapunov matrix P. Then there exists $\lambda \in [0,1]$ and $\bar{\mathrm{x}}_s = \lambda \mathrm{x}_s + (1-\lambda)\hat{\mathrm{x}}_s$ such that:

(1) $\mathrm{x}_s \in O_\infty(\bar{\mathrm{x}}_s)$.
(2) For all $T > P$ and $\mathrm{x}_s \neq \hat{\mathrm{x}}_s$ then

$$\|\mathrm{x}_s - \bar{\mathrm{x}}_s\|_P^2 + \|\bar{\mathrm{x}}_s - \hat{\mathrm{x}}_s\|_T^2 < \|\mathrm{x}_s - \hat{\mathrm{x}}_s\|_T^2$$

Next, a (standard) lemma is presented:

Lemma 2. Consider system (1) subject to contraints (2) and (3). Let $u = K{\cdot}x$ be a stabilizing controller with an associated Lyapunov matrix P such that

$$(A+BK)^T P(A+BK) - P = -(Q + K^T RK)$$

where $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$ are definite positive matrices. Consider any $x \in X_f = Proj_x(\mathcal{X}_f^w)$. Consider an admissible steady state x_s such that $x \in O_\infty(\mathrm{x}_s)$. Then we have that

$$V_N^*(x,s) \le \|\mathrm{x} - \mathrm{x}_s\|_P^2 + \|\mathrm{x}_s - \hat{\mathrm{x}}_s\|_T^2$$

where $\hat{\mathrm{x}}_s$ is an steady state associated to s.

Based on these lemmas, the following one can be proved:

Lemma 3. Consider a given reference s and the selected associated steady state $\hat{\mathrm{x}}_s$; assume that for a given state x the optimal solution of $P_N(x,s)$ is such that $\|x - \mathrm{x}_s^*\|_Q = 0$ (i.e. $x = \mathrm{x}_s^*$), then $\|x - \hat{\mathrm{x}}_s\|_Q = 0$.

Now the main result of the paper is presented:

Theorem 4. Consider a system (1) subject to contraints (2) and (3). Let $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$ be definite positive matrices. Let $u = K{\cdot}x$ be a stabilizing controller with an associated Lyapunov matrix P such that

$$(A+BK)^T P(A+BK) - P = -(Q + K^T RK)$$

Consider a matrix $T > P$. Assume that $\mathcal{X}_f^w = O_{\infty,\lambda}^w$, computed for a given $\lambda \in (0,1)$, and for the gain matrix K (see section 4). Then for any feasible initial state $x_0 \in X_N$ and for any admissible set point $s \in \lambda S$, the proposed MPC controller steers asymptotically the system to s in an admissible way.

Remark 5. The set of the admissible steady states that can be tracked without offset is λX_s. Since $\lambda X_s \subset O_\infty$ and since the evolution of the system remains in X_N, the system can be steered to any admissible reference. Then, a sequence of piecewise admissible references can be tracked without offset.

If the desired reference s (and hence the associated steady state $\hat{\mathrm{x}}_s$) is not admissible, then the controller steers the system to an admissible steady state x_s in such a way that the distance $\|\mathrm{x}_s - \hat{\mathrm{x}}_s\|_T$ is minimized.

Remark 6. The stability theorem can be extended to consider $\mathcal{X}_f^w$ as any admissible invariant set for tracking. In this case, the set of references that can be tracked without offset is $\hat{S} = \{s \in \mathbb{R}^p : \exists (x,s,r) \in \mathcal{X}_f^w, x = M_x{\cdot}s + N_x{\cdot}r\}$.

Remark 7. The proposed controller with $s = 0$, i.e the origin as the desired steady state, provides a larger domain of attraction and a better performance (lower optimal cost) than the MPC formulated for regulation.

7. EXAMPLE

Consider a LTI system given by:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}, \text{ and } \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The system is constrained to $\|x\|_\infty \le 5$ and $\|u\|_\infty \le 3$. The steady state and input are characterized by the matrices

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T, \quad N = \begin{bmatrix} 0 & 0.4472 & -0.8944 & 0 \end{bmatrix}^T.$$

The weighting matrices have been chosen as $Q = 0.01{\cdot}I_2$ and $R = I_2$. The local controller gain and the

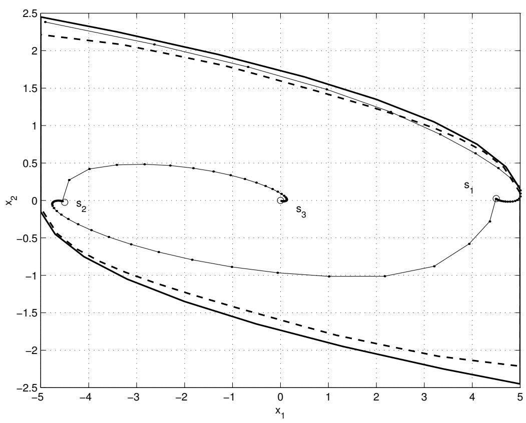

Fig. 1. State portrait of the system.

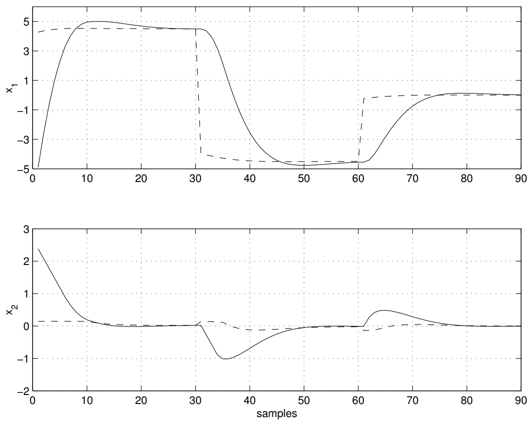

Fig. 2. Evolution of the states of the system

Lyapunov matrix P has been computed using an LQR. The computed maximal invariant set for tracking is shown in figure 1 in dashed line.

The system is controlled by the proposed MPC with a control horizon $N = 10$ and a deviation penalization matrix $T = 1.1 \cdot P$. The domain of attraction of the proposed controller is depicted in figure 1 in solid line.

To illustrate the proposed controller, a piece-wise constant sequence of references has been tracked. The first value of the set point is $s_1 = 4.5$, the second $s_2 = -4.5$ and the third $s_3 = 0$. The evolution of the state is shown in figure 1. In figure 2 the evolution of the states is plotted. As it can be seen, the system evolution is admissible for all time and for drastic changes of the set point. Notice also that the controller steers the system to the desired set points. In this figure the artificial reference $\mathrm{x}_s^*(k)$ of each state is drawn in dashed line.

Appendix A. PROOFS

Proof of Lemma 1

Fact (1): Let P be a Lyapunov matrix of the closed loop system $x^+ = (A + BK) \cdot x$. Since $\mathrm{x}_s \in int(X)$, there exists $\alpha > 0$ such that $\{x : \|x - \mathrm{x}_s\|_P \leq \alpha\} \subseteq X$. Analogously, there exists an $\beta > 0$ such that $\{u : \|u - \mathrm{u}_s\|_P \leq \beta\} \subseteq U$.

Consider a positive constant

$$\varepsilon = min(\frac{\alpha}{2}, \frac{\beta}{2\|K\|_P})$$

where $\|K\|_P$ denotes the induced weighted euclidean norm of matrix K. Consider $\lambda_1 \in [0,1]$ such that $(1 - \lambda_1)\|\mathrm{x}_s - \hat{\mathrm{x}}_s\|_P \leq \varepsilon$. Consider $\lambda_2 \in [0,1]$ such that $(1 - \lambda_2)\|\mathrm{u}_s - \hat{\mathrm{u}}_s\|_P \leq \beta/2$. Consider $\lambda \geq max(\lambda_1, \lambda_2)$ and denote $\bar{\mathrm{x}}_s = \lambda \mathrm{x}_s + (1 - \lambda)\hat{\mathrm{x}}_s$ and $\bar{\mathrm{u}}_s = \lambda \mathrm{u}_s + (1 - \lambda)\hat{\mathrm{u}}_s$. Then:

- For all $\{x : \|x - \bar{\mathrm{x}}_s\|_P \leq \varepsilon\}$, we have that $u = K(x - \bar{\mathrm{x}}_s) + \bar{\mathrm{u}}_s$ is admissible.

$$\begin{aligned} \|u - \mathrm{u}_s\|_P &= \|K(x - \bar{\mathrm{x}}_s) + \bar{\mathrm{u}}_s - \mathrm{u}_s\|_P \\ &\leq \|K\|_P \cdot \|x - \bar{\mathrm{x}}_s\|_P + \|\bar{\mathrm{u}}_s - \mathrm{u}_s\|_P \end{aligned}$$

  We have that $\bar{\mathrm{u}}_s - \mathrm{u}_s = (1 - \lambda)(\hat{\mathrm{u}}_s - \mathrm{u}_s)$ and hence

$$\begin{aligned} \|u - \mathrm{u}_s\|_P &\leq \|K\|_P \cdot \|x - \bar{\mathrm{x}}_s\|_P + (1 - \lambda)\|\hat{\mathrm{u}}_s - \mathrm{u}_s\|_P \\ &\leq \|K\|_P \cdot \varepsilon + (1 - \lambda)\|\hat{\mathrm{u}}_s - \mathrm{u}_s\|_P \\ &\leq \beta/2 + (1 - \lambda)\|\hat{\mathrm{u}}_s - \mathrm{u}_s\|_P \leq \beta \end{aligned}$$

  and then $u \in U$.
- $\{x : \|x - \bar{\mathrm{x}}_s\|_P \leq \varepsilon\} \subseteq X$.
  In effect $\|x - \mathrm{x}_s\|_P \leq \|x - \bar{\mathrm{x}}_s\|_P + \|\bar{\mathrm{x}}_s - \mathrm{x}_s\|_P = \|x - \bar{\mathrm{x}}_s\|_P + (1 - \lambda)\|\hat{\mathrm{x}}_s - \mathrm{x}_s\|_P \leq 2\varepsilon \leq \alpha$, and hence is contained in X.

From these two facts and the property of Lyapunov matrix P we derive that $\{x : \|x - \bar{\mathrm{x}}_s\|_P \leq \varepsilon\}$ is an admissible invariant set, and hence it is contained in $O_\infty(\bar{\mathrm{x}}_s)$.

Since $\|\mathrm{x}_s - \bar{\mathrm{x}}_s\|_P = (1 - \lambda)\|\mathrm{x}_s - \hat{\mathrm{x}}_s\|_P \leq \varepsilon$, we have that $\mathrm{x}_s \in \{x : \|x - \bar{\mathrm{x}}_s\|_P \leq \varepsilon\}$ and hence it is contained in $O_\infty(\bar{\mathrm{x}}_s)$.

Fact (2): In virtue of the previous fact we have that there exists a $\lambda \in [0,1]$ such that for $\bar{\mathrm{x}}_s = \lambda \mathrm{x}_s + (1 - \lambda)\hat{\mathrm{x}}_s$ we have that $\mathrm{x}_s \in O_\infty(\bar{\mathrm{x}}_s)$. Considering the expression of $\bar{\mathrm{x}}_s$ it is easy to see that $\mathrm{x}_s - \bar{\mathrm{x}}_s = (1 - \lambda)(\mathrm{x}_s - \hat{\mathrm{x}}_s)$ and $\bar{\mathrm{x}}_s - \hat{\mathrm{x}}_s = \lambda(\mathrm{x}_s - \hat{\mathrm{x}}_s)$. Then we get that $\|\mathrm{x}_s - \bar{\mathrm{x}}_s\|_P = (1 - \lambda)\|\mathrm{x}_s - \hat{\mathrm{x}}_s\|_P$ and $\|\bar{\mathrm{x}}_s - \hat{\mathrm{x}}_s\|_T = \lambda\|\mathrm{x}_s - \hat{\mathrm{x}}_s\|_T$.

From this result and the assumption that $T > P$, we can state that

$$\begin{aligned} &\|\mathrm{x}_s - \bar{\mathrm{x}}_s\|_P + \|\bar{\mathrm{x}}_s - \hat{\mathrm{x}}_s\|_T = \\ &(1 - \lambda)\|\mathrm{x}_s - \hat{\mathrm{x}}_s\|_P + \lambda\|\mathrm{x}_s - \hat{\mathrm{x}}_s\|_T < \\ &(1 - \lambda)\|\mathrm{x}_s - \hat{\mathrm{x}}_s\|_T + \lambda\|\mathrm{x}_s - \hat{\mathrm{x}}_s\|_T = \|\mathrm{x}_s - \hat{\mathrm{x}}_s\|_T \end{aligned}$$

Since $a + b < c$ implies that $a^2 + b^2 < c^2$ for all $a, b, c > 0$, we have that $\|\mathrm{x}_s - \bar{\mathrm{x}}_s\|_P^2 + \|\bar{\mathrm{x}}_s - \hat{\mathrm{x}}_s\|_T^2 < \|\mathrm{x}_s - \hat{\mathrm{x}}_s\|_T^2$, which proves the fact.

Proof of lemma 2 It suffices to note that the sequence of control inputs obtained from the control law $u =$

$K(x - x_s) + u_s$, denoted as $\mathbf{u}_s$, is a feasible solution for the MPC optimization problem since $x \in O_\infty(x_s)$.

Consider that $A_K = A + BK$ and $Q^* = Q + K^T \cdot R \cdot K$, then from Lyapunov equation $P - A_K^T \cdot P \cdot A_K = Q^*$ we derive that $\|x(i) - x_s\|_P^2 - \|x(i+1) - x_s\|_P^2 = \|x(i) - x_s\|_{Q^*}^2$ for all $x(i)$. Summing this terms we have that

$$\sum_{i=0}^{N-1} \|x(i) - x_s\|_{Q^*}^2 = \|x - x_s\|_P^2 - \|x(N) - x_s\|_P^2$$

Therefore,

$$\begin{aligned} V_N^*(x,s) &\leq \sum_{i=0}^{N-1} \overbrace{\|x(i) - x_s\|_Q^2 + \|K \cdot (x(i) - x_s)\|_R^2}^{\|x(i)-x_s\|_{Q^*}^2} \\ &\quad + \|x(N) - x_s\|_P^2 + \|x_s - \hat{x}_s\|_T^2 \\ &= \|x - x_s\|_P^2 + \|x_s - \hat{x}_s\|_T^2 \end{aligned}$$

Proof of lemma 3 It is proved by contradiction. Assume that $x = x_s^*$ and $x \neq \hat{x}_s$. Since $x = x_s^*$ is a steady state of the system, the control sequence given by the steady input is the optimal solution of $P_N(x_s^*, s)$ and hence $V_N^*(x_s^*, s) = \|x_s^* - \hat{x}_s\|_T^2$.

Since $x_s^* \neq \hat{x}_s$, in virtue of lemma 1 it is inferred that there exists a steady state $\bar{x}_s$ (and an input $\bar{u}_s$), described by $\bar{t}$ and $\bar{r}$, such that $x_s^* \in O_\infty(\bar{x}_s)$. Then the sequence $\bar{\mathbf{u}}$ derived from the control law $u = K(x - \bar{x}_s) + \bar{u}_s$ is admissible and hence, from lemma 2 we have that

$$V_N(x_s^*, s, \bar{\mathbf{u}}, \bar{t}, \bar{r}) \leq \|x_s^* - \bar{x}_s\|_P^2 + \|\bar{x}_s - \hat{x}_s\|_T^2$$

In virtue of lemma 1 we have that

$$V_N(x_s^*, s, \bar{\mathbf{u}}, \bar{t}, \bar{r}) < \|x_s^* - \hat{x}_s\|_T^2 = V_N^*(x_s^*, s)$$

which contradicts the fact of the optimality of $V_N^*(x_s^*, s)$, and then $x = x_s^* = \hat{x}_s$.

Proof of theorem 4 In what follows we denote the optimal solution to the optimization problem by the superscript $*$. Thus, $(\mathbf{u}^*(k), t^*(k), r^*(k))$ denotes the optimal solution obtained in the optimization problem solved at sampling time k. Moreover, $x_s^*(k)$ and $u_s^*(k)$ denote the optimal steady state and input associated to $t^*(k)$ and $r^*(k)$. $x^*(i;k)$ denotes the optimal predicted evolution of the system.

Feasibility: Assume that the state at the current state k, x_k is such that $x_k \in X_N$ and assume that the optimal solution is $(\mathbf{u}^*(k), t^*(k), r^*(k))$ with an optimal cost $V_N^*(x_k, s)$. Let x_{k+1} be the state at the next sampling time. Consider $t(k+1) = t^*(k)$, $r(k+1) = r^*(k)$ and a control sequence

$$\begin{aligned} \mathbf{u}(k+1) = \{&u^*(1;k), \cdots, u^*(N-1;k), \\ &K(x^*(N;k) - x_s^*(k)) + u_s^*(k)\} \end{aligned}$$

Then, it is easy to see that$(\mathbf{u}(k+1), t(k+1), r(k+1))$ is feasible due to the feasibility of the optimal solution at k and the positively invariance of $\mathcal{X}_f^w$. Consequently, $x_{k+1} \in X_N$.

Convergence: Consider the feasible solution at time $k+1$ previously presented. Following standard steps in the stability proofs of MPC (Mayne et al. 2000), we get that

$$\begin{aligned} V_N^*(x_{k+1}, s) &\leq V_N(x_{k+1}, s, \mathbf{u}(k+1), t(k+1), r(k+1)) \\ &\leq V_N^*(x_k, s) - \|x_k - x_s^*(k)\|_Q^2 \end{aligned}$$

Due to the definite positiveness of the optimal cost and its non-increasing evolution, we infer that

$$\lim_{k \to \infty} \|x_k - x_s^*(k)\|_Q = 0$$

and in virtue of lemma 3 we have that

$$\lim_{k \to \infty} \|x_k - \hat{x}_s\|_Q = 0$$

Consequently, the system output evolves to s.

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