STATE AND UNKNOWN INPUTS ESTIMATION FOR A CLASS OF NONLINEAR SYSTEMS

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Abstract: A high gain observer is proposed for a class of multi-output nonlinear systems with unknown inputs in order to simultaneously estimate the whole state as well as the unknown inputs. The gain of this observer does not require the resolution of any dynamical system and is analytically given. Moreover, its tuning is reduced to the choice of two real numbers. The performances of the proposed observer are demonstrated in simulation through an illustrative example.

Keywords: Nonlinear system, High gain observer, Unknown inputs.

1. INTRODUCTION

Over the last twenty years, many researches have focused on the observer design for linear systems with unknown inputs (Johnson, 1975; Kudva *et al.*, 1980; Hou and Müller, 1992; Guan and Saif, 1991; Darouach *et al.*, 1994). In most cases, the objective was to estimate the non measured state variables and the proposed observers do not provide any information on the unknown inputs. In a relatively recent paper (Corless and Tu, 1998), the authors proposed a LMI based observer in order to jointly estimate the missing states and the unknown inputs. However, strong conditions are assumed to ensure the convergence of the inputs estimates. In a more recently paper (Xiong and Saif, 2003), the authors proposed reduced order observers to simultaneously estimate state and the unknown inputs when the latter vary slowly. Other results on unknown observers synthesis for some particular classes of nonlinear systems can be found in (Xiong and Saif, 2001; Farza *et al.*, 2004; Ha and Trinh, 2004).

In this paper, one proposes a full order high gain observer for the simultaneous estimation of the non measured states and the unknown inputs. The proposed approach does not necessitate the output differentiation and it only assumes that the dynamics of these inputs are bounded without making any hypothesis on how these inputs vary.

This paper is organized as follows. In the next section, the class of nonlinear systems which is the basis of the observer design is introduced. Section 3 is devoted to the observer synthesis. For sake of clarity, only relevant results are given and corresponding proofs are reported in appendices. Section 4 is devoted to a simulation example in order to highlight the performance of the proposed observer.

2. PROBLEM FORMULATION

Consider MIMO systems of the form:

$$\begin{cases} \dot{x} = f(w, x) \\ y = \bar{C}x = x^1 \end{cases} \tag{1}$$

with $x = \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^q \end{pmatrix}$; $f(w,x) = \begin{pmatrix} f^1(w,x^1,x^2) \\ f^2(w,x^1,x^2,x^3) \\ \vdots \\ f^{q-1}(w,x) \\ f^q(w,x) \end{pmatrix}$;

$$\bar{C} = [I_{n_1}, 0_{n_1\times n_2}, 0_{n_1\times n_3}, \ldots, 0_{n_1\times n_q}] \quad (2)$$

where the state $x \in \mathbb{R}^n$ with $x^k \in \mathbb{R}^{n_k}$, $k = 1,\ldots,q$ and $p = n_1 \geq n_2 \geq \ldots \geq n_q$, $\sum_{k=1}^{q} n_k = n$; the input $w(t) \in \mathcal{U}$ the set of bounded absolutely continuous functions with bounded derivatives from $\mathbb{R}^+$ into W a compact subset of $\mathbb{R}^s$; the output $y \in \mathbb{R}^p$ and $f(u,x) \in \mathbb{R}^n$ with $f^k(u,x) \in \mathbb{R}^{n_k}$. The functions f^k are assumed to satisfy the following hypothesis:

(H1) Each function $f^k(w,x)$, $k = 1,\ldots,q-1$ satisfies the following rank condition:

$Rank(\frac{\partial f^k}{\partial x^{k+1}}(w,x)) = n_{k+1} \;\; \forall x \in \mathbb{R}^n; \forall w \in W$

Moreover, one assumes that $\exists \alpha_f, \beta_f > 0$ such that for all $k \in \{1,\ldots,q-1\}$, $\forall x \in \mathbb{R}^n$, $\forall w \in W$, $\alpha_f^2 I_{n_{k+1}} \leq \left(\frac{\partial f^k}{\partial x^{k+1}}(w,x)\right)^T \frac{\partial f^k}{\partial x^{k+1}}(w,x) \leq \beta_f^2 I_{n_{k+1}}$

(H2) For $1 \leq k \leq q-1$; the map $x^{k+1} \mapsto f^k(w,x^1,\ldots,x^k,x^{k+1})$ from $\mathbb{R}^{n_{k+1}}$ into $\mathbb{R}^{n_k}$ is one to one.

System (1) has been considered in (Hammouri and Farza, 2003) and it characterizes a subclass of U-uniformly observable systems. In this paper, one shall suppose that a subset of the inputs is unknown. More precisely, one shall suppose that the vector $w(t)$ can be partitioned as follows $w(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$ where $v(t) \in \mathbb{R}^m$ is completely unknown and $u(t) \in \mathbb{R}^{s-m}$ is fully known. The objective then consists in synthesizing an observer to simultaneously estimate the vector of unknown inputs $v(t)$ and the non measured states without assuming any model for the unknown inputs. The synthesis of such observer necessitates the adoption of some hypothesis which will be stated in due courses. At this step, one assumes the following:

(H3) For $k = 1,\ldots,q$, each function f^k has the following structure:

• $f^k(u,v,x) = \widetilde{f}^k(u,x^1,\ldots,x^{k+1}) + G^k(u(t),s(t))v$ where $s(t)$ is a known signal with a bounded time derivative; $G^k(u(t),s(t)) \in \mathbb{R}^{n_k}$ and $G^1(u(t),s(t))$ satisfies the following rank condition:

$Rank(G) = Rank(G^1) = m, \;\; \forall\, u \in U$ and $\forall\, t \geq 0$

Moreover, one supposes that $\exists \alpha_G, \beta_G > 0$ such that $\forall u \in U, \forall t \geq 0, \;\; 0 < \alpha_G^2 I_m \leq (G^1)^T G^1 \leq \beta_G^2 I_m$.

Notice that hypothesis (H1) and (H2), satisfied by f^k, still be satisfied by $\widetilde{f}^k$.

(H4) The output x^1 can be partitioned as follows: $x^1 = \begin{pmatrix} x_1^1 \\ x_2^1 \end{pmatrix}$ with $x_1^1 \in \mathbb{R}^{m_1}$, $x_2^1 \in \mathbb{R}^{p-m_1}$ and $m \leq m_1 < p$. Such a partition induces the following ones $\widetilde{f}^1(u,x^1,x^2) = \begin{pmatrix} \widetilde{f}_1^1(u,x^1,x^2) \\ \widetilde{f}_2^1(u,x^1,x^2) \end{pmatrix}$ and $G^1(u,s) = \begin{pmatrix} G_1^1(u,s) \\ G_2^1(u,s) \end{pmatrix}$. The following two rank conditions are assumed to be satisfied:

(i) $Rank\left(G_1^1(u,s)\right) = m$,
for all $u \in U$ and for all $t \geq 0$

$(ii) Rank \begin{pmatrix} \frac{\partial \widetilde{f}_1^1}{\partial x^2}(u,x^1,x^2) & G_1^1(u,s) \\ \frac{\partial \widetilde{f}_2^1}{\partial x^2}(u,x^1,x^2) & G_2^1(u,s) \end{pmatrix} = n_2 + m$
for all $x \in \mathbb{R}^n$, $u \in U$ and $t \geq 0$

(H5) The time derivative of the unknown input $v(t)$ is a completely unknown function, $\varepsilon(t)$, which is uniformly bounded that is $\sup_{t\geq 0} \|\varepsilon(t)\| \leq \beta_\varepsilon$ where $\beta_\varepsilon > 0$ is a real number.

To summarize, the nonlinear system which will be considered with view to observer synthesis can be written under the following condensed form:

$$\begin{cases} \dot{X}^1 = \widetilde{f}_X^1(u,X_1^1,X_1^2,X_2^2) + G_X^1(u,s)X_2^1 + \bar{\varepsilon}(t) \\ \dot{X}^2 = \widetilde{f}_X^2(u,X_1^1,X^2) + G_X^2(u,s)X_2^1 \\ y = \begin{pmatrix} y_1 = X_1^1 = x_1^1 \\ y_2 = X_1^2 = x_2^1 \end{pmatrix} \end{cases} \quad (3)$$

where $X = \begin{pmatrix} X^1 \\ X^2 \end{pmatrix} \in \mathbb{R}^{n+m}$; $X^1 = \begin{pmatrix} X_1^1 = x_1^1 \\ X_2^1 = v \end{pmatrix} \in \mathbb{R}^{m_1+m}$; $X^2 = \begin{pmatrix} X_1^2 = x_2^1 \\ X_2^2 = x^2 \\ \vdots \\ X_q^2 = x^q \end{pmatrix} \in \mathbb{R}^{n-m_1}$; $\bar{\varepsilon}(t) = \begin{pmatrix} 0 \\ \varepsilon(t) \end{pmatrix}$; $\widetilde{f}_X^1(u,X_1^1,X_1^2,X_2^2) = \begin{pmatrix} \widetilde{f}_1^1(u,x^1,x^2) \\ 0 \end{pmatrix}$, $\widetilde{f}_X^2(u,X_1^1,X^2) = \begin{pmatrix} \widetilde{f}_2^1(u,x^1,x^2) \\ \widetilde{f}^2(u,x^1,x^2,x^3) \\ \vdots \\ \widetilde{f}^q(u,x) \end{pmatrix}$; $G_X^1(u,s) = \begin{pmatrix} G_1^1(u,s) \\ 0 \end{pmatrix}$; $G_X^2(u,s) = \begin{pmatrix} G_2^1(u,s) \\ G^2(u,s) \\ \vdots \\ G^q(u,s) \end{pmatrix}$.

3. OBSERVER SYNTHESIS

Before giving the equations of the proposed observer, one shall introduce some notations and preliminary results.

• Let $\theta_1, \theta_2 > 0$ be two real numbers and let $\Delta_1(\theta_1)$ and $\Delta_2(\theta_2)$ be the following two block diagonal matrices:

$$\Delta_1(\theta_1) = diag(I_{m_1}, \frac{1}{\theta_1} I_{m_1}) \quad (4)$$

$$\Delta_2(\theta_2) = diag(I_{p-m_1}, \frac{1}{\theta_2} I_{p-m_1}, \ldots, \frac{1}{\theta_2^{q-1}} I_{p-m_1})$$

• For $i = 1, 2$, let S_i be the unique solution of the algebraic Lyapunov equation :

$$S_i + A_i^T S_i + S_i A_i - C_i^T C_i = 0 \tag{5}$$

where $A_1 = \begin{bmatrix} 0 & I_{m_1} \\ 0 & 0 \end{bmatrix}$ and

$$A_2 = \begin{bmatrix} 0 & I_{p-m_1} & 0 & 0 \\ \vdots & & I_{p-m_1} & \\ 0 & & \ddots & I_{p-m_1} \\ 0 & \dots & 0 & 0 \end{bmatrix} \tag{6}$$

are respectively $2m_1 \times 2m_1$ and $q(p-m_1) \times q(p-m_1)$ square matrices and

$$C_1 = [I_{m_1} 0_{m_1}] \,;\, C_2 = [I_{p-m_1} 0_{p-m_1} \dots 0_{p-m_1}] \tag{7}$$

are respectively $m_1 \times 2m_1$ and $(p-m_1) \times q(p-m_1)$ rectangular matrices. It can be shown that S_1 and S_2 are symmetric positive definite and that one has:

$$S_1^{-1} C_1^T = \begin{bmatrix} 2I_{m_1} \\ I_{m_1} \end{bmatrix} \;;\; S_2^{-1} C_2^T = \begin{bmatrix} C_q^1 I_{p-m_1} \\ C_q^2 I_{p-m_1} \\ \vdots \\ C_q^q I_{p-m_1} \end{bmatrix} \tag{8}$$

where $C_j^i = \dfrac{j!}{i!(j-i)!}$

• Let $D(u,s)$ be the following $(n-m_1) \times m_1$ rectangular matrix:

$$D(u,s) = \begin{pmatrix} G_2^1(u,s) \left(G_1^1(u,s)\right)^+ \\ G^2(u,s) \left(G_1^1(u,s)\right)^+ \\ \vdots \\ G^q(u,s) \left(G_1^1(u,s)\right)^+ \end{pmatrix} \tag{9}$$

where the notation $(\cdot)^+$ means the left inverse of $(\cdot)$.

• $\forall \xi_1^1 \in \mathbb{R}^{m_1}$, $\forall \xi_1^2 \in \mathbb{R}^{p-m_1}$, $\forall \xi_2^2 \in \mathbb{R}^{n_2}$, set:

$$\begin{aligned} \bar{f}_1^2(u,s,\xi_1^1,\xi_1^2,\xi_2^2) &= \widetilde{f}_2^1(u,\xi_1^1,\xi_1^2,\xi_2^2) \\ &- G_2^1(u,s)\left(G_1^1(u,s)\right)^+ \widetilde{f}_1^1(u,\xi_1^1,\xi_1^2,\xi_2^2) \end{aligned} \tag{10}$$

One states the following (see the appendix for the proof):

Lemma 1 *Under hypothesis (H1) and (H4), one has:*

$$Rank\left(\frac{\partial \bar{f}_1^2}{\partial \xi_2^2}(u,s,\xi_1^1,\xi_1^2,\xi_2^2)\right) = n_2 \tag{11}$$

• Set $\Lambda_1(u,s) = diag\left(I_{m_1}, G_1^1(u,s)\right)$ (12)

• $\forall \xi_1^1 \in \mathbb{R}^{m_1}$, $\forall \xi_1^2 \in \mathbb{R}^{p-m_1}$, $\forall \xi_k^2 \in \mathbb{R}^{n_k}$, $k = 2, \dots, q$, let Λ_2 be the following block diagonal matrix:

$$\Lambda_2(u,s,\xi_1^1,\xi_1^2,\xi_2^2,\dots,\xi_q^2) = diag\Bigg(I_{p-m_1}, \frac{\partial \bar{f}_1^2}{\partial \xi_2^2}(u,s,\xi_1^1,\xi_1^2,\xi_2^2), \tag{13}$$
$$\frac{\partial \bar{f}_1^2}{\partial \xi_2^2}(u,s,\xi_1^1,\xi_1^2,\xi_2^2)\frac{\partial \widetilde{f}^2}{\partial \xi_3^2}(u,\xi_1^1,\xi_1^2,\xi_2^2,\xi_3^2),\dots,$$
$$\frac{\partial \bar{f}_1^2}{\partial \xi_2^2}(u,s,\xi_1^1,\xi_1^2,\xi_2^2)\prod_{k=2}^{q-1}\frac{\partial \widetilde{f}^k}{\partial \xi_{k+1}^2}(u,\xi_1^1,\xi_1^2,\dots,\xi_{k+1}^2)\Bigg)$$

Notice that according to Hypothesis (H1),(H4) and lemma 1, the matrices Λ_1 and Λ_2 are of full rank.

Now, consider the following dynamical system:

$$\begin{cases} \dot{\hat{X}}^1 = \widetilde{f}_X^1(u,\hat{X}_1^1,\hat{X}_1^2,\hat{X}_2^2) + G_X^1(u,s)\hat{X}_2^1 \\ \quad -\theta_1 \Lambda_1^+(u,s)\Delta_1^{-1}(\theta_1)S_1^{-1}C_1^T(\hat{X}_1^1 - y_1) \\ \dot{\hat{X}}^2 = \widetilde{f}_X^2(u,\hat{X}_1^1,\hat{X}^2) + G_X^2(u,s)\hat{X}_2^1 \\ \quad -\theta_2\Lambda_2^+\left(u,s,X_1^1,\tilde{X}^2\right)\Delta_2^{-1}(\theta_2)S_2^{-1}C_2^T(\hat{X}_1^2 - y_2) \\ \quad -2\theta_1 D(u,s)(\hat{X}_1^1 - y_1) \end{cases} \tag{14}$$

where $\hat{X} = \begin{pmatrix} \hat{X}^1 \\ \hat{X}^2 \end{pmatrix} \in \mathbb{R}^{n+m}$ with $\hat{X}^1 = \begin{pmatrix} \hat{x}_1^1 \\ \hat{v} \end{pmatrix} \in \mathbb{R}^{m_1+m}$, $\hat{x}_1^1 \in \mathbb{R}^{m_1}$, $\hat{v} \in \mathbb{R}^m$, $\hat{X}^2 = \begin{pmatrix} \hat{x}_2^1 \\ \hat{x}^2 \\ \vdots \\ \hat{x}^q \end{pmatrix} \in \mathbb{R}^{n-m_1}$, $\hat{x}_1^2 \in \mathbb{R}^{p-m_1}$, $\hat{x}^k \in \mathbb{R}^{n_k}$, $k = 2, \dots, q$; $\tilde{X}^2 = \hat{X}^2 - D(u,s)(\hat{X}_1^1 - y_1)$, Λ_1 and Λ_2 are respectively given by (12) and (13); $\Delta_k(\theta_k)$ and $S_k^{-1}C_k^T$, $k = 1, 2$ are respectively given by (4) and (8); the matrix D is given by (9); $\theta_1, \theta_2 > 0$ are real numbers.

One now states the main result (a sketch of the proof is given in the appendix):

Theorem 1: *Suppose that system (3) satisfies hypothesis (H1) to (H5). Then,*

$$\exists \theta_{1,0} > 0;\ \exists \theta_{2,0} > 0;\ \forall \theta_1 > \theta_{1,0};\ \forall \theta_2 > \theta_{2,0};$$

$$\exists \lambda > 0;\ \exists \mu(\theta_1,\theta_2) > 0;\ \exists M(\theta_1,\theta_2) > 0;$$

$$\forall u \in U;\ \forall \hat{X}(0) \in R^{n+m};\ \textit{one has}\ :$$

$$\|e(t)\| \leq \lambda\theta_1 exp\left(-\mu(\theta_1,\theta_2)t\right)\|e(0)\| + M(\theta_1,\theta_2)\beta_\varepsilon$$

where $e(t) = \hat{X}(t) - X(t)$ with $X(t)$ is the unknown trajectory of system (3) associated to the input u, $\hat{X}(t)$ is any trajectory of system (14) associated to the input u and the outputs y_1 and y_2; β_ε is the upper bound of $\|\varepsilon(t)\|$ given in hypothesis (H5). Moreover, one has: $\lim\limits_{\theta_1,\theta_2 \to \infty} \mu(\theta_1,\theta_2) = +\infty$ and $\lim\limits_{\theta_1,\theta_2\to\infty} M(\theta_1,\theta_2) = 0$.

Remark: Observe that for $\varepsilon(t) = 0$, i.e. when the unknown inputs are constant, the convergence of the estimation error is exponential. In the case where $\|\varepsilon(t)\| \neq 0$ but bounded by β_ε, the asymptotic estimation error can be made as small as desired by choosing values of θ_1 and θ_2 high enough.

4. EXAMPLE

Consider the following dynamical system:

$$\begin{cases} \dot{x}_1 &= (a - x_3)x_4 - x_3 v(t) - x_1 \\ \dot{x}_2 &= x_3 x_4 + (a - x_3)v(t) - x_2 \\ \dot{x}_3 &= x_4(1 + x_4^2) - x_3^3 - 10sin(t)v(t) \\ \dot{x}_4 &= x_5 - x_4^3 - 2cos(t)v(t) \\ \dot{x}_5 &= 5sin(2t)v(t) \\ y &= [x_1 \; x_2 \; x_3]^T \end{cases} \tag{15}$$

where $x = [x_1 \; x_2 \; x_3 \; x_4 \; x_5]^T \in \mathbb{R}^5$ with $x_i \in \mathbb{R}$, $v(t)$ is the unknown input and $a \neq 0$ is a real number. To simplify, no known input has been considered. For simulation purposes, the following expression (unknown by the observer) has been used for the unknown input:

$$v(t) = 5sin(5t) \tag{16}$$

It is easy to see that system (15) is under form (3) with:

$$x^1 = [x_1 \; x_2 \; x_3]^T; \; x^2 = x_4; \; x^3 = x_5;$$

$$\widetilde{f}^1(x^1, x^2) = \begin{pmatrix} (a - x_3)x_4 - x_1 \\ x_3 x_4 - x_2 \\ x_4(1 + x_4^2) - x_3^3 \end{pmatrix}; \widetilde{f}^2(x) = x_5 - x_4^3;$$

$$\widetilde{f}^3(x) = 0; \; G^1(s(t)) = \begin{pmatrix} -x_3(t) \\ a - x_3(t) \\ -10sin(t) \end{pmatrix};$$

$$G^2(s(t)) = -2cos(t); \; G^3(s(t)) = 5sin(2t)$$

Concerning the partition of x^1 needed in hypothesis (H4), one can consider the following one (the only possible partition in this example): $x_1^1 = [x_1 \; x_2]^T$ and $x_2^1 = x_3$. Now, one can easily check hypothesis (H1) to (H5) and an observer under form (14) can be used in order to achieve the required estimations.

4.1 simulation Results

An observer of the form (14) has been used in order to estimate x_4, x_5 and v. This observer has been simulated using data issued from simulation. In order to simulate practical situations, each of the measurements of x_1, x_2 and x_3 has been corrupted by a uniformly distributed random signal produced by SIMULINK with zero mean value and a standard deviation respectively equal to 10^{-3}, 10^{-3} and $3.2 \; 10^{-4}$. In figure 1, the true time evolutions of x_4, x_5 and v (issued from model simulation) are compared with their respective estimates provided by the observer. Notice that corresponding curves are almost superimposed. The employed values of θ_1 and θ_2 are respectively equal to 60 and 15. The initial conditions for the model and the observer are: $x_1(0) = \hat{x}_1(0) = 1$; $x_2(0) = \hat{x}_2(0) = 1$; $x_3(0) = \hat{x}_3(0) = 1$; $x_4(0) = 2$; $x_5(0) = 10$; $\hat{x}_4(0) = 0$; $\hat{x}_5(0) = 0$; $\hat{v}(0) = -1$. The obtained results clearly show the good agreement between the estimated and simulated variables. Recall that the expression of the unknown input (equation (16)) introduced for simulation purposes is ignored by the observer.

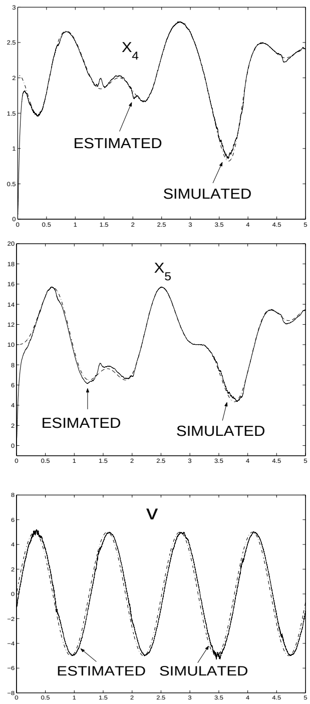

Fig. 1. Estimation of x_4, x_5 and v

Conclusion: A high gain observer has been designed for a class of nonlinear systems. The appealing features of the proposed observer are its implementation and calibration simplicity. The performances of the proposed observer have been demonstrated in simulation through an example. The use of the proposed observers in real experiments related to bioreactors and induction motors will be treated in upcoming works.

Appendix: Proofs

Proof of Lemma 1. Let

$$P(t) = \begin{pmatrix} I_{m_1} & 0 \\ -G_2^1(u(t), s(t)) \left(G_1^1(u(t), s(t))\right)^+ & I_{p-m_1} \end{pmatrix}$$

Since $rank(P(t)) = p$ for all $t \geq 0$ and $p \geq n_2 + m$ (according to (ii) of (H4)), one has:

$$\begin{aligned} n_2 + m &= Rank \\ &\left(P(t) \cdot \begin{pmatrix} \frac{\partial \widetilde{f}_1^1}{\partial \xi_2^2}(u, \xi_1^1, \xi_1^2, \xi_2^2) & G_1^1(u, s) \\ \frac{\partial \widetilde{f}_2^1}{\partial \xi_2^2}((u, \xi_1^1, \xi_1^2, \xi_2^2) & G_2^1(u, s) \end{pmatrix} \right) \\ &= Rank \begin{pmatrix} \frac{\partial \widetilde{f}_1^1}{\partial \xi_2^2}(u, \xi_1^1, \xi_1^2, \xi_2^2) & G_1^1(u, s) \\ \frac{\partial \bar{f}_1^2}{\partial \xi_2^2}(u, s, \xi_1^1, \xi_1^2, \xi_2^2) & 0 \end{pmatrix} \\ &= Rank \left(\frac{\partial \bar{f}_1^2}{\partial \xi_2^2}(u, s, \xi_1^1, \xi_1^2, \xi_2^2) \right) + m \end{aligned}$$

This leads to (11).

Sketch of the proof of Theorem 1. One shall introduce two changes of coordinates. Consider the following first one: $T(u, s) : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ $X \mapsto \bar{x} = \begin{pmatrix} \bar{x}^1 \\ \bar{x}^2 \end{pmatrix} = T(u, s)X$ with $\bar{x}^1 = \begin{pmatrix} \bar{x}_1^1 \\ \bar{x}_2^1 \end{pmatrix} \in$

$\mathbb{R}^{m_1+m}$; $\bar{x}_1^1 \in \mathbb{R}^{m_1}$; $\bar{x}_2^1 \in \mathbb{R}^m$; $\bar{x}^2 = \begin{pmatrix} \bar{x}_1^2 \\ \vdots \\ \bar{x}_q^2 \end{pmatrix} \in \mathbb{R}^{n-m_1}$, $\bar{x}_1^2 \in \mathbb{R}^{p-m_1}$, $\bar{x}_k^2 \in \mathbb{R}^{n_k}$, $k = 2, \dots, q$ and where $T(u,s)$ is the following $(n+m)\times(n+m)$ non singular square matrix :

$$T(u,s) = \begin{pmatrix} I_{m_1} & 0 & 0 & \dots & \dots & 0 \\ 0 & I_m & 0 & \dots & \dots & 0 \\ -\gamma_1^2(u,s) & 0 & I_{p-m_1} & 0 & \dots & \vdots \\ -\gamma_2^2(u,s) & 0 & 0 & I_{n_2} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ -\gamma_q^2(u,s) & 0 & \dots & & 0 & I_{n_q} \end{pmatrix} \quad (17)$$

with $\gamma_1^2(u,s) \triangleq G_2^1(u,s)\left(G_1^1(u,s)\right)^+$ and $\gamma_k^2(u,s) \triangleq G^k(u,s)\left(G_1^1(u,s)\right)^+$, $k = 2, \dots, q$.

The objective of this transformation is to generate a subsystem that does not depend on the unknown input ($v = \bar{x}_2^1$). Indeed, one can show that this transformation puts system (3) under the following form:

$$\begin{cases} \dot{\bar{x}}^1 = \bar{f}^1(u,s,\bar{x}_1^1,\bar{x}_1^2,\bar{x}_2^2) + G(u,s)\bar{x}^1 + \bar{\varepsilon}(t) \\ y_1 = \bar{h}_1(\bar{x}^1) \triangleq \bar{C}_1\bar{x}^1 \\ \dot{\bar{x}}^2 = \bar{f}^2(u,s,\bar{x}_1^1,\bar{x}^2) + \bar{g}^2(u,s,\bar{x}_1^1,\bar{x}^2) \\ y_2 = \bar{h}_2(\bar{x}^1,\bar{x}^2) \triangleq \bar{x}_1^2 + \gamma_1^2(u,s)\bar{x}_1^1 \\ \qquad = \bar{C}_2\bar{x}^2 + \gamma_1^2(u,s)\bar{C}_1\bar{x}^1 \end{cases} \quad (18)$$

where $\bar{f}^1(u,s,\bar{x}_1^1,\bar{x}_1^2,\bar{x}_2^2) = \begin{pmatrix} \bar{f}_1^1(u,s,\bar{x}_1^1,\bar{x}_1^2,\bar{x}_2^2) \\ 0 \end{pmatrix}$

$$\bar{f}^2(u,s,\bar{x}_1^1,\bar{x}^2) = \begin{pmatrix} \bar{f}_1^2(u,s,\bar{x}_1^1,\bar{x}_1^2,\bar{x}_2^2) \\ \bar{f}_2^2(u,s,\bar{x}_1^1,\bar{x}_1^2,\bar{x}_2^2,\bar{x}_3^2) \\ \vdots \\ \bar{f}_q^2(u,s,\bar{x}_1^1,\bar{x}^2) \end{pmatrix};$$

$G(u,s) = \begin{pmatrix} 0 & G_1^1(u,s) \\ 0 & 0 \end{pmatrix}$; $\bar{\varepsilon}(t) = \begin{pmatrix} 0 \\ \varepsilon(t) \end{pmatrix}$;

$\bar{g}^2(u,s,\bar{x}_1^1,\bar{x}^2) = \begin{pmatrix} \bar{g}_1^2(u,s,\bar{x}_1^1) \\ \bar{g}_2^2(u,s,\bar{x}_1^1,\bar{x}_1^2,\bar{x}_2^2) \\ \vdots \\ \bar{g}_q^2(u,s,\bar{x}_1^1,\bar{x}_1^2,\bar{x}_2^2) \end{pmatrix}$ with

$$\begin{aligned} &\bar{f}_1^1(u,s,\bar{x}_1^1,\bar{x}_1^2,\bar{x}_2^2) \triangleq \widetilde{f}_1^1(u,x^1,x^2) \\ &\bar{f}_1^2(u,s,\bar{x}_1^1,\bar{x}^2) \triangleq \widetilde{f}_2^1(u,x^1,x^2) \\ &-\gamma_1^2(u,s)\widetilde{f}_1^1(u,x^1,x^2) \\ &\bar{f}_k^2(u,s,\bar{x}_1^1,\bar{x}^2) \triangleq \widetilde{f}^k(u,x^1,\dots,x^{k+1}) \\ &k = 2,\dots,q-1 \\ &\bar{f}_q^2(u,s,\bar{x}_1^1,\bar{x}^2) \triangleq \widetilde{f}^q(u,x) \\ &\bar{g}_1^2(u,s,\bar{x}_1^1) \triangleq -\frac{d\gamma_1^2}{dt}(u,s)\bar{x}_1^1 \\ &\bar{g}_k^2(u,s,\bar{x}_1^1,\bar{x}^2) \triangleq -\gamma_k^2(u,s)\widetilde{f}_1^1(u,x^1,x^2) \\ &-\frac{d\gamma_k^2}{dt}(u,s)\bar{x}_1^1, k = 2,\dots,q \end{aligned}$$

$$\bar{C}_1 = \begin{pmatrix} I_{m_1} & 0_{m_1\times m} \\ 0_{m\times m_1} & 0_{m\times m} \end{pmatrix} \quad (19)$$

$$\bar{C}_2 = \begin{pmatrix} I_{p-m_1} & 0_{(p-m_1)\times(n-p)} \\ 0_{(n-p)\times(p-m_1)} & 0_{(n-p)\times(n-p)} \end{pmatrix} \quad (20)$$

where I_{m_1} is the $m_1 \times m_1$ identity matrix and the notation $0_{i\times j}$ means the $i \times j$ null matrix.

Now, consider the second change of coordinates: $\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} : \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^{2m_1+q(p-m_1)}$, $\bar{x} = \begin{pmatrix} \bar{x}^1 \\ \bar{x}^2 \end{pmatrix} \mapsto z = \begin{pmatrix} z^1 \\ z^2 \end{pmatrix}$. More precisely, one has:

- $\Phi_1 : \mathbb{R}^{m_1+m} \longrightarrow \mathbb{R}^{2m_1}$
  $\bar{x}^1 = \begin{pmatrix} \bar{x}_1^1 \\ \bar{x}_2^1 \end{pmatrix} \mapsto z^1 = \begin{pmatrix} z_1^1 \\ z_2^1 \end{pmatrix} = \Phi_1(u,s,\bar{x}^1) \triangleq \Lambda_1(u,s)\bar{x}^1 = \begin{pmatrix} \bar{x}_1^1 \\ G_1^1(u,s)\bar{x}_2^1 \end{pmatrix}$ with $z_k^1 \in \mathbb{R}^{m_1}$, $k = 1, 2$ and where the matrix Λ_1 is given by equation (12).
- $\Phi_2 : \mathbb{R}^{n-m_1} \longrightarrow \mathbb{R}^{q(p-m_1)}$

$$\bar{x}^2 = \begin{pmatrix} \bar{x}_1^2 \\ \bar{x}_2^2 \\ \vdots \\ \bar{x}_q^2 \end{pmatrix} \longrightarrow z^2 = \begin{pmatrix} z_1^2 \\ z_2^2 \\ \vdots \\ z_q^2 \end{pmatrix} = \Phi_2(u,s,y_1,\bar{x}^2) \triangleq$$

$$\begin{pmatrix} x_1^2 \\ \bar{f}_1^2(u,s,y_1,\bar{x}^2) \\ \frac{\partial \bar{f}_1^2}{\partial \bar{x}_2^2}(u,s,y_1,\bar{x}^2)\bar{f}_2^2(u,s,y_1,\bar{x}^2) \\ \vdots \\ \left(\prod_{k=1}^{q-2} \frac{\partial \bar{f}_k^2}{\partial \bar{x}_{k+1}^2}(u,s,y_1,\bar{x}^2)\right)\bar{f}_{q-1}^2(u,s,y_1,\bar{x}^2) \end{pmatrix} \text{ with}$$

$z_k^2 \in \mathbb{R}^{p-m_1}$, $k = 1, \dots, q$. One can show that this transformation puts system (18) under the following form (see e.g. (Hammouri and Farza, 2003)):

$$\begin{cases} \dot{z}^1 = A_1 z^1 + \psi^1(u,s,z^1,z^2) + \Lambda_1(u,s)\bar{\varepsilon}(t) \\ y_1 = h_1(z^1) = C_1 z^1 = z_1^1 \\ \dot{z}^2 = A_2 z^2 + \psi^2(u,s,y_1,z_1^1,z^2) \\ y_2 = h_2(z^1,z^2) = C_2 z^2 + \gamma_1^2 C_1 z^1 \end{cases} \quad (21)$$

where C_1 and C_2 are given by (7) and each of the functions $\psi^i(u,s,z^1,z^2)$ is triangular with respect to z^i, $i = 1, 2$. Now, consider the following dynamical system:

$$\begin{aligned} &\dot{\hat{z}}^1 = A_1\hat{z}^1 + \psi^1(u,s,\hat{z}^1,\hat{z}^2) \\ &-\theta_1\Delta_1^{-1}(\theta_1)S_1^{-1}C_1^T(h(\hat{z}^1) - y_1) \\ &\dot{\hat{z}}^2 = A_2\hat{z}^2 + \psi^2(u,s,y_1,\hat{z}_1^1,\hat{z}^2) \\ &+\frac{\partial \Phi_2}{\partial \bar{x}^2}(u,s,y_1,\Phi_2^c(\hat{z}^2))\frac{\partial \bar{f}^2}{\partial \bar{x}_1^1}(u,s,\xi_1^1,\Phi_2^c(\hat{z}^2))(\hat{z}_1^1 - z_1^1) \\ &-\theta_2\Delta_2^{-1}(\theta_2)S_2^{-1}C_2^T(h(\hat{z}^1,\hat{z}^2) - y_2) \\ &-\frac{\partial \Phi_2}{\partial \bar{x}^2}(u,s,y_1,\Phi_2^c(\hat{z}^2)) \\ &\left(\Lambda_2^+(u,s,y_1,\Phi_2^c(\hat{z}^2)) - \left(\frac{\partial \Phi_2}{\partial \bar{x}^2}(u,s,y_1,\Phi_2^c(\hat{z}^2))\right)^+\right) \\ &\theta_2\Delta_2^{-1}(\theta_2)S_2^{-1}C_2^T(h(\hat{z}^1,\hat{z}^2) - y_2) \end{aligned} \quad (22)$$

where $\hat{z} = \begin{pmatrix} \hat{z}^1 \\ \hat{z}^2 \end{pmatrix} \in \mathbb{R}^{2m_1+q(p-m_1)}$; $\hat{z}^1 = \begin{bmatrix} \hat{z}_1^1 \\ \hat{z}_2^1 \end{bmatrix} \in \mathbb{R}^{2m_1}$, $\hat{z}_i^1 \in \mathbb{R}^{m_1}$; $\hat{z}^2 = \begin{bmatrix} \hat{z}_1^2 \\ \hat{z}_2^2 \\ \vdots \\ \hat{z}_q^2 \end{bmatrix} \in \mathbb{R}^{q(p-m_1)}$ with $\hat{z}_k^2 \in \mathbb{R}^{p-m_1}$, $k = 1, \ldots, q$; Φ_i^c denotes the converse function of Φ_i; $\theta_1, \theta_2 > 0$ are two real numbers and the variable $\xi_1^1 \in \mathbb{R}^{m_1}$ is given by the Mean value Theorem:

$\bar{f}^2(u, s, \hat{\bar{x}}_1^1, \hat{\bar{x}}^2) - \bar{f}^2(u, s, \bar{x}_1^1, \hat{\bar{x}}^2) =$
$\dfrac{\partial \bar{f}^2}{\partial \bar{x}_1^1}(u, s, \xi_1^1, \hat{\bar{x}}^2)(\hat{\bar{x}}_1^1 - \bar{x}_1^1) =$
$\dfrac{\partial \bar{f}^2}{\partial \bar{x}_1^1}(u, s, \xi_1^1, \Phi_2^c(\hat{z}^2))(\hat{z}_1^1 - z_1^1)$.

One shall show that system (22) can be written under form (14) in the original coordinates x. Indeed, proceeding as in (Farza *et al.*, 2004), one can show that system (22) can be written in the coordinates $\bar{x}$ as follows:

$$\begin{cases} \dot{\hat{\bar{x}}}^1 = \bar{f}^1(u, s, \hat{\bar{x}}_1^1, \hat{\bar{x}}_1^2, \hat{\bar{x}}_2^2) + G(u, s)\hat{\bar{x}}^1 \\ \qquad - \theta_1 \Lambda_1^+(u, s)\Delta_1^{-1}(\theta_1) S^{-1} C_1^T (\bar{C}_1 \hat{\bar{x}}^1 - y_1) \\ \dot{\hat{\bar{x}}}^2 = \bar{f}^2(u, s, \hat{\bar{x}}_1^1, \hat{\bar{x}}^2) + \bar{g}^2(u, s, \hat{\bar{x}}_1^1, \hat{\bar{x}}^2) \\ \quad - \theta_2 \Lambda_2^+(u, s, \bar{x}_1^1, \hat{\bar{x}}^2)\Delta_2^{-1}(\theta_2) \\ \quad S_2^{-1} C_2^T (\hat{\bar{x}}_1^2 + \gamma_1^2(u, s)\hat{\bar{x}}_1^1 - y_2) \end{cases} \tag{23}$$

where $\hat{\bar{x}} = \begin{pmatrix} \hat{\bar{x}}^1 \\ \hat{\bar{x}}^2 \end{pmatrix} \in \mathbb{R}^{n+m}$, $\hat{\bar{x}}^1 = \begin{pmatrix} \hat{\bar{x}}_1^1 \\ \hat{\bar{x}}_2^1 \end{pmatrix} \in \mathbb{R}^{m_1+m}$, $\hat{\bar{x}}_1^1 \in \mathbb{R}^{m_1}$, $\hat{\bar{x}}_2^1 \in \mathbb{R}^m$, $\hat{\bar{x}}^2 = \begin{pmatrix} \hat{\bar{x}}_1^2 \\ \vdots \\ \hat{\bar{x}}_q^2 \end{pmatrix} \in \mathbb{R}^{n-m_1}$, $\hat{\bar{x}}_1^2 \in \mathbb{R}^{n-m_1}$; $\hat{\bar{x}}_k^2 \in \mathbb{R}^{n_k}$, $k = 2, \ldots, q$.

Now, according to the expression of $T(u, s)$ (equation (17)), one can easily show that system (23) can be written in the original coordinates x under form (14). To end the proof, it suffices to demonstrate theorem 1 by considering system (18) on one hand and system (22) on the other hand. Indeed, for $= 1, 2$, set $e^i(t) = \hat{z}^i(t) - z^i(t)$, $\bar{e}^i = \Delta_i(\theta_i)e^i$, $V_i(\bar{e}^i) = \bar{e^i}^T S_i \bar{e}^i$ and let $V(\bar{e}^1, \bar{e}^2) = V_1(\bar{e}^1) + V_2(\bar{e}^2)$ be the Lyapunov candidate function. Using classical computations (see e.g.(Gauthier *et al.*, 1992; Hammouri and Farza, 2003; Farza *et al.*, 2004)), one can show that:

$$\dot{V}_1 \leq -(\theta_1 - c_1)V_1 + c_2\theta_2^{q-1}\sqrt{V_1}\sqrt{V_2} + \frac{c_3\beta_\varepsilon}{\theta_1}\sqrt{V_1}$$
$$\dot{V}_2 \leq -(\theta_2 - c_4)V_1 + c_5\theta_2\sqrt{V_1}\sqrt{V_2} \tag{24}$$

where c_i, $i = 1, \ldots, 5$ are positive constant parameters which do not depend on θ_1 nor θ_2 and β_ε is the upper bound of $\|\varepsilon(t)\|$ as given in Hypothesis (H5).
Combining inequalities of (24) and taking $\theta_2 \geq 1$, one obtains: $\dot{V} \leq -(\theta_1 - c_1)V_1 - (\theta_2 - c_4)V_2 + c\theta_2^{q-1}\sqrt{V_1}\sqrt{V_2} + \dfrac{c_3\beta_\varepsilon}{\theta_1}\sqrt{V_1}$ where $c = c_2 + c_5$. Now, for $\theta_1 > c_1$, $\theta_2 > c_4$, choose θ_1 such that $\theta_1 > \max(\theta_2^{q-1}, \theta_2 - c_4 + c_1, c_1 + \dfrac{c^2\theta_2^{2(q-1)}}{(\theta_2 - c_4)})$. One can show that: $\dot{V} \leq -\eta(\theta_2 - c_4)V + \dfrac{c_3}{\theta_1}\beta_\varepsilon\sqrt{V}$ where $\eta = 1 - \dfrac{c\theta_2^{q-1}}{\sqrt{(\theta_1 - c_1)(\theta_2 - c_4)}} > 0$. Using the fact that $\|\bar{e}(t)\| \leq \|e(t)\| \leq \theta_1\|\bar{e}(t)\|$, one can show that: $\|e(t)\| \leq \theta_1\sqrt{\dfrac{\lambda_2}{\lambda_1}} \exp\left[-\dfrac{\eta}{2}(\theta_2 - c_4)t\right] \|e(0)\| + \dfrac{c_3}{\eta\sqrt{\lambda_1}(\theta_2 - c_4)}\beta_\varepsilon$. This ends the proof of Theorem 1.

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