UNCERTAINTY MODELLING AND MIXED SENSITIVITY $\mathcal{H}_\infty$ DESIGN FOR A PILOT-SCALE FLOTATION COLUMN

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Abstract:
A methodology for modelling and designing $\mathcal{H}_\infty$ controllers for an experimental pilot-scale flotation column plant is described. Experimental results, including the modelling procedure and an $\mathcal{H}_\infty$ design, are presented. For this purpose, different models, for different operating conditions, are determined, by means of system identification techniques. A strategy to get the uncertainty model, using a multiplicative nonstructured norm bounded uncertainty is also defined. The complete model is used for an $\mathcal{H}_\infty$ design using the mixed sensitivity approach. All the results have been tested in a real pilot-scale flotation column operating in a water air system.

Keywords:
Robust Control, Uncertainty Systems, System Identification, Modelling, Mixed Sensitivity, Flotation Column

1. INTRODUCTION

Robust Control analysis and synthesis should consider different modelling errors, disturbances signals and uncertainties types. Among the classes of uncertainty models that have been used extensively are the parametric uncertainty models, non-parametric unstructured (additive or multiplicative) or coprime factor uncertainty, to mention a few. These uncertainties can arise from many sources, including parameters variation, unmodelled frequency, perturbations, lack of knowledge of the system or identification error (Skogestad and Postlethwaite, 1996), (Balas *et al.*, 1992).

In this paper, a procedure to identify a nominal and an uncertainty model for a real multivariable (MIMO) control system is described. The resulting model is used to design an $\mathcal{H}_\infty$ control, by using the well-known loopshaping approach.

This procedure will be applied to a pilot-scale flotation column, which is, from a control technique standpoint, a multivariable process originally modelled as a 3×3 MIMO system (Persechini *et al.*, 2000),(Persechini *et al.*, 2004).

The pilot-scale flotation column used in the experiments performed here is composed of a transparent acrylic tube with 5.1 cm internal diameter, 720 cm height, operating in a water-air system. All the necessary instruments are connected to a data acquisition system. The data acquisition

[1] Partially supported by CAPEs and CDTN
[2] Partially supported by FAPEMIG

system, in turn, is connected via a standard RS-232 serial port to a microcomputer in which the $\mathcal{H}_\infty$ controller runs under a real-time operational system.

The **controlled variables** for the flotation column considered in this paper are the *froth layer height* (h) and *the air holdup in the collection zone* (ϵ_{gcz}), expressed as the percentage of the total volume occupied by the air in that zone. There is a third variable, named *bias*, but this variable is not measured on line. The **manipulated variables** are the control signals sent, respectively, to a peristaltic pump for the non-floated (U_T), to a wash water valve (U_W) and to an air valve (U_g). This system is initially modelled as a 3×3 system (Persechini *et al.*, 2000); subsequently, its structure is changed to a 2×2 uncertainty model plus an 2×1 perturbation at the plant output (Persechini, 2001). This alternative structure has been validated with experimental data and proved to be better than the original structure when the bias is not considered (Persechini, 2001).

For the design in the mixed sensitivity framework, it is possible to make use of the non-standard $\mathcal{H}_\infty$ structure (Skogestad and Postlethwaite, 1996).

The procedure to get the nominal model, as proposed here, is accomplished in five steps, as detailed in section 2. In its turn, the uncertainty model is obtained following a three step procedure (as proposed in section 3). These procedures are correlated to each other (in the sense that one depend on another). They have been implemented in (Miranda, 2000) and (Persechini, 2001) and applied to different pilot-scale plants, following the ideas presented in (Skogestad and Postlethwaite, 1996).

1.1 Model Structure

Since one of the specific control objectives in the column flotation process is to keep $h(t)$ and $\epsilon_{gcz}(t)$ at desired values, by manipulating the signals $u_t(t)$, $u_g(t)$ and having $u_w(t)$ as a perturbation, the column flotation is often modelled as a multivariable interacting process. The process can be represented as (Persechini, 2001):

$$\begin{bmatrix} H(s) \\ \\ E_{gcz}(s) \end{bmatrix} = \begin{bmatrix} g_{11}(s) & g_{12}(s) \\ \\ g_{21}(s) & g_{22}(s) \end{bmatrix} \begin{bmatrix} U_t(s) \\ \\ U_g(s) \end{bmatrix} + \begin{bmatrix} g_{13}(s) \\ g_{23}(s) \end{bmatrix} U_w(s), \qquad (1)$$

where each g_{ij} is the transfer functions for the respective input-output pair.

2. NOMINAL MODEL IDENTIFICATION

Details on how to get the nominal model from a data set are given in this section. Considering an experimental data set, under different operating conditions, it is possible to determine the best model for each experiment. From the analysis of these models, a nominal model is selected so as to minimize the representation of the uncertainty.

In order to better introduce the model representation, the following notation has been used. For each experimental run, each transfer function of the plant model (equation 1) are denoted by g_{ij}^k, where k refers to a specific experiment, $k = 1, 2, ..24$, and $i = 1, 2$ and $j = 1, 2, 3$. The identification of the nominal model, thus, consists in determining the coefficients of g_{0ij}.

2.1 Selection of the experimental data set

First step: *Select the experimental data set that represents typical operational conditions.*

As stated before, 24 experiments have been performed in the pilot-scale plant, which aimed at determining the nominal and the uncertainty models. In these experiments, the operating conditions were set to: 50 cm $< h < 120\ cm$ and 15% $< \epsilon_{gcz}$ $<$ 22%. So, the data set corresponds to different experimental runs, in which the manipulated variables, $U_g(s)$, $U_t(s)$ and the perturbation U_w were modified to vary $H(s)$ and $E_{gcz}(s)$ around their operational range values.

2.2 Definition of the structure

Second step: *For each transfer function $g_{ij}(s)$, choose the structure that best represents the system (1). Thus, the "definition of the structure", consists in selecting the degrees of the numerator and the denominator polynomials, i.e., the number of parameters to be estimated for each transfer function $g_{ij}(s)$ (equation (1)).*

To determine the transfer functions $g_{11}(s)$, $g_{12}(s)$ and $g_{13}(s)$ that relates $H(s)$ to its manipulated variable and to the perturbations, it is necessary, first, define the transfer functions structure. From a detailed analysis of the experimental data,the structure for these three transfer functions has been defined as being an integrator with a gain plus a transport delay. From the experimental modelling, it became evident the presence of a non-minimum phase zero in the transfer function between $H(s)$ and $U_g(s)$ (Persechini *et al.*, 2000). Therefore:

$$H(s) = \frac{K_{h_{uW}} e^{-10s} U_w(s) - K_{h_{uT}} e^{-10s} U_t(s)}{s} \qquad (2)$$
$$+ \frac{-K_{h_{ug1}} U_g}{s} + K_{h_{ug2}} e^{-60s} U_g'(s),$$

where

$$U_g'(s) = \frac{-681.88s + 1}{(80.68s + 1)(486.46s + 1)} U_g(s). \qquad (3)$$

The transfer function U_g' eq.(3) was experimentally determined in (Persechini, 2001).

The transfer functions $g_{21}(s)$ $g_{22}(s)$ and $g_{23}(s)$ which define the relations between $E_{gcz}(s)$ and $U_t(s)$, $U_g(s)$ and $U_w(s)$, respectively have also to be estimated. In this case, a good representation for these transfer functions (as experimentally determined in (Persechini *et al.*, 2004)), is a first order system with a transport delay. Therefore:

$$E_{gcz}(s) = \frac{k_{\epsilon_{ug}}\ e^{-90s}}{\tau_{\epsilon_{ug}} s + 1} U_g + \qquad (4)$$
$$\frac{k_{\epsilon_{uT}}\ e^{-20s}}{\tau_{\epsilon_{uT}} s + 1} U_t + \frac{k_{\epsilon_{uW}}\ e^{-20s}}{\tau_{\epsilon_{uW}} s + 1} U_w$$

For both equations (2) and (4) it was also defined the transport delay for each transfer function. These delays have been selected to represent the mean value between the greatest and the smallest delay found in each experimental run. The gains and the time constants in equations (2) and (4) are to be estimated (see next step).

2.3 Parameter identification

Third step: *Get, by means of parameter identification techniques, the numerical values for the transfer function's coefficients and validate them using experimental data.*

The coefficient estimation for the equations (4) and (2) was done by means of a recursive least square algorithm for parameter estimation, with a forgetting factor equals 0.99.

2.4 Determination of the median response

Fourth step: *Determine, for each frequency, the median response between the greatest and the smallest amplitude, for each transfer function g_{ij}.*

Once the parameters of the transfer functions g_{11}, g_{12} and g_{13} have been estimated, their Bode diagrams were plot, for each experimental run (that is it, for each g_{ij}^k, $k = 1, \ldots, 24$), as shown in Fig. 1, where the dotted lines represent the frequency responses of the entries g_{ij}^k and the solid line is the frequency response of the nominal model estimated from the median response (see section 2.5), g_{0ij}.

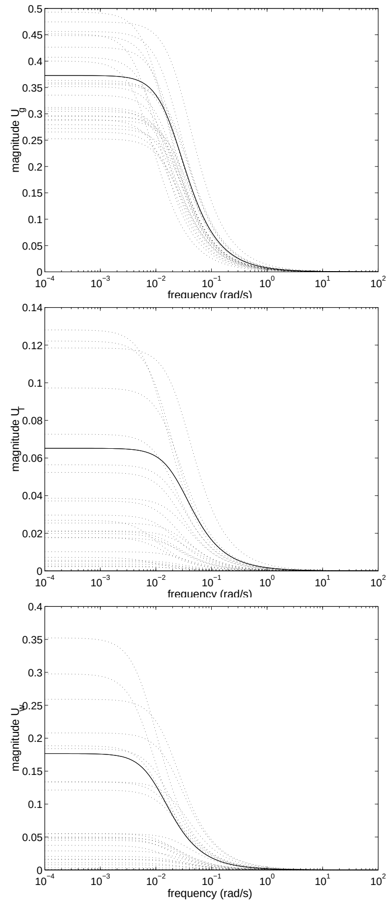

Fig. 1. Frequency response for the transfer functions g_{21}, g_{22} and g_{23}, associated with the relation of $E_{gcz}(s)$ to the variables U_g, U_t and U_w for each one of the experiments (dotted line) and for the nominal model (solid line).

For the froth layer height, it is necessary to determine only the gains, K, as its structure is defined as being an integrator, K/s. From experimental data $K_{h_{uW}}$ varies from -0.007 to -0.063; $K_{h_{uT}}$ varies from 0.029 to 0.003; $K_{h_{ug1}}$ varies from -0.002 to -0.028; and $K_{h_{ug2}}$ varies from 8.550 to 0.271.

2.5 Determination of the nominal model

Fifth setp: *Determine the nominal model as being the most suitable transfer function that fits the median response.*

Once the structure has been defined and all the transfer functions coefficients have been estimated, the final nominal model is determined.

The nominal model for the holdup, $E_{0_{gcz}}(s)$, is determined from the median response, in each frequency, as shown in Fig. 1. The final nominal model is given by:

$$E_{0_{gcz}}(s) = \frac{0.37\ e^{-60s}}{48.26s+1}U_g(s) + \frac{0.07\ e^{-20s}}{38.11s+1}U_t(s) + \frac{0.18\ e^{-20s}}{94.91s+1}U_w(s). \quad (5)$$

The nominal model for the froth layer height, H_0, is given by:

$$H_0(s) = \frac{-0.034e^{-10s}U_w(s) + 0.016U_t(s) - 0.015U_g(s)}{s} + 4.414e^{-60s}U_g'(s), \quad (6)$$

where the gains $K_{h_{uW}} = -0.035$, $K_{h_{uT}} = 0.016$, $K_{h_{ug1}} = -0.015$ and $K_{h_{ug2}} = 4.414$ were determined by the arithmetic mean between the greatest and the smallest estimated value.

3. UNCERTAINTY MODEL DEFINITION

The uncertainty model will be represented as:

$$G_p(s) = G_0(s)(I + \Delta(s)W(s)) \quad (7)$$

where $\Delta(s)$ is any matrix such that $||\Delta||_\infty \leq 1$ and $W(s)$ is a diagonal matrix that represents the weights of the uncertainties as function of frequency.

To determine $W(s)$ from the experimental data, it is necessary to find, for each frequency and for all models g_{ij}^k, the smallest circle with origin on the nominal model, $g_{0ij}(j\omega)$ and radius $l(\omega)$ such that:

$$l_{ij}(w) = \max_{g_{ij}^k \in G_p} \left| \frac{g_{ij}^k(jw) - g_{0ij}(jw)}{g_{0ij}(jw)} \right|, \quad (8)$$

and

$$|W_{ij}(jw)| \geq l_{ij}(w),\ \forall w. \quad (9)$$

3.1 Determination of $l_{ij}(w)$ corresponding to each g_{ij}

First step: *Apply the equation (8) to all estimated model, g_{ij}^k, and get the amplitude value of $l_{ij}(j\omega)$, for each frequency value.*

By using equation (8) on the set of estimated models (dotted curves of Fig. 1) and the nominal model given by equation (5), it is possible to get the frequency response that represents the difference between the nominal model and each one of the estimated models.

3.2 Determination of the biggest deviation

Second step: *Determine, for each frequency, the value of $W_{ij}(j\omega)$, using the equation (9), that corresponds to the greatest deviation from the nominal model.*

The weighting functions $W_{ij}(s)$ are defined in equation (9) and correspond to the greatest deviation from the nominal model and are represented by means of the dotted curves of figures 2, 3 and 4, associated to, respectively, the inputs U_g, U_T and U_W.

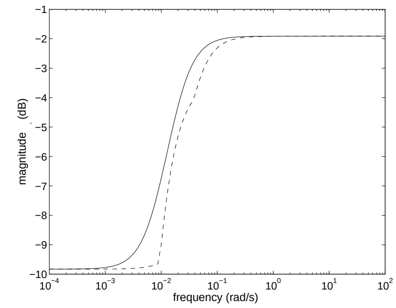

Fig. 2. Uncertainty of ϵ_{gcz} associated with the variable U_g.Frequency response of $W_{\epsilon_g u_g}(s)$ (solid line); greatest deviation from the nominal model (dotted line)

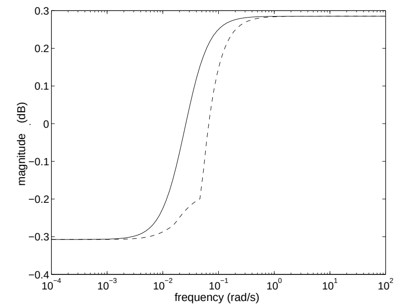

Fig. 3. Uncertainty of ϵ_{gcz} associated with the variable U_T.Frequency response of $W_{\epsilon_g u_T}(s)$ (solid line); greatest deviation from the nominal model (dotted line)

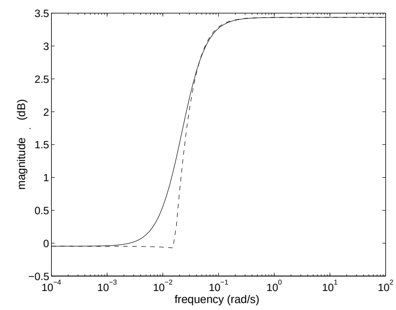

Fig. 4. Uncertainty of ϵ_{gcz} associated with the variable U_w.Frequency response of $W_{\epsilon_g u_w}(s)$ (solid line); greatest deviation from the nominal model (dotted line)

3.3 Determination of the $W(s)$ Structure

Third step: *Get the $W_{ij}(s)$ representation, which can be expressed by:*

$$W_{ij}(s) = \frac{s/w_B + A}{s/(w_B M) + 1}, \quad (10)$$

where A is the greatest deviation from the lowest frequency, M is the greatest deviation from the highest frequency, and ω_B must be adjusted such as the frequency response of equation (10) ever be bigger or equal to the biggest deviation.

Using the equation (10) the weights $W_{\epsilon_g u_g}(s)$, $W_{\epsilon_g u_t}(s)$ and $W_{\epsilon_g u_w}(s)$, associated, respectively to inputs signals U_g, U_T e U_W are given by:

$$W_{22} = W_{\epsilon_g u_g}(s) = \frac{40s + 0.32}{49.84s + 1}, \quad (11)$$

$$W_{21} = W_{\epsilon_g u_t}(s) = \frac{40s + 0.96}{38.71s + 1}, \quad (12)$$

$$W_{23} = W_{\epsilon_g u_w}(s) = \frac{55s + 0.99}{37.03s + 1}. \quad (13)$$

The frequency responses of equations (11), (12) and (13) are represented by the solid lines of figures 2*(b)*, 3*(b)* and 4*(b)*, respectively.

For $H(s)$, the uncertainties are associated only to the gains $K_{h_{uW}}$, $K_{h_{uT}}$, $K_{h_{ug1}}$ and $K_{h_{ug2}}$, given by equation (6).

The corresponding W_{ij} are calculated from the variations between the gain of the nominal model and the estimated gains for each one of the estimated models.

The calculated W_{ij} are given below:

$$W_{13} = W_{h_{uW}} = 0.799, \quad (14)$$

$$W_{11} = W_{h_{uT}} = 0.847, \quad (15)$$

$$W_{12} = W_{h_{ug}} = 0.901, \quad (16)$$

where equations (14) and (15) correspond, respectively, to the weighting functions associated to the input signal U_W and U_T and the equation (16) represent a weight to the different values of $K_{h_{ug1}}$ and $K_{h_{ug1}}$.

4. $\mathcal{H}_\infty$ DESIGN AND EXPERIMENTAL RESULT

After have gotten the complete model the $\mathcal{H}_\infty$ design using the mixed sensitivity approach was done.

The main control objective for the flotation process is to keep h and ϵ_{gcz} on a fix value despite perturbations on the variable U_W; or despite set-points variations with U_T and U_g as manipulated variables.

The feedback control scheme can be represented by the block diagram of Fig. 5, which is a generic

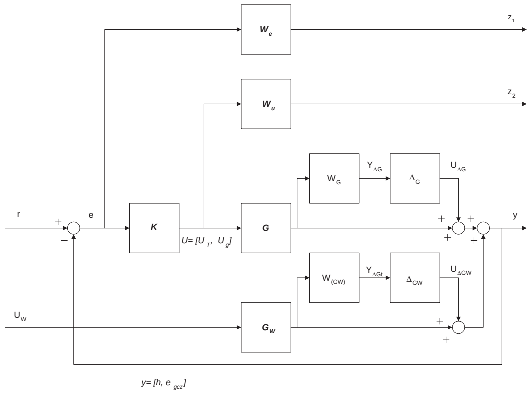

Fig. 5. Block diagram for the $\mathcal{H}_\infty$ design

formulation for $\mathcal{H}_\infty$ design. On this figure, G and G_W are, respectively, the nominal model of the process and for the perturbation. These models can be determined from equations (5) and (6), considering the output signal $y = [h, \epsilon_{gcz}]$ and input signal $U = [U_T, U_g]$ e U_W.

For this configuration, the generalized plant (Skogestad and Postlethwaite, 1996), P is given as:

$$\begin{bmatrix} Y_{\Delta_G} \\ Y_{\Delta_{GW}} \\ z_1 \\ z_2 \\ e \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & W_G G \\ 0 & 0 & 0 & W_{GW} G_W & 0 \\ -W_e & -W_e & W_e & -W_e G_W & -W_e G \\ 0 & 0 & 0 & 0 & W_u \\ -I & -I & I & -G_W & G \end{bmatrix} \begin{bmatrix} U_{\Delta_G} \\ U_{\Delta_{GW}} \\ r \\ U_W \\ U \end{bmatrix} \quad (17)$$

and the controller is calculated from the minimization problem:

$$\left\| \begin{bmatrix} -W_G T & -W_G T & W_G T & -W_G T G_W \\ 0 & 0 & 0 & W_T G_W \\ -W_e S & -W_e S & W_e S & -W_e S G_W \\ -W_u K S & -W_u K S & W_u K S & -W_u K S G_W \end{bmatrix} \right\|_\infty . \quad (18)$$

The weighting functions W_e and W_u were defined from the desired closed-loop time response (Persechini, 2001).

The uncertainty representation W_G was obtained from combinations between equations 12 and 11 and equations 15 and 16 while the matrix W_{GW} was obtained from equations 14 and 13:

$$W_G = \begin{bmatrix} \frac{40s + 0.418}{47.37s + 1} & 0 \\ 0 & 0.878 \end{bmatrix}, \quad (19)$$

and

$$W_{GW} = \begin{bmatrix} 0.799 & 0 \\ 0 & \frac{55s + 0.99}{37.03s + 1} \end{bmatrix}. \quad (20)$$

The design of the controller has been accomplished with the aid of the Matlab Robust Control and Mu-Analysis toolboxes (Balas *et al.*, 1992), (Chiang and Safonov, 1992). The resulting controller had 17 states; it has been implemented as a discrete controller with 12 states, which has been obtained using a model reduction procedure (Balas *et al.*, 1992).

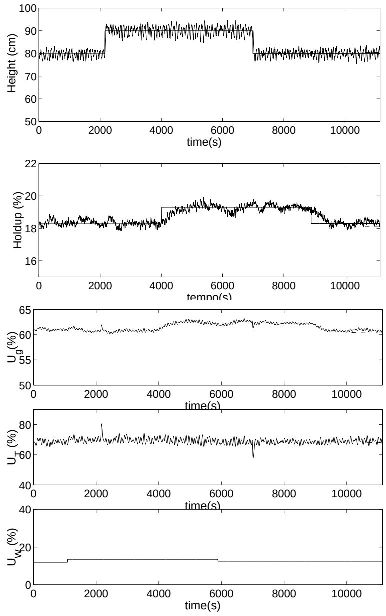

Fig. 6. Closed-loop experiment for water-air system. Controlled variables h and ϵ_{gcz}; Manipulated variables u_g and u_t; perturbation u_w.

In Fig. 6, it is presented the time response of an experimental run where the objective was to control the froth layer height, h, using the manipulated variable U_T and the hold up, ϵ_{gcz}, using the manipulated variable U_g. As it can be seen in Fig. 6, the close-loop response presented low overshot and small variations in the manipulated variables. It is also possible to see that, from the time domain analysis, the system is robust against setpoint variation and perturbations. For instance, at time t = 2150, the froth layer height, h, was varied from 80 to 90 and the hold up keep in its original value (18.3%) and, at t = 5890, even in the presence of a perturbation (U_w varying from 13.5 to 12%), both h and the hold up, ϵ_{gcz}, stayed in their original values.

From the comparison of the closed loop responses (Fig. 6) with the ones obtained with a PI controller (see, for example Persechini *et al.* (2004)), it is possible to conclude that both controllers presented similar results. Nevertheless, the design of the PI controller has not been made considering the variations in the process characteristics, whereas the $\mathcal{H}_\infty$ design implicitly took them into account. Therefore, it is expected that, even under significant parameter variations, the $\mathcal{H}_\infty$ control will guarantee a certain level of performance, due to its inherent robustness.

5. CONCLUSIONS

The modelling procedure considers a data set that represents the different operating conditions and a corresponding set of estimated parameter that represents a process model. The least square algorithm proved to be an efficient tool for the identification of the transfer function coefficients. The use of the median response in the frequency domain to obtain the nominal model allowed the determination of the uncertainty under the considered operation condition.

This procedure proved to be useful to obtain the nominal model and the associated uncertainty that represents the real process.

Based on the derived model, an $\mathcal{H}_\infty$ control has been designed using the mixed sensitivity framework. The generalized plant considered the uncertainty in the model and weighting functions according to a desired closed-loop time response.

The validity of the model and the controller has been verified through experimental results. Therefore, using this $\mathcal{H}_\infty$ controller, the closed loop process presented a good tracking response and a good disturbance rejection.

This methodology was applied to a real process and the same procedure could be used in similar process where the experimental data is the main information and model structure is only a complementary information.

REFERENCES

Balas, Gary J., Jonh C. Doyle, Keith Glover, Andy Packard and Roy Smith (1992). *μ-Analysis and Synthesis Toolbox - User's Guide*. The MathWorks Inc.

Chiang, R. Y. and M. G. Safonov (1992). *Robust Control Toolbox, User's Guide*. The MathWorks Inc.

Miranda, Mrcio Fantini (2000). Controle Multivarivel na Presena de Incertezas. Tese de doutorado. Universidade Federal de Minas Gerais. Belo Horizonte, Minas Gerais, Brasil.

Persechini, M. A. M., F. G. Jota and A. E. C. Peres (2000). Dynamic model of a flotation column. *Minerals Engineering* **13**(14-15), 1465–1481.

Persechini, Maria Auxiliadora Muanis (2001). Estratégias de Controle para Processo de Flotação em Coluna. PhD thesis. Universidade Federal de Minas Gerais.

Persechini, Maria Auxiliadora Muanis, Antônio Eduardo Clark Peres and Fábio Gonçalves Jota (2004). Control strategy for a column flotation process. *Control Engineering Practice* **12**, 963–976.

Skogestad, Sigurd and Ian Postlethwaite (1996). *Multivariable Feedback Control - Analysis and Design*. John Wiley and Sons.