STATE AND UNKNOWN INPUT ESTIMATION FOR LINEAR DISCRETE-TIME SYSTEMS

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Abstract: This paper deals with a new type of estimator for discrete-time linear systems with unknown inputs. A constructive algorithm is given in order to analyze the state observability and the left invertibility of the system (i.e the possibility to recover the unknown inputs with the outputs) and then an estimator is designed.

Keywords: Estimation, discrete-time linear systems, unknown inputs

1. INTRODUCTION

In many applications, like fault detection and identification, cryptography or parameter identifiability, the design of observers for linear systems with unknown inputs is of importance and lots of works can be found in the literature (see e.g. Darouach et al. (1994), Hou and Müller (1991), Kudva et al. (1980)). In all the works previously mentioned, it is possible to construct a linear observer under a necessary existence condition stating that one can recover from the available outputs the part of the state which is directly coupled with the unknown inputs. In Floquet and Barbot (2004), the authors designed an algorithm that allows to overcome this restrictive condition in the continuous-time case by using a sliding mode observer.

Here, it is aimed at designing a delayed estimator of the state variables and the unknown inputs for discrete-time systems, which is quite different than the observer design for continuous time one (while the delayed outputs play a similar role than the output derivatives in continuous time systems, they are drastically easier to obtain). The problem is to recover the state and the unknown inputs after a finite number of delays. Thus, it is a left invertibility with delays problem. Obviously, the design of delayed estimator has many advantages: simplicity of implementation, finite time convergence, structural stability, and it introduces less delay than a discrete-time observer. However it has also some drawbacks. For example, as it is the case for all systems with transient time, the delayed estimator introduces a structural delay which can be prejudicial for some fault detection and isolation problems (such that for the rolling-mill, Gu and Poon (2003)). Nevertheless, this is not a real constraint for some other applications as cryptography (as in the case of chaotic synchronization, Barbot et al. (2003)) or off-line diagnostic.

Finally, it is important to mention that a discrete-time estimator can be used for discrete-time system but also for systems under sampling. This is the reason why the design of discrete-time estimator is more and more popular and more appropriate when dealing with real applications (as e.g. applications in signal processing).

Consider a linear discrete-time system of the form:

$$x(k+1) = Ax(k) + Bu(k) + Dw(k) \quad (1)$$
$$y(k) = Cx(k) \quad (2)$$

where $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}^{p_1}$ is the output vector, $u \in \mathbb{R}^q$ represents the known inputs and $w \in \mathbb{R}^m$ stands for the unknown inputs. A, B, C and D are known constant matrices of appropriate dimension. It is assumed that $m \leq p_1$ and, without loss of generality, that rank $C = p_1$ and rank $D = m$.

In this paper, we propose a simple algorithm in order to analyze if the system is or not left invertible with delays and to put the system in a particular observability triangular form. This form is well suited to design a delayed estimator that provides the states and unknown inputs of the linear discrete-time systems after some finite number of sampling delays. The main contribution of this paper is that the previously mentioned necessary condition for the design of linear observer, i.e.

$$\text{rank } CD = \text{rank } D = m$$

is not required anymore. Furthermore, there is no assumption on the unknown inputs (boundedness or statistical properties) and a bound on the number of sampling delays necessary to recover the information is known.

2. OUTPUT INFORMATION ALGORITHM

Iteration 1: Consider the vector of outputs $y^1 \triangleq Cx$.

a. Without loss of generalities, one can reorder the components of y^1 as follows:

$$y^1 = \begin{bmatrix} C_1^T & \cdots & C_{\eta_1}^T & C_{\eta_1+1}^T & \cdots & C_{p_1}^T \end{bmatrix}^T x$$

where C_1**,...,** C_{η_1} satisfy for all $j \leq \eta_1$

$$C_j A^k D = 0, \text{ for all } k \in \mathbb{N} \quad (3)$$

and where C_{η_1+1},..., C_{p_1} are such that for $1 \leq j \leq p_1 - \eta_1$, there exists an integer r_j^1 such that:

$$C_{\eta_1+j} A^k D = 0, \text{ for all } k < r_j^1 - 1$$
$$C_{\eta_1+j} A^{r_j^1 - 1} D \neq 0. \quad (4)$$

and such that $r_1^1 \leq \ldots \leq r_{p_1-\eta_1}^1$. Note that only the outputs $y_j^1 = C_j x$, $\eta_1 + 1 \leq j \leq p_1$, are affected by the unknown inputs.

b. Define the set of covectors

$$\Phi^1 = \text{span}\left\{C_1, ..., C_1 A^{n-1}, C_2, ..., C_2 A^{n-1}, ..., C_{\eta_1}, ..., C_{\eta_1} A^{n-1}\right\}$$

and note $\varphi^1 = \text{rank } \Phi^1$.

Find η_1 integers $\varphi_1^1, \ldots, \varphi_{\eta_1}^1$ such that

$$\text{rank } I_1 = \begin{bmatrix} C_1 \\ \vdots \\ C_1 A^{\varphi_1^1 - 1} \\ \vdots \\ C_{\eta_1} \\ \vdots \\ C_{\eta_1} A^{\varphi_{\eta_1}^1 - 1} \end{bmatrix} = \varphi^1$$

(i.e. $\left\{C_1, ..., C_1 A^{\varphi_1^1 - 1}, ..., C_{\eta_1}, ..., C_{\eta_1} A^{\varphi_{\eta_1}^1 - 1}\right\}$ is a basis of Φ^1). One has $\varphi^1 = \varphi_1^1 + \ldots + \varphi_{\eta_1}^1$. If $\varphi^1 = n$, it is obviously easy to design an observer for the system (1-2) and we stop the algorithm. Actually, this is the case when the state is not affected by any disturbance, i.e. $D = 0$.

c. Define the set of covectors

$$\Upsilon^1 = \text{span}\left\{C_{\eta_1+1}, ..., C_{\eta_1+1} A^{r_1^1 - 1}, ..., C_{p_1}, ..., C_{p_1} A^{r_{p_1-\eta_1}^1 - 1}\right\}$$

and the integer ρ^1 such that rank $\left(\Phi^1 \cup \Upsilon^1\right) = \varphi^1 + \rho^1$.

Find $p_1 - \eta_1$ integers $\rho_1^1, ..., \rho_{p_1-\eta_1}^1$ such that, the matrix $\begin{bmatrix} I_1 \\ D_1 \end{bmatrix}$, where $D_1 = \begin{bmatrix} C_{\eta_1+1} \\ \vdots \\ C_{\eta_1+1} A^{\rho_1^1 - 1} \\ \vdots \\ C_{p_1} \\ \vdots \\ C_{p_1} A^{\rho_{p_1-\eta_1}^1 - 1} \end{bmatrix}$, has rank $\varphi^1 + \rho^1$. One has $\rho^1 = \rho_1^1 + .. + \rho_{p_1-\eta_1}^1$. If $\varphi^1 + \rho^1 = n$, quit the algorithm.

d. Define the matrix

$$\Gamma_1 = \begin{bmatrix} C_{\eta_1+1} A^{r_1^1 - 1} D \\ \vdots \\ C_{p_1} A^{r_{p_1-\eta_1}^1 - 1} D \end{bmatrix}$$

and note $d_1 = \text{rank } \Gamma_1$. If $d_1 < p_1 - \eta_1$, one can find a matrix $\Lambda_1 \in \mathbb{R}^{p_2 \times (p_1 - \eta_1)}$, where $p_2 = p_1 - \eta_1 - d_1$, such that $\Lambda_1 \Gamma_1 = 0$.

Define the auxiliary variable (or fictitious output)

$$y^2 = \Lambda_1 \begin{bmatrix} C_{\eta_1+1} A^{r_1^1} \\ \vdots \\ C_{p_1} A^{r_{p_1-\eta_1}^1} \end{bmatrix} x \triangleq C^2 x, \; C^2 = \begin{bmatrix} C_1^2 \\ \vdots \\ C_{p_2}^2 \end{bmatrix}.$$

Note that C^2 is not necessarily full rank.

Iteration 2: The Output Information Algorithm is applied to the new vector of fictitious outputs $y^2 \in \mathbb{R}^{p_2}$.

a. After possible reordering of the components of y^2, by analogy with Iteration 1.a, one can define the integers η_2 and r_j^2, $1 \leq j \leq p_2 - \eta_2$.

b. Assume that rank $\left(\Phi^1 \cup \Upsilon^1 \cup \Phi^2\right) = \varphi^1 + \rho^1 + \varphi^2$ where

$$\Phi^2 = \text{span}\left\{C_1^2, ..., C_1^2 A^{n-1}, C_2^2, ..., C_2^2 A^{n-1}, ..., C_{\eta_2}^2, ..., C_{\eta_2}^2 A^{n-1}\right\}.$$

Then define the integers φ_j^2, $1 \leq j \leq \eta_2$ and the matrix I_2 such that $\begin{bmatrix} I_1 \\ D_1 \\ I_2 \end{bmatrix}$ has rank $\varphi^1 + \rho^1 + \varphi^2$.

If $\varphi^1 + \rho^1 + \varphi^2 = n$, the algorithm is stopped.

c. By analogy with Iteration 1.c, one can define the set Υ^2 and the matrix D_2 and the related integers ρ^2 and $(\rho_1^2, ..., \rho_{p_2-\eta_2}^2)$. The algorithm is stopped if $\varphi^1 + \rho^1 + \varphi^2 + \rho^2 = n$ or if $\varphi^1 + \rho^1 + \varphi^2 + \rho^2 < n$ and $D_2 = \emptyset$.

d. Define the matrix

$$\Gamma_2 = \begin{bmatrix} \Gamma_1 \\ C_{\eta_2+1}^2 A^{r_1^2-1} D \\ \vdots \\ C_{p_2}^2 A^{r_{p_2-\eta_2}^2-1} D \end{bmatrix}$$

and note $d_2 = \text{rank}\, \Gamma_2$.

If $d_2 < (p_1 - \eta_1) + (p_2 - \eta_2)$, one can find a matrix $\Lambda_2 \in \mathbb{R}^{p_3 \times ((p_1-\eta_1)+(p_2-\eta_2))}$, where $p_3 = (p_1 - \eta_1) + (p_2 - \eta_2) - d_2$, such that $\Lambda_2 \Gamma_2 = 0$. Then the Output Information Algorithm is applied to the new fictitious outputs

$$y^3 = \Lambda_2 \begin{bmatrix} C_{\eta_1+1} A^{r_1^1} \\ \vdots \\ C_{p_1} A^{r_{p_1-\eta_1}^1} \\ C_{\eta_2+1}^2 A^{r_1^2} \\ \vdots \\ C_{p_2}^2 A^{r_{p_2-\eta_2}^2} \end{bmatrix} x \triangleq C^3 x.$$

Repeating this procedure, one has:

Iteration k: The fictitious output $y^k \in \mathbb{R}^{p_k}$, that has been defined in Iteration $k-1$, is considered.

a. The integers η_k and r_j^k, $1 \leq j \leq p_k - \eta_k$, are determined.

b. Compute the set of covectors

$$\Phi^k = \text{span}\left\{C_1^k, ..., C_1^k A^{n-1}, C_2^k, ..., C_2^k A^{n-1}, ..., C_{\eta_k}^k, ..., C_{\eta_k}^k A^{n-1}\right\}$$

and assume that rank $\left(\left(\bigcup_{i=1}^{k-1} \Phi^i \cup \Upsilon^i\right) \cup \Phi^k\right) = \sum_{i=1}^{k-1} \left(\varphi^i + \rho^i\right) + \varphi^k$.

Find η_k integers $\varphi_1^k, \ldots, \varphi_{\eta_k}^1$, such that

$$\text{rank} \begin{bmatrix} I_1 \\ D_1 \\ \vdots \\ I_k \end{bmatrix} = \sum_{i=1}^{k-1} \left(\varphi^i + \rho^i\right) + \varphi^k, \text{ where}$$

$$I_k = \left[\left(C_1^k\right)^T, ..., \left(C_1^k A^{\varphi_1^k - 1}\right)^T, ..., \left(C_{\eta_k}^k\right)^T, ..., \left(C_{\eta_k}^k A^{\varphi_{\eta_k}^k - 1}\right)^T\right]^T.$$

c. Compute the set of covectors

$$\Upsilon^k = \text{span}\left\{C_{\eta_k+1}^k, ..., C_{\eta_k+1}^k A^{r_1^k-1}, ..., C_{p_k}^k, ..., C_{p_k}^k A^{r_{p_k-\eta_k}^k - 1}\right\}$$

and assume rank $\left(\bigcup_{i=1}^{k} \Phi^i \cup \Upsilon^i\right) = \sum_{i=1}^{k} \left(\varphi^i + \rho^i\right)$.

Find $p_k - \eta_k$ integers $\rho_1^k, ..., \rho_{p_k-\eta_k}^k$ such that

$$\text{rank} \begin{bmatrix} I_1 \\ D_1 \\ \vdots \\ I_k \\ D_k \end{bmatrix} = \sum_{i=1}^{k} \left(\varphi^i + \rho^i\right), \text{ where}$$

$$D_k = \left[\left(C_{\eta_k+1}^k\right)^T, ..., \left(C_{\eta_k+1}^k A^{\rho_1^k - 1}\right)^T, ..., \left(C_{p_k}^k\right)^T, ..., \left(C_{p_k}^k A^{\rho_{p_k-\eta_k}^k - 1}\right)^T\right]^T.$$

d. Define

$$\Gamma_k = \begin{bmatrix} \Gamma_1 \\ \vdots \\ \Gamma_{k-1} \\ C_{\eta_k+1}^k A^{r_1^k-1} D \\ \vdots \\ C_{p_k}^k A^{r_{p_k-\eta_k}^k-1} D \end{bmatrix}$$

and note $d_k = \text{rank}\, \Gamma_k$. If $d_k < \sum_{s=1}^{k} (p_s - \eta_s)$, let us set $p_{k+1} = \sum_{s=1}^{k} (p_s - \eta_s) - d_k$. One can find a matrix $\Lambda_k \in \mathbb{R}^{p_{k+1} \times \left(\sum_{s=1}^{k} (p_s - \eta_s)\right)}$ such that $\Lambda_k \Gamma_k = 0$.

Define a new fictitious output:

$$y^{k+1} = \Lambda_k \begin{bmatrix} C_{\eta_1+1} A^{r_1^1} \\ \vdots \\ C_{p_1} A^{r_{p_1-\eta_1}^1} \\ \vdots \\ C_{\eta_k+1}^k A^{r_1^k} \\ \vdots \\ C_{p_k}^k A^{r_{p_k-\eta_k}^k} \end{bmatrix} x \triangleq C^{k+1} x.$$

Stop the algorithm if:

1. there exists $\mu \in \mathbb{N}$, such that

$$\varphi^1 + \rho^1 + \ldots + \varphi^\mu + \rho^\mu < n \text{ and } \left\{D_\mu = \emptyset \text{ or } d_\mu = \sum_{s=1}^{\mu} (p_s - \eta_s)\right\},$$

2. there exists $k^\star \in \mathbb{N}$ such that $\sum_{i=1}^{k^\star} \left(\varphi^i + \rho^i\right) = n$.

In case 1, it is not possible to estimate the state of system (1-2) with the method described in this work. In case 2, one obtains a set of covectors $S_{k^\star} = I_1 \cup D_1 \cup \ldots \cup I_{k^\star} \cup D_{k^\star}$ where $\dim S_{k^\star} = n$. Obviously, the number of iterations is finite ($< n$).

Note that the fictitious outputs play a quite similar role that the non-degenerate solution of the algorithm given in Ljung and Glad (1994).

Proposition 1. If there exists $k^\star \in \mathbb{N}$ such that $\sum_{i=1}^{k^\star} \left(\varphi^i + \rho^i\right) = n$ then rank $\Gamma_{k^*} = m$.

Proof: From (3), (4), the definitions of the matrices I_i and D_i, and since $\sum_{i=1}^{k^\star} \left(\varphi^i + \rho^i\right) = n$:

$$\operatorname{rank} \Gamma_{k^\star} = \operatorname{rank} \begin{bmatrix} C_{\eta_1+1} A^{r_1^1 - 1} \\ \vdots \\ C_{p_1} A^{r^1_{p_1-\eta_1}-1} \\ \vdots \\ C^{k^\star}_{\eta_{k^\star}+1} A^{r_1^{k^\star}-1} \\ \vdots \\ C^{k^\star}_{p_{k^\star}} A^{r^{k^\star}_{p_{k^\star}-\eta_{k^\star}}-1} \end{bmatrix} D$$

$$= \operatorname{rank} \begin{bmatrix} I_1 \\ D_1 \\ \vdots \\ I_{k^\star} \\ D_{k^\star} \end{bmatrix} D = m.$$

As a straightforward consequence of this Proposition, all the components of the state and all the unknown inputs can be estimated after a finite number of delays.

3. SYSTEM TRANSFORMATION

After applications of the algorithm, the following $(n \times n)$ nonsingular matrix can be defined:

$$T = \begin{bmatrix} I_1 \\ D_1 \\ \vdots \\ I_{k^\star} \\ D_{k^\star} \end{bmatrix}$$

Under the coordinate transformation

$$x = T^{-1} Z = T^{-1} \begin{bmatrix} \sigma^1 \\ \chi^1 \\ \vdots \\ \sigma^{k^\star} \\ \chi^{k^\star} \end{bmatrix}$$

where $\sigma^i = \begin{bmatrix} \sigma_1^i \\ \vdots \\ \sigma_{\eta_i}^i \end{bmatrix}$ and $\chi^i = \begin{bmatrix} \chi_1^i \\ \vdots \\ \chi^i_{p_i-\eta_i} \end{bmatrix}$, for $1 \le i \le k^\star$, with $\sigma_j^i = \begin{bmatrix} \left(\sigma_j^i\right)_1 \\ \vdots \\ \left(\sigma_j^i\right)_{\varphi_j^i} \end{bmatrix}$, for $1 \le j \le \eta_i$, and $\chi_j^i = \begin{bmatrix} \left(\chi_j^i\right)_1 \\ \vdots \\ \left(\chi_j^i\right)_{\rho_j^i} \end{bmatrix}$, for $1 \le j \le p_i - \eta_i$, the system (1-2) becomes:

$$\sigma_j^i(k+1) = \Delta^\sigma_{i,j} \sigma^i_j(k) + \Xi^\sigma_{i,j} x(k) + B^\sigma_{i,j} u(k) \tag{5}$$

$$\chi_j^i(k+1) = \Delta^\chi_{i,j} \chi^i_j(k) + \Xi^\chi_{i,j} x(k) + \Theta^\chi_{i,j} w(k) + B^\chi_{i,j} u(k) \tag{6}$$

$$\Delta^\sigma_{i,j} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{\varphi_j^i \times \varphi_j^i}, \quad \Xi^\sigma_{i,j} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ C_j A^{\varphi_j^i} \end{bmatrix}_{\varphi^i_j \times n}$$

$$\Delta^\chi_{i,j} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{\rho_j^i \times \rho_j^i},$$

$$\Xi^\chi_{i,j} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ C_{\eta_i+j} A^{\rho_j^i} \end{bmatrix}_{\rho^i_j \times n}, \quad \Theta^\chi_{i,j} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ C_{\eta_i+j} A^{\rho_j^i-1} D \end{bmatrix}_{\rho^i_j \times m}.$$

$B^\sigma_{i,j}$ and $B^\chi_{i,j}$ are a $\left(\varphi_j^i \times q\right)$ and a $\left(\rho_j^i \times q\right)$-matrix, respectively. The system is put in a block triangular observable form. Note that $C_{\eta_i+j} A^{\rho_j^i-1} D \neq 0$ if and only if $\rho_j^i = r_j^i$.

4. ESTIMATOR DESIGN

The estimator designed in this section is called a step-by-step delayed reconstructor, see Belmouhoub et al. (2003).

4.1 Reconstruction of the state:

First step: one consider the subsystem of (5)-(6) related to the available measurements, that is to say $i = 1$. For the outputs that are not affected by the unknown inputs, one has $y_j^1(k) = (\sigma_j^1)_1(k)$ and:

$$(\sigma_j^1)_1(k+1) = (\sigma_j^1)_2(k) + \sum_{l=1}^{q} \left(B^\sigma_{1,j}\right)_{1,l} u_l(k),$$

$$(\sigma_j^1)_2(k+1) = (\sigma_j^1)_3(k) + \sum_{l=1}^{q} \left(B^\sigma_{1,j}\right)_{2,l} u_l(k),$$

$$\vdots$$

$$(\sigma_j^1)_{\varphi_j^1-1}(k+1) = (\sigma_j^1)_{\varphi_j^1}(k) + \sum_{l=1}^{q} \left(B^\sigma_{1,j}\right)_{\varphi_j^1-1,l} u_l(k),$$

$$(\sigma_j^1)_{\varphi_j^1}(k+1) = C_j A^{\varphi_j^1} x(k) + \sum_{l=1}^{q} \left(B^\sigma_{1,j}\right)_{\varphi_j^1,l} u_l(k),$$

where $1 \leq j \leq \eta_1$, and where $\left(B^{\sigma}_{1,j}\right)_{h,l}$ are the elements of the $h-th$ row of the matrix $B^{\sigma}_{1,j}$. Those equations can be rewritten as follows:

$$(\sigma^1_j)_1(k) = y^1_j(k)$$

$$(\sigma^i_j)_2(k-1) = y^1_j(k) - \sum_{l=1}^{q}\left(B^{\sigma}_{1,j}\right)_{1,l} u_l(k-1)$$

$$(\sigma^i_j)_3(k-2) = y^1_j(k) - \sum_{l=1}^{q}\left(B^{\sigma}_{1,j}\right)_{1,l} u_l(k-1) - \sum_{l=1}^{q}\left(B^{\sigma}_{1,j}\right)_{2,l} u_l(k-2)$$

$$\vdots$$

$$(\sigma^1_j)_{\varphi^1_j}\left(k-\varphi^1_j+1\right) = y^1_j(k) - \sum_{s=1}^{\varphi^1_j-1}\sum_{l=1}^{q}\left(B^{\sigma}_{1,j}\right)_{s,l} u_l(k-s)$$

Consequently, using delays, all the state σ^1 can be estimated at the time $\left(k-\bar{\varphi}^1+1\right)$ where $\bar{\varphi}^1 = \max\limits_{1\leq j\leq \eta_1} \varphi^1_j$.

In a similar way, one gets an estimation for the state χ^1 at the time $\left(k-\bar{\rho}^1_j+1\right)$, where $\bar{\rho}^1 = \max\limits_{1\leq j\leq p_1-\eta_1} \rho^1_j$.

Second step: in order to get an estimation of the remaining states, one uses the fictitious outputs.

$$y^2 = \Lambda_1 \begin{bmatrix} C_{\eta_1+1}A^{r^1_1} \\ \vdots \\ C_{p_1}A^{r^1_{p_1-\eta_1}} \end{bmatrix} x = \Lambda_1 \begin{bmatrix} y^1_{\eta_1+1}(k+r^1_1) \\ \vdots \\ y^1_{p_1}(k+r^1_{p_1-\eta_1}) \end{bmatrix}.$$

Since $r^1_{p_1-\eta_1} = \max\limits_{1\leq j\leq p_1-\eta_1} r^1_j$, $y^2\left(k-r^1_{p_1-\eta_1}\right)$ is available. From the definition of Λ_1 (see iteration 1.d in the algorithm), y^2 is not affected by the unknown inputs.

Then, in a similar manner as in the first step, σ^2 is estimated at time $\left(k-r^1_{p_1-\eta_1}-\bar{\varphi}^2+1\right)$ and the state χ^2 is known at time $\left(k-r^1_{p_1-\eta_1}-\bar{\rho}^2+1\right)$, where $\bar{\varphi}^2 = \max\limits_{1\leq j\leq \eta_1} \varphi^1_j$ and $\bar{\rho}^1 = \max\limits_{1\leq j\leq p_1-\eta_1} \rho^1_j$.

Following this procedure, one obtains recursively the whole state. For $2 \leq \alpha \leq k^\star$:

σ^α is known at time $\left(k - \sum\limits_{i=1}^{\alpha-1} r^i_{p_i-\eta_i} - \bar{\varphi}^\alpha + 1\right)$,

χ^α at time $\left(k - \sum\limits_{i=1}^{\alpha-1} r^i_{p_i-\eta_i} - \bar{\rho}^\alpha + 1\right)$.

Thus, one gets the estimation of the state variables with a finite number of delays less than $\tau = \max\limits_{0\leq\alpha\leq k^\star-1}\left\{\sum\limits_{i=0}^{\alpha} r^i_{p_i-\eta_i} + \bar{\varphi}^{\alpha+1} - 1; \sum\limits_{i=0}^{\alpha} r^i_{p_i-\eta_i} + \bar{\rho}^{\alpha+1} - 1\right\}$ where $r^0_{p_0-\eta_0} \triangleq 0$.

4.2 Estimation of ω

The last rows of each subsystem of (6) provide an estimation of ω. Indeed, for $1 \leq i \leq k^\star$, $1 \leq j \leq p_i - \eta_i$:

$$C_{\eta_i+j}A^{\rho^i_j-1}Dw(k) = (\chi^i_j)_{\rho^i_j}(k+1) - C_{\eta_i+j}A^{\rho^i_j}T^{-1}x(k) - \sum_{l=1}^{q}\left(B^{\chi}_{i,j}\right)_{\rho^1_j,l} u_l(k)$$

or, in compact form

$$\Theta^{\chi} w(k) = \Pi\left(\chi(k+1), \sigma(k), \chi(k), u(k)\right) \quad (7)$$

where $\Theta^{\chi} = \begin{bmatrix} C_{\eta_1+1}A^{\rho^1_1-1}D \\ \vdots \\ C_{p_1}A^{\rho^1_{p_1-\eta_1}-1}D \\ \vdots \\ C_{\eta_{k^\star}+1}A^{\rho^{k^\star}_1-1}D \\ \vdots \\ C_{p_{k^\star}}A^{\rho^{k^\star}_{p_{k^\star}-\eta_{k^\star}}-1}D \end{bmatrix}$.

Remark 2. Following the same argument as in Proposition 1, one has rank $\Theta^{\chi} = m$.

Since Π is known, at least, at time $(k-\tau-1)$ and since rank $\Theta^{\chi} = m$, the relation (7) provides an estimation of the unknown inputs:

$$w(k-\tau-1) = \left(\Theta^{\chi}\right)^{+}\Pi$$

where $\left(\Theta^{\chi}\right)^{+}$ is the pseudo-inverse of Θ^{χ}.

5. EXAMPLE

$$x^{+} = \frac{1}{10}\begin{bmatrix} -2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ -2 & 3 & 0 & 0 & -4 \end{bmatrix} x + \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \\ -1 & 0 & 2 \end{bmatrix} w$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} x = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} x$$

$$\Gamma_1 = \begin{bmatrix} C_1D \\ C_2D \\ C_3D \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 2 & 1 & 0 \end{bmatrix}.$$

The matrix $\Lambda_1 = \begin{bmatrix} -2 & 1 & 1 \end{bmatrix}$ defined by is such that $\Lambda_1\Gamma_1 = 0$. Then, one can choose the fictitious output as $y^2 = \Lambda_1 \begin{bmatrix} C_1A \\ C_2A \\ C_3A \end{bmatrix} x = C^2_1 x$ or $y^2 = -2y^+_1 + y^+_2 + y^+_3$.

It can be checked that:

$$\text{rank } T = \text{rank} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C^2_1 \\ C^2_1 A \end{bmatrix} = 5.$$

Under the change of coordinates $z = Tx$, the system becomes:

$$z_1^+ = -0.3z_2 + 0.2z_3 - 3z_4 - 10z_5 + w_1 \quad (8)$$

$$z_2^+ = -0.1z_1 - 0.6z_2 + 0.5z_3 - 5z_4 - 20z_5 - w_2 \quad (9)$$

$$z_3^+ = 0.1z_1 - 0.1z_3 + 2w_1 + w_2 \quad (10)$$

$$z_4^+ = z_5 \quad (11)$$

$$\begin{aligned} 100z_5^+ &= 0.4z_1 + 0.5z_2 - 0.6z_3 + 4z_4 - 10z_5 \\ &\quad + 5w_1 + 5w_2 - w_3 \end{aligned} \quad (12)$$

$$y = \left[\, z_1 \ z_2 \ z_3 \, \right].$$

It is clear that one has the knowledge of the states z_1, z_2, and z_3 from 0 till instant k. From the choice of the new input y^2, one gets the value of z_4 and z_5 after one and two sampling periods, respectively, since $\left(y^2\right)^- = z_4^- = z_5^{--} = -2y_1 + y_2 + y_3$.

Then, the unknown input can be estimated as follows. From equations (8) and (9):

$$w_1^{--} = z_1^- + 0.3z_2^{--} - 0.2z_3^{--} + 3z_4^{--} + 10z_5^{--}$$

$$\begin{aligned} w_2^{--} &= -z_2^- - 0.1z_1^{--} - 0.6z_2^{--} + 0.5z_3^{--} - 5z_4^{--} \\ &\quad - 20z_5^{--} \end{aligned}$$

Since z_5 is only known at time $(k-2)$, w_3 is obtained from equation (12) at time $(k-3)$:

$$\begin{aligned} w_3^{(3-)} &= -100z_5^{--} + 0.4z_1^{(3-)} + 0.5z_2^{(3-)} - 0.6z_3^{(3-)} \\ &\quad + 4z_4^{(3-)} - 10z_5^{(3-)} + 5w_1^{(3-)} + 5w_2^{(3-)}. \end{aligned}$$

Simulation results are given in Figures 1 and 2, where it can be seen that the state and the unknown inputs are estimated after three sampling periods.

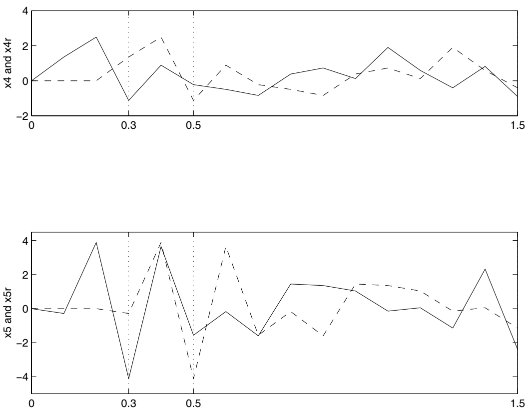

Fig. 1. Reconstructed states

6. CONCLUSION

In this paper has been considered the problem of state estimation and unknown input identification for discrete-time linear systems. An algorithm has been given in order to introduce fictitious outputs that allow to recover both the state and the unknown inputs after a finite number of sampling delays. Straightforward applications can be found in fault detection and identification or cryptography.

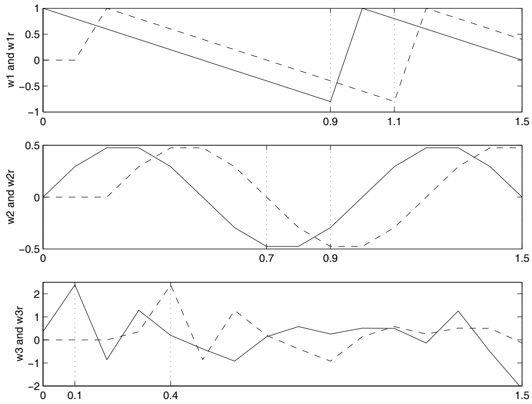

Fig. 2. Estimated unknown inputs

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