INPUT DESIGN FOR IDENTIFICATION OF ZEROS

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Abstract: The objective of this contribution is input design for accurate identification of non-minimum phase zeros in linear systems. Recently, several variance results regarding estimation of non-minimum phase zeros have been presented. Based on these results, we will show how to design the input that has the least energy content required to keep the variance of an estimated zero below a certain limit. Both analytical and numerical results are presented. A striking fact of the analytical results is that the variance of an estimated zero is independent of the model order when the optimal input is applied.
We will also quantify the benefits of using the optimal design compared to using a white input signal or a square-wave. Robustness issues will also be covered in this presentation. The optimal design depends on the location of the true unknown zero and is therefore infeasible. This is typically circumvented by replacing the true zero by an estimate. The sensitivity of the solution to this estimate is investigated.

Keywords: non-minimum phase zeros, identification, input design, convex optimization

1. INTRODUCTION

A model is often used in control design for both analysis and synthesis purposes. Consequently, system identification with focus on control design has been a research area with a lot of activity. The overall objective of identification for control is to deliver models suitable for control design. See (Gevers, 1993), (Van den Hof and Schrama, 1995) and (Hjalmarsson, 2004) for overviews of the area.

For scalar linear systems, the model should be accurate in the frequency bands important for the control design and it is generally acknowledged that the region around the cross-over frequency of the loop gain is of particular importance. Since the loop gain depends on the controller yet to be designed, the cross-over frequency is in generally unknown. However, for systems that contain performance limitations *e.g.* non-minimum phase zeros and time-delays the achievable bandwidth is restricted. For example, a real single non-minimum phase zero at z restricts the achievable bandwidth to approximately $z/2$ (Skogestad and Postlethwaite, 1996). Therefore knowledge of a non-minimum phase zero is very useful since it gives valuable information of what control specifications that can be defined.

This information is also valuable in the identification step since it would simplify the task of deciding on model structure, model order, noise model and pre-filters since it specifies an important frequency range.

Spurred by this observation, expressions for the variance of an estimated non-minimum zero have been derived in (Lindqvist, 2001) for FIR models and in (Hjalmarsson and Lindqvist, 2002) for ARX models. This work is generalized to include general linear single input/single output (SISO) model structures in (Mårtensson and Hjalmarsson, 2003). A key result in these contributions is that the variance of estimated non-minimum phase zeros is not subject to the usual increase in the variance when the model order is increased. Based on these variance results, we will in this contribution consider input design for accurate identification of non-minimum phase zeros. The input design problem is formulated as an optimization problem where the objective is to minimize the input effort

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required to keep the variance of the non-minimum zero below a certain limit.

The objective of classical input design has been to minimize some scalar function of the asymptotic parameter covariance subject to power constraints on the input or the output, see *e.g.* (Goodwin and Payne, 1977). From a control design point of view, variance of frequency functions are typically of more importance than the parametric covariance itself. There are several contributions on input design for control that are based on frequency domain variance expressions which are asymptotic in the model order and data, see *e.g.* (Gevers and Ljung, 1986), (Hjalmarsson *et al.*, 1996), (Forssell and Ljung, 2000), (Lindqvist and Hjalmarsson, 2000) and (Zhu and van den Bosch, 2000). However these results cannot handle frequency wise constraints, which *e.g.* implies that, in control applications, robust stability with a prespecified probability can not be guaranteed by the experiment design. This has been the inspiration leading to the contributions (Hildebrand and Gevers, 2003), (Bombois *et al.*, 2004) and (Jansson and Hjalmarsson, 2004), in which input design for robust control is one of the leading stars. This contribution can be seen as a continuation of these efforts.

The paper is organized as follows. Section 2 contains information about system assumptions and the used identification framework. Asymptotic variance expressions for an estimated zero are given in Section 3. Based on these variance expressions the optimal input design problem is formulated and both analytical and numerical solutions to this problem are presented in Section 4 and Section 5. Sensitivity and benefits of optimal input design for identification of zeros are discussed illustrated in Section 6. The paper is concluded in Section 7.

2. PARAMETER ESTIMATION

The model of our single input/single output system is defined by

$$y(t) = G(q,\theta)u(t) + H(q,\theta)e(t) \tag{1}$$

where G and H are parameterized by the real valued parameter vector θ. Furthermore, y is the output and u is the input and e is zero mean white noise with variance λ. It is assumed that G and H have the rational forms

$$G(q,\theta) = \frac{q^{-n_k}B(q,\theta)}{A(q,\theta)}, \quad H(q,\theta) = \frac{C(q,\theta)}{D(q,\theta)} \tag{2}$$

where

$$A(q,\theta) = 1 + a_1 q^{-1} + \cdots + a_{n_a} q^{-n_a} \tag{3}$$

$$B(q,\theta) = b_0 + b_1 q^{-1} + \cdots + b_{n_b} q^{-n_b} \tag{4}$$

$$C(q,\theta) = 1 + c_1 q^{-1} + \cdots + c_{n_c} q^{-n_c} \tag{5}$$

$$D(q,\theta) = 1 + d_1 q^{-1} + \cdots + d_{n_d} q^{-n_d} \tag{6}$$

with q being the delay operator. We will assume that there exists a description of the true system within the model class defined by $\theta = \theta_o$ and $\lambda = \lambda_o$. The one-step-ahead predictor for the model (1) is

$$\begin{aligned}\hat{y}(t,\theta) =& H^{-1}(q,\theta)G(q,\theta)u(t) \\ &+ \left(1 - H^{-1}(q,\theta)\right) y(t)\end{aligned} \tag{7}$$

and the prediction error is $\varepsilon(t,\theta) = y(t) - \hat{y}(t,\theta)$.The parameters are estimated with the prediction error method using a least mean square criterion to minimize the prediction error. The parameter estimate is

$$\widehat{\theta}_N = \arg\min_{\theta} \frac{1}{2N} \sum_{t=1}^{N} \varepsilon^2(t,\theta) \tag{8}$$

where N denotes the number of the data that is used for the estimation. Under mild assumptions the parameter estimate has an asymptotic distribution (Ljung, 1999) that obeys

$$\begin{aligned}&\sqrt{N}\left(\widehat{\theta}_N - \theta_o\right) \in \mathrm{AsN}(0, \lambda_o P) \\ &P = \left(\mathbf{E}\{\psi(t,\theta_o)\psi^T(t,\theta_o)\}\right)^{-1} \\ &\psi(t,\theta_o) = -\left.\frac{\partial}{\partial\theta}\varepsilon(t,\theta)\right|_{\theta=\theta_o} = \left.\frac{\partial}{\partial\theta}\widehat{y}(t|\theta)\right|_{\theta=\theta_o}\end{aligned} \tag{9}$$

Using (7) we obtain

$$\psi(t,\theta_o) = F_u(q,\theta_o)u(t) + F_e(q,\theta_o)e_o(t) \tag{10}$$

where

$$F_u(q,\theta) = \frac{1}{H(q,\theta)}\frac{\partial G(q,\theta)}{\partial\theta} \tag{11}$$

and

$$F_e(q,\theta) = \frac{1}{H(q,\theta)}\frac{\partial H(q,\theta)}{\partial\theta} \tag{12}$$

Under the assumption of open loop operation, *i.e.* that u and e are uncorrelated, we can write

$$P^{-1} = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_u(\theta_o)\Phi_u F_u^*(\theta_o)d\omega + R_o \tag{13}$$

where Φ_u is the spectrum of the input and where

$$R_o = \frac{\lambda_o}{2\pi}\int_{-\pi}^{\pi} F_e(\theta_o)F_e^*(\theta_o)d\omega. \tag{14}$$

The expression (13) is very useful for input design purposes since it shows exactly the influence of the input spectrum on the asymptotic parameter covariance matrix. In fact, in open-loop, the only quantity that can be used to shape P is actually the input spectrum Φ_u. This connection between the asymptotic covariance and the input spectrum will be further exploited for input design for identification of zeros. But first we will review some results regarding the accuracy of identified zeros.

3. ESTIMATION OF ZEROS

Consider identification of a system defined by (1) and (2). Let $\theta_b^T = [b_0, \ldots, b_{n_b}]$ and introduce the polynomial

$$p(z,\theta_b) = b_0 z^{n_b} + b_1 z^{n_b-1} \cdots + b_{n_b} \tag{15}$$

A zero $z_i(\theta)$ of the system (1) is defined by

$$p(z_i,\theta_b) = 0.$$

All zeros are assumed to be unique. Now we consider one particular zero, $z_k(\theta)$. Introduce the notation

$$\hat{z}_k = z_k(\widehat{\theta}_N),\ z_k^o = z_k(\theta_o),$$

$$\widetilde{B}(q,\theta) = \frac{B(q,\theta)}{1 - z_k(\theta)q^{-1}} \text{ and let}$$

$$\Gamma_b(q) = \begin{bmatrix} 1 & q^{-1} & \cdots & q^{-n_{n_b}} \end{bmatrix}^T \tag{16}$$

Furthermore, let

$$\alpha^2 = \frac{\lambda_o |z_k^o|^2}{N|\widetilde{B}(z_k^o)|^2} \tag{17}$$

In (Lindqvist, 2001) it is established that the variance of an estimated zero is

$$\lim_{N\to\infty} \mathbf{E}(\hat{z}_k - z_k^o)^2 = \alpha^2 \Gamma_b^*(z_k^o) P_b \Gamma_b(z_k^o) \tag{18}$$

where $P_b = \mathbf{E}(\theta_b - \theta_b^o)(\theta_b - \theta_b^o)^T$, *i.e.* the covariance matrix of θ_b. If we consider non-minimum phase zeros and increase the model order we can simplify the expression (18). Let $u(t) = Q(q)v(t)$ where $v(t)$ is a white noise sequence with variance 1 and $Q(q)$ is a minimum phase filter. Then according to (Mårtensson and Hjalmarsson, 2003) we have

$$\lim_{n_b\to\infty} \lim_{N\to\infty} \mathbf{E}(\hat{z}_k - z_k^o)^2 = \frac{\alpha^2 |H(z_k^o)|^2 |A(z_k^o)|^2}{(1 - |z_k^o|^{-2})|Q(z_k^o)|^2} \tag{19}$$

4. INPUT DESIGN - ANALYTICAL RESULTS

In this section we will use the variance expressions (18) and (19) in order to determine suitable inputs for accurate identification of zeros. The input design will be formulated as an optimization problem where we seek the input spectrum with least energy that keeps the variance below a certain limit. This can be stated as follows:

$$\min_{\Phi_u} \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_u(e^{i\omega}) d\omega \quad \text{s.t.} \quad \operatorname{Var} \hat{z}_k \le \gamma \tag{20}$$

The choice of optimization variable is natural because the only quantity, asymptotically in N, that can be used to shape the variance is the spectrum of the input, *cf* (13), (18) and (19).

4.1 Input design for finite model orders

The first step to solve (20) is to rewrite the original problem formulation into a convex program *wrt* the input spectrum. The objective function is already convex but the constraint is not. Let

$$\Gamma_{b0} = \begin{pmatrix} \Gamma_b^T & 0 \end{pmatrix}^T \tag{21}$$

The variance constraint in (20) using (18) now becomes

$$\frac{\gamma}{\alpha^2} - \Gamma_{b0}^*(z_k^o) P \Gamma_{b0}(z_k^o) \ge 0 \tag{22}$$

which by Schur complements is equivalent to

$$P^{-1} - \frac{\alpha^2}{\gamma} \Gamma_{b0} \Gamma_{b0}^* \ge 0. \tag{23}$$

Since the inverse of the covariance matrix is affine in Φ_u, the constraint (23) is convex *wrt* Φ_u. Thus, the convex formulation of (20) is

$$\begin{aligned} &\min_{\Phi_u} \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_u(e^{i\omega}) d\omega \\ &\text{s.t.} \quad P^{-1} - \frac{\alpha^2}{\gamma} \Gamma_{b0} \Gamma_{b0}^* \ge 0 \end{aligned} \tag{24}$$

This means that if (24) is feasible it has a global optimal solution. Even though (24) is convex, it is in general infinite-dimensional which calls for special care when undertaking the optimization. But as will be shown in Section 5, by imposing certain parameterizations of the input spectrum it is possible to reformulate (24) as a finite-dimensional convex optimization problem. Today, there exist several numerical optimization routines that solve such problems to any demanded accuracy. But first we will show that it is possible to derive analytical solutions to (24) for FIR and for ARX model structures.

Theorem 4.1. Consider the FIR-system

$$y(t) = q^{-n_k} B(q, \theta_b) u(t) + e(t). \tag{25}$$

For a *non-minimum phase zero*, z_k^o, the input design problem (24) is solved by filtering unit variance white noise with the first order AR-filter

$$Q(q) = \frac{\alpha}{\sqrt{\gamma}} \frac{\sqrt{1 - (z_k^o)^{-2}}}{(1 - (z_k^o)^{-1} q^{-1})}, \tag{26}$$

i.e. by placing a pole in $(z_k^o)^{-1}$. The minimal required input energy is α^2/γ.

Proof: Let the auto-correlations of the input u be denoted r_m. Then the energy of the input is equal to r_0 and

$$P^{-1} = \begin{pmatrix} r_0 & \cdots & r_{n_b} \\ \vdots & \ddots & \vdots \\ r_{n_b} & \cdots & r_0 \end{pmatrix} = R_u \tag{27}$$

In this case the constraint in (24) becomes

$$\begin{pmatrix} r_0 & \cdots & r_{n_b} \\ \vdots & \ddots & \vdots \\ r_{n_b} & \cdots & r_0 \end{pmatrix} - \frac{\alpha^2}{\gamma} \begin{pmatrix} 1 & \cdots & (z_k^o)^{-n_b} \\ \vdots & \ddots & \vdots \\ (z_k^o)^{-n_b} & \cdots & (z_k^o)^{-2n} \end{pmatrix} \ge 0 \tag{28}$$

To satisfy (28) we need that $r_0 \ge \alpha^2/\gamma$. If we can find a covariance function r_m with $r_0 = \alpha^2/\gamma$ that satisfies (28) we have a solution. In the following we prove that the covariance function

$$r_m = \frac{\alpha^2}{\gamma} (z_k^o)^{-m} \tag{29}$$

is such a solution. First we note that this particular choice of r_m gives $R_u \ge 0$ and that the first row and column of (28) is zero. Now we need to show that

$$\begin{pmatrix} 1 & \cdots & z_k^{1-n_b} \\ \vdots & \ddots & \vdots \\ z_k^{1-n_b} & \cdots & 1 \end{pmatrix} - \begin{pmatrix} z_k^{-1} \\ \vdots \\ z_k^{-n_b} \end{pmatrix} \begin{pmatrix} z_k^{-1} & \dots & z_k^{-n_b} \end{pmatrix} \ge 0$$

Using Schur complements this is equivalent to

$$\frac{\alpha^2}{\gamma} \begin{pmatrix} 1 & \cdots & z_k^{-n_b} \\ \vdots & \ddots & \vdots \\ z_k^{-n_b} & \cdots & 1 \end{pmatrix} = R_u \ge 0, \tag{30}$$

which is true as noted before. A signal with the covariance function (29) can be generated by filtering unit variance white noise with the filter (26). This proves Theorem 4.1. □.

Remark: The filter (26) is constructed such that the variance of the estimated zero will be γ. Thus, the variance constraint in (20) is tight. Notice that the optimal filter is independent of the model order. From this it is easy to conclude that the variance of the estimated zero also will be independent of the model order [2] when optimal input design is used.

Theorem 4.2. Consider the ARX-system

$$y(t) = q^{-n_k}\frac{B(q,\theta_b)}{A(q,\theta_a)}u(t) + \frac{1}{A(q,\theta_a)}e(t). \quad (31)$$

For a non-minimum phase zero, the input design problem (24) has the same solution as for a FIR-system, see Theorem 4.1.

Proof: Similar calculations as for Theorem 4.1. For a complete proof see (Jansson, 2004). □.

Remark: As for the FIR models, the variance of the estimated zeros will be independent of the model order when we use optimal input design. Furthermore, the solution in Theorem 4.2 gives a tight bound of the variance constraint with a filter that is independent of the A-polynomial. Hence, it is easy to conclude that the variance of the zero is independent of the A-polynomial as well. However, it is important to estimate the A-polynomial for the asymptotic properties (9) to hold.

4.2 Input design for high-order systems

For general linear SISO models it is possible to derive an analytical solution of (20) based on the asymptotic variance expression (19).

Theorem 4.3. The input design problem (20) where the variance of a non-minimum phase zero is defined by (19) is solved by filtering unit variance white noise with the first order AR-filter

$$Q(q) = \frac{\alpha|H(z_k^o)A(z_k^o)|}{\sqrt{\gamma}}\frac{\sqrt{1-(z_k^o)^{-2}}}{(1-(z_k^o)^{-1}q^{-1})} \quad (32)$$

Proof: See (Jansson, 2004). □.

Notice that the optimal filter coincides with (26) for FIR and ARX models. This in complete line with the observation that the optimal filter for any finite model order is actually given by (26) for these model structures. The solution for other model structures is in principle the same, *i.e.* a pole placed in $(z_k^o)^{-1}$, when the model order is sufficiently large. The only difference is the gain of the filters.

Remark: In this section the optimal input is presented in terms of filtered white noise. A signal with auto-correlations $r_m = \beta\eta^{-|m|}$ can *e.g.* also be realized by a binary signal, see (Tulleken, 1990).

5. INPUT DESIGN - NUMERICAL SOLUTION

We have so far presented analytical solutions to (24) for FIR and ARX model structures and for general linear model structures if we let the model order tend to ∞. Here we will show how to solve (24) for a Box-Jenkins model structure defined by (1)-(6). The key is to rewrite (24) to a finite-dimensional convex program which indeed can be obtained by a suitable parametrization of the input spectrum. For an overview of different parameterizations of the input spectrum, we refer to (Jansson and Hjalmarsson, 2004). Here we will illustrate one such approach introduced in (Stoica and Söderström, 1982). Define L and $\{l_k\}$ as

$$\begin{aligned} L(\mathrm{e}^{j\omega},\theta) &= |C(\mathrm{e}^{j\omega},\theta)|^2|A(\mathrm{e}^{j\omega},\theta)|^4 \\ &\triangleq \sum_{k=-n_l}^{n_l} l_k\left(\mathrm{e}^{kj\omega}+\mathrm{e}^{-kj\omega}\right) \end{aligned} \quad (33)$$

where $n_l = 2n_a + n_c - 1$. Furthermore, introduce the auto-correlations c_k defined by

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\Phi_u(e^{i\omega})}{L(\mathrm{e}^{j\omega},\theta_o)}e^{i\omega k}\,d\omega \quad (34)$$

and let $n_p = n_a + n_b + n_d - 1$.

Lemma 5.1. Let $L(\mathrm{e}^{j\omega},\theta)$ be defined by (33). Furthermore assume that the polynomials A, B, C and D in the Box-Jenkins model are coprime. Then there exist matrices $M_k \in \mathbb{R}^{n_a+n_b}$ such that the inverse covariance matrix P^{-1} defined by (13) can be expressed as

$$P^{-1}(\theta_o) = \sum_{k=-n_p}^{n_p} c_k(\theta_o)M_k(\theta_o) + R_o(\theta_o) \quad (35)$$

Proof: See (Stoica and Söderström, 1982) and (Jansson and Hjalmarsson, 2004) □.

With this particular parametrization it is possible to express the input power as a linear function.

Lemma 5.2. The power of the input $u(t)$ with power spectrum $\Phi_u(\omega)$ can be expressed as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_u(e^{i\omega})d\omega = \sum_{k=-n_l}^{n_l} c_k l_k \quad (36)$$

Proof: See (Stoica and Söderström, 1982) and (Jansson and Hjalmarsson, 2004) □.

Let $m = max(n_l, n_p)$. Now it is possible to rewrite the original input design formulation (24).

Theorem 5.1. Under the assumptions stated in Lemma 5.1 and Lemma 5.2, the input design problem (24) is equivalent to the following finite-dimensional convex program

$$\begin{aligned} &\min_{c_0,\ldots,c_m} \sum_{k=-n_l}^{n_l} c_k l_k \\ &\text{s.t.} \quad \sum_{k=-n_p}^{n_p} c_k M_k + R_o - \alpha^2\Gamma_{b0}\Gamma_{b0}^* \geq 0 \\ &\qquad \begin{bmatrix} c_0 & \cdots & c_m \\ \vdots & \ddots & \vdots \\ c_m & \cdots & c_0 \end{bmatrix} \geq 0 \end{aligned} \quad (37)$$

[2] The model order must be equal or greater than the true system order.

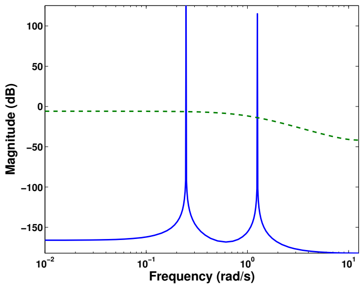

Fig. 1. Optimal spectra for $n_b = 2$ (solid) and for $n_b = 3$ (dashed).

Proof: Direct application of the results in Lemma 5.1 and Lemma 5.2 to (24). The constraint on the Toeplitz matrix in (37) assures that the optimization variables $c_0, \ldots, c_m$ are indeed auto-correlations to a quasi-stationary process. $\square$.

The input design problem (37) is now convex and finite-dimensional and there are several efficient numerical optimization methods that solve such problems. Let us illustrate the results of this section. We will assume that the dynamics of the system is defined by the continuous time system

$$G_c(s) = \frac{1-s}{(s+1)(2s+1)} \tag{38}$$

i.e. there is a continuous time non-minimum phase zero in 1. With a zero-order hold discretization with sampling time $T_s = 0.25\ s$ this corresponds to the discrete non-minimum phase zero $z_d = 1.29$. Furthermore, we will assume that the input/output relation is defined by an output-error (OE) model structure, the data length is $N = 500$ and $\lambda_o = 0.1$. When the model order equals the true system order, *i.e.* the order is two, the solution to (37) is basically a sum of two sinusoids. When the order of the B-polynomial, n_b, is increased, the solution coincides with the first order AR-filter defined in (32) This is illustrated in Figure 1 where the optimal spectra for $n_b = 2$ and for $n_b = 3$ are shown. Notice that there is a quite dramatic difference between the optimal spectra.

6. SENSITIVITY AND BENEFITS

Based on (19) we will in this section try to quantify possible benefits of using an optimal or sub-optimal design instead of using a white input. There will be a comparison of the obtained variance levels when the input power is normalized to one for all the designs. We will also study how the location of the zero affects the result.

In the first comparison, the optimal input filter with unit power *i.e.*

$$Q_{opt}(q, z_k^o) = \frac{\sqrt{1-(z_k^o)^{-2}}}{1-(z_k^o)^{-1}q^{-1}} \tag{39}$$

is compared with $Q = 1$. From (19) we have that

$$\frac{\text{Var }\ \hat{z}_k(Q_{opt})}{\text{Var }\ \hat{z}_k(Q=1)} = \frac{1}{|Q_{opt}(z_k^o)|^2} = 1-(z_k^o)^{-2} \tag{40}$$

The thick solid line in Figure 2 corresponds to $1 - (z_k^o)^{-2}$ as a function of the zero location. Thus there

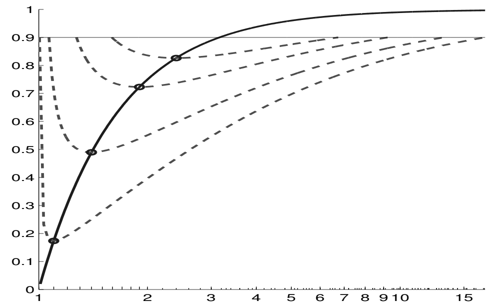

Fig. 2. The thick solid line represents the optimal variance reduction as a function of the zero location, see (40). The dashed lines corresponds to (42) and illustrates the variance reduction with a sub-optimal design.

is a substantial decrease in variance close to the unit circle when the optimal input design is used instead of a white input. This comparison also indicates that when the zero is located far from the unit circle ($|z_k^o| \gtrapprox 4$), there is no benefit in using optimal input design. A white input performs almost as good as the optimal design. One interpretation of this relates to the location of the discrete zero *wrt* to the sampling time. Consider the continuous system (38), which for $T_s = 0.25$ has a discrete zero in 1.29. If the sampling time is increased the discrete zero will move away from the unit circle, and hence the effect of the non-minimum phase zero will *e.g.* be less visible in the discrete measurements of a step response. Consequently, the benefits of optimal input design are reduced.

In a practical situation the location of the true zero is unknown and an estimate of the zero may be used for input design. Given the optimal filter (39) and an estimate of the zero, $\hat{z}_k$, a natural choice of input filter is

$$Q_{app}(q, \hat{z}_k) = \frac{\sqrt{1-(\hat{z}_k)^{-2}}}{1-(\hat{z}_k)^{-1}q^{-1}} \tag{41}$$

A reasonable question is how the uncertainty in the zero location will affect the estimation accuracy. This is also illustrated in Figure 2. The dashed lines corresponds to the quotient

$$\frac{\text{Var }\ \hat{z}_k(Q_{app})}{\text{Var }\ \hat{z}_k(Q=1)} = \frac{\left(1-(\hat{z}_k)^{-1}(z_k^o)^{-1}\right)^2}{1-(\hat{z}_k)^{-2}} \tag{42}$$

as a function of $\hat{z}_k$ for four different locations of the true zero (corresponding to the circles in the figure). These curves show that there is a quite large tolerance with respect to the estimated zero location.

Let us now illustrate some of the derived results. The dynamics of the system is defined by the continuous system (38). The sampling time is 0.25 s and the data length is 500 samples but here we will assume that the true system is of ARX type with a noise variance of 0.0025.

Now we will compare, by means of an example, the obtained accuracy when using four different types of inputs. The first input is a Pseudo-Random-Binary-Signal (PRBS) which has white-noise-like properties. The second input is the optimal one and hence the optimal input filter is given by (26). We know from

Table 1. Comparison of variance of estimated non-minimum phase zero.

Model order	PRBS	Q_{opt}	Q_{app}	Square-wave
2	0.0022	0.0011	0.0012	0.0017
5	0.0027	0.0011	0.0012	0.0023

(40) that the optimal gain in accuracy when using the optimal input compared to a white input is approximately a factor 2.5 when the model order tends to infinity. These two input signals will be compared to a sub-optimal input given by (41) with the zero estimate $\hat{z}_k = 1.6$ and a square-wave signal where the signal is constant in 10 s before switching level. This square-wave signal, that takes the values ± 1, is constructed such that the typical dip of the step response for a system with a non-minimum phase zero is clearly visible. The power of all inputs are equalized to one. We have used a model structure of order two (the true order) and one of order five, *i.e.* an over-parametrization. The result of 10000 Monte-Carlo simulations is given in Table1. The theoretical value of the variance for the optimal input is, asymptotically in data, 0.0010, independently of the model order provided it is larger than the true system order. For model order two, the accuracy gain of the optimal input is approximately a factor 1.5 to 2. When the order is increased to five, this factor increases to 2 to 2.5, *i.e.* close to the predicted value. Notice that the performance of the square-wave deteriorates for high model orders, but it remains constant for the optimal and the sub-optimal design.

7. CONCLUSIONS

Analytical solutions have been derived for FIR and ARX model structures that presents the most efficient input, in terms of input energy, to estimate a discrete non-minimum phase zero z_k^o. The optimal input can be characterized by a first order AR-filter with a pole in $(z_k^o)^{-1}$. This solution is independent of the model order. Thus, the variance of the estimated non-minimum phase zero will be independent of the model order when the optimal input is applied. A similar analytic solution is obtained for general linear models based on a variance expression that is asymptotic in model order and data. A numerical solution is presented for general linear SISO models of finite orders. It is illustrated that the optimal input may be very different depending on model structure and order.

Possible benefits of optimal design are presented. It is shown that the variance can be reduced significantly compared to white inputs and square-waves, especially when the model is over-parameterized. It is also shown that a solution based on the optimal AR-filter, in which the true zero is replaced by an estimated zero, is quite robust *wrt* the estimated zero location.

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