SIMULATION OF UNCERTAIN DYNAMIC SYSTEMS
DESCRIBED BY INTERVAL MODELS: A SURVEY

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Abstract: In this paper, a survey of approaches to interval simulation of uncertain systems is presented. The kind of uncertain systems considered are those described by a model with parameters bounded in intervals. These last years the research of algorithms for simulating these type of systems has been a very active. Many researchers coming from different research areas and using different types of approaches have developed different algorithms. In this paper, the main problems and approaches when performing this kind of simulation are presented. The main goal of the paper is to present for the first time all existent approaches together emphasising that since all researchers face the same problem but in different contexts, they are finding the same kind of problems in spite of their different formalisms and methodologies. *Copyright © 2005 IFAC*

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1. INTRODUCTION

When modelling a physical dynamic system with a mathematical model there is always some mismatch between the model and the reality because of the modelling errors or the tolerance of the components. This mismatch is called **uncertainty** or **tolerance**. The uncertainty can be located in the parameters, in this case it is called **structured** or **parametric**, or in the structure of the model, then it is called **unstructured**. Uncertainties in the parameters can appear in the model of the actual system because of one or several of the following reasons: nominal values of system parameters are the result of the system design step, but their actual value is often different from the nominal one; parameters whose value is estimated, thus resulting in some confidence interval within which they lie.; real parameters can vary in time becuase of drifting, non-linearities, etc.; low-order or simplified models are used to model complex systems. In the case of structured uncertainty, the exact value of the parameters is unknown but bounded in an interval. A system modelled using this type of model uncertainty is called an **interval dynamic system**. The aim of this paper is to review existing approaches to interval simulation coming from the different research communities signalling the main features and identifying common problems. In *Section 2*, interval simulation is introduced and the main approaches are enumerated. In *Section 3*, the main problems associated with interval simulation are described. In *Section 4*, algorithms coming from the solution of validated initial value problems are presented. In *Section 5*, algorithms coming automatic control and fault detection fields are presented. In *Section 6*, algorithms coming for circuit analysis community are presented. In *Section 7*, algorithms coming from artificial intelligence communities are presented. Finally, in *Section 8* the conclusions of this survey are introduced.

2. INTERVAL SIMULATION

2.1 Statement of the problem

Considering a non-linear dynamic system in continuous time (or discrete-time[1]) and the modelling uncertainty that affect the behaviour of the system, the state-space relationship can be written as

$$\dot{x}(t) = f(x(t), u(t), \theta) \qquad (1)$$

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[1] In case of a discrete-time sytem (1) should be substituted by

$$x(k+1) = f(x(k), u(k), \theta)$$

where: $\boldsymbol{x} \in \Re^{nx}$, $\boldsymbol{u} \in \Re^{nu}$ and $\boldsymbol{y} \in \Re^{ny}$ are state, input and output vectors of dimension nx, nu and ny respectively; $\boldsymbol{g}$ is the state space function; $\boldsymbol{\theta}$ is the vector of uncertain parameters of dimension p with their values bounded by a compact set $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ of box type, i.e., $\boldsymbol{\Theta} = \{ \boldsymbol{\theta} \in \Re^p \mid \underline{\boldsymbol{\theta}} \leq \boldsymbol{\theta} \leq \overline{\boldsymbol{\theta}} \}$; $\boldsymbol{x}_o$ is the vector of uncertain initial conditions of dimension nx with their values bounded also by a box type compact set $\boldsymbol{x}_o \in X_o$, i.e,

$$X_o = \{ \boldsymbol{x}_o \in \Re^{nx} \mid \underline{\boldsymbol{x}_o} \leq \boldsymbol{x}_o \leq \overline{\boldsymbol{x}_o} \} .$$

Definition 1. *The **solution set** of a system, whose model is described by (1) for the time interval $[0,t_f]$ consists of*

$$X(0,t_f) = \left\{ \boldsymbol{x}(t,\boldsymbol{u},\boldsymbol{x}_o,\boldsymbol{\theta}) : t \in [0,t_f], \boldsymbol{\theta} \in \boldsymbol{\Theta}, \boldsymbol{x}_o \in X_o \right\} , \quad (2)$$

*where $\boldsymbol{x}(t,\boldsymbol{u},\boldsymbol{\theta},\boldsymbol{x}_o)$ denotes the solution of (1) at time t for some vector of parameters $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ and some initial condition $\boldsymbol{x}_o \in X_o$ at time $t=0$. The set of values for a fixed time interval $[0,t_f]$ will be referred to as the **reachability set** at time t and denoted by*

$$X(t) = \left\{ \boldsymbol{x}(t,\boldsymbol{u},\boldsymbol{x}_o,\boldsymbol{\theta}) : \boldsymbol{\theta} \in \boldsymbol{\Theta}, \boldsymbol{x}_o \in X_o \right\} . \quad (3)$$

Herein, it will be assumed that the uncertain system is stable for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. This assumption will allow $X(t)$ to be a bounded region for each $t \in [0,\infty)$.

Definition 2. *The **interval simulation** of a system, whose model is described by (1), for the time interval $[0,t_f]$ consists in computing the interval hull of the reachability set $X(t)$, i.e., the smallest interval vector containing it:*

$$\square X(t) = \left[\underline{\boldsymbol{x}}(t), \overline{\boldsymbol{x}}(t) \right] , \quad (4)$$

*where $\square$ is used to denote the **interval hull** of X, for all $t \in [0,t_f]$. The sequence of interval vectors $\square X(t)$ with $t \in [0,t_f]$ will be called the **interval solution** or **envelopes** of (1) (Fig. 1).*

From the definition of interval hull, $\square X(k)$ can be determined by solving the following optimisation problems:
$$\overline{\boldsymbol{x}}(t) = max\ \boldsymbol{x}(t,\boldsymbol{u},\boldsymbol{x}_o,\boldsymbol{\theta})$$
and
$$\underline{\boldsymbol{x}}(t) = min\ \boldsymbol{x}(t,\boldsymbol{u},\boldsymbol{x}_o,\boldsymbol{\theta})$$
subject to:
$$\boldsymbol{\theta} \in \boldsymbol{\Theta}$$
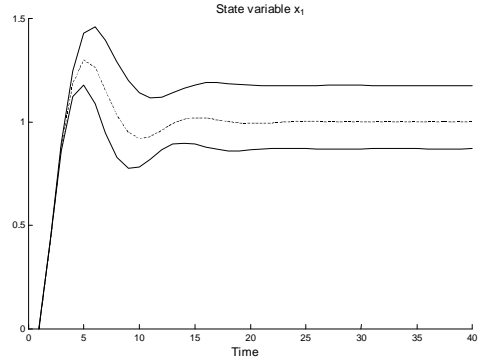$$\boldsymbol{x}_o \in X_o \quad (5)$$

Fig. 1 Envelopes produced by the interval simulation of a system described by (1)

This is a dynamic optimisation problem that can be solved using the classical tools that have been used for optimal control of dynamics systems. However, in general optimization problems (5) are **non-convex**. Then, traditional numerical optimisation techniques based on derivatives and the method of the steepest descent could not guarantee the global optimum, so they are not suitable to solve these optimisation problems. Instead, **global optimisation** techniques that can rigorously guarantee the global optimum must be used (Papamichail, 2002; 2004). Since the exact computation of the interval simulation in general needs a lot of time, some approximate solutions have been introduced in the literature (Armengol, 2001) through the introduction of the inner $[\check{\boldsymbol{x}}(t)]$ and outer $[\hat{\boldsymbol{x}}(t)]$ solutions.

Definition 3. *The interval vector $[\check{\boldsymbol{x}}(t)]$ is called the **inner solution** for the time t of the interval simulation problem if and only if $\underline{\check{x}}_i(t) \geq \underline{x}_i(t)$ and $\overline{\check{x}}_i(t) \leq \overline{x}_i(t)$, $i=1,..,n$ where $\square X(t) = \left[\underline{\boldsymbol{x}}(t), \overline{\boldsymbol{x}}(t) \right]$ is the interval hull of the solution set.*

According to Kolev (1993), the following approximate solution to the interval simulation problem for the time interval $[0,t_f]$ that provide an inner solution can be obtained by determining the interval vector $[\check{\boldsymbol{x}}(t)] = \left[\underline{\check{\boldsymbol{x}}}(t), \overline{\check{\boldsymbol{x}}}(t) \right]$ by solving the following global optimisation problems:

$$\overline{\check{\boldsymbol{x}}}(t) = max\ \boldsymbol{x}(t,\boldsymbol{x}_o,\boldsymbol{\theta},\boldsymbol{u})$$
and
$$\underline{\check{\boldsymbol{x}}}(t) = min\ \boldsymbol{x}(t,\boldsymbol{x}_o,\boldsymbol{\theta},\boldsymbol{u})$$

subject to:

$$\boldsymbol{\theta} \in V(\boldsymbol{\Theta})$$
$$\boldsymbol{x}_o \in V(X_o) \quad (6)$$

where $V(\boldsymbol{\Theta})$ and $V(X_o)$ denotes the set of vertices of the uncertain parameters and initial states sets, respectively. This interval simulation algorithm is known as the **vertices algorithm**.

According to Walter (1970), if the state space function $\boldsymbol{f}$ in (1) is **quasi-isotone**[2], i.e., the kth component of state vector function $\boldsymbol{f}$ is monotone increasing with respect the kth component of state vector $\boldsymbol{x}$. Then, approximate solution provided by vertices algorithm (6) provide the exact solution to the interval simulation problem. Such a systems are known also as **positive** (Luenberger, 1979) or **cooperative** (Gouzé, 2000).

Definition 4. *The interval vector $[\hat{\boldsymbol{x}}(t)]$ is called the* ***outer solution*** *for the time t of the interval simulation problem if and only if $\hat{\underline{x}}_i(t) \leq \underline{x}_i(t)$ and $\bar{\hat{x}}_i(t) \geq \bar{x}_i(t)$, $i=1,..,n$ where $\Box X(t) = [\underline{\boldsymbol{x}}(t), \bar{\boldsymbol{x}}(t)]$ is the interval hull of the solution set.*

Interval simulation algorithms based on interval methods provide outer solutions of the exact interval simulation since the range evaluation of a function $f(\boldsymbol{x})$ with $\boldsymbol{x} \in [\boldsymbol{x}]$ using the interval natural extension $F([\boldsymbol{x}])$ satisfies the following inclusion: $f(\boldsymbol{x}) \subseteq F([\boldsymbol{x}])$ with $\boldsymbol{x} \in [\boldsymbol{x}]$.

2.2 Existing approaches

When implementing algorithms for interval simulation of continuous time systems such as (1), usually some kind of discretisation should be introduced. For these reason most of the existent algorithms consider directly the model in discrete-time. However, some approaches that initially consider the model in continuous time when discretising the model in order to numerically simulate it, even consider the errors of discretisation as is the case of approaches coming from the solution validated initial value problems. In the literature, several algorithms for the interval simulation of systems described by (1) taken from different research areas have been proposed. These approaches are taken from:

- **Qualitative Reasoning** (Kuipers,1994) (Vescovi,1995) (Kay,1995)
- **Fuzzy Dynamical Systems** (Bonarini, 1994) (Hüllermeier, 1997) (Keller,1999)

- **Constraint Satisfaction** (Deville, 1998) (Janssen,2001)(Jaulin,2001)
- **Validated Initial Value Problems** (Moore,1966) (Moore,1979) (Neumaier,1993) (Lohner,1987) (Berz,1998) (Kühn,1998) (Nedialkov,1999)
- **Automatic Control** (Barmish,1979) (Chernousko,1994) (Tibken,1993) (Kurzhanski,1997) (Norton, 1999) (ElGhaoui,1999) (Kieffer,2002)
- **Fault Detection** (Horak,1988) (Puig,1999) (Armengol,2001)
- **Circuit Tolerance Analysis** (Oppenheimer,1988) (Kolev,1993) (Femia,1999) (Femia,2000)

All these algorithms can be classified according to if they compute the interval hull of the set of estimated state $\Box \hat{X}(k)$ using one step-ahead iteration based on previous approximations of the reachable set (**region based approaches**), or a set of point-wise trajectories generated by selecting particular values of $\theta \in \boldsymbol{\Theta}$ using heuristics or optimisation (**trajectory based approaches**). In the first case, the set of states $X(t)$ is bounded by its interval hull at each iteration and some propagation algorithm is used to produce the interval hull of next set of states $X(t+\Delta t)$. This approach is affected by several problems, in particular, the wrapping effect, range evaluation of an interval function (in this case, the state space function is $\dot{x}(t) = f(x(t), u(t), \theta))$ and the uncertain parameter time dependency that will be reviewed in *Section 2*. However, in the second case, the interval hull of $X(t)$ is built following real trajectories generated by selecting particular values of $\theta \in \boldsymbol{\Theta}$. Consequently, this approach overcomes the wrapping effect and preserves the uncertain parameter time dependency, but the problem of the interval function (in this case the trajectory function $x(t, u, x_o, \theta)$) range evaluation still remains. However, region based approaches present a lower computationally complexity than trajectory based approaches being this their main interest.

3. PROBLEMS ASSOCIATED TO INTERVAL SIMULATION

3.1 Range evaluation of an interval function

When the behaviour of a dynamic system described by (1) is interval simulated, the interval hull of the region of system states should be evaluated using either the state space function $\dot{x}(t) = f(x(t), u(t), \theta)$, or the trajectory function $x(t, u, x_o, \theta)$. The determination of this interval hull at each time step implies the computation of the range of previous interval functions. One possibility for evaluating this range is to apply directly **interval**

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[2] In case of a discrete-time system, the isotone property is requiered (Cugueró, 2002). This property is satisfied if the state vector function $\boldsymbol{f}$ is monotone increasing with respect all component of state vector $\boldsymbol{x}$ (Cugueró, 2002).

arithmetic substituting operations between real numbers by operations between intervals (Moore, 1966). But, although the ranges of basic interval arithmetic operations are exactly the ranges of the corresponding real operations, this is not the case if the operations are composed since multi instances of the same variable are not taken into account. For instance: let us consider $x \in [-1,1]$, then the interval for $z = x - x$ should be 0, but when applying interval arithmetic is $[-2,2]$. This phenomenon is termed as *interval dependence* or *multi-incidence problem* (Moore,1966). One possibility to avoid this problem is to combine the use of interval arithmetic with a branch and bound algorithm (Hansen, 1992). Another possibility to evaluate the range of an interval function is to solve two optimisation problems (a minimisation and a maximisation) using numerical methods. But, classical numerical optimisation algorithms can only guarantee *local optimums* since they are gradient based. Global optimums can only be obtained if the optimisation problems associated with the range evaluation are convex. However, in general, to guarantee global optimums in non-convex optimisation problems, global optimisation algorithms based on branch and bound should be used (Puig, 1999).

3.2 Wrapping effect

The problem of wrapping is related to the use of a crude approximation (its interval hull) of the solution set and its iteration using one-step ahead recursion of the state space function $\dot{x}(t) = f(x(t), u(t), \theta)$, i.e., using region based approaches. This problem does not appear if instead the trajectory function $x(t, u, x_0, \theta)$ is used. When using the one-step ahead recursion approach, at each iteration, the true solution set $X(t)$ is wrapped into a superset feasible to construct and to represent the real region on a computer (interval box, ellipsoid, zonotopes*)*. Since the overestimation of the wrapped set is proportional to its radius, an spurious growth of the enclosures can result if the composition of wrapping and mapping is iterated (Kühn,1998) as it is shown in Fig. 2. This *wrapping effect* can be completely unrelated to the stability properties of the interval system, and even stable systems are shown to exhibit exponentially fast growing enclosures that are useless for practical purposes. Not all the interval systems exhibit this problem, as is the case of cooperative (or positive) systems introduced in *Section 1* .
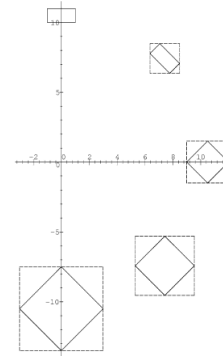

Fig. 2 Wrapping effect

The wrapping effect was first observed with the advent of interval methods and its application to the validated solution of initial value problems in the early 60s (Moore,1966). Since then, several approaches have been proposed to avoid it when one step-ahead iteration of the interval system is used: either rotating the state space of the interval system as it was proposed first by Moore (1966) using a change of coordinates and then by Lohner (1987) using a QR-factorisation, or approximating the state space region using a better approximation than interval hull (box) approximation, for example, using ellipsoids (Neumaier, 1993) or using zonotopes (Kühn, 1998).

3.3 Time variation of uncertain parameters

An additional issue should be taken into account when an interval model, as (1), is used: uncertain parameter time-invariance is not naturally preserved using one-step ahead recursion algorithms. If one-step recursion scheme is used (ElGhaoui,1999), the set for system states $X(t+\Delta t)$ is approximated by a set computed using previous sets approximating system state region $X(t)$ and the set for uncertain parameters Θ. Then, the relation between parameters and states is not preserved since every parameter contained in the parameter uncertainty region Θ is combined with every state in the set approximating state region $X(t)$ when determining the new set approximating state region $X(t+\Delta t)$. Thus, recursive schemes based on one-step are intrinsically time varying (Adrot, 2003)(Puig, 2003). Time-invariance in parameters can only be guaranteed if the relation between parameters and states is preserved at every iteration. One possibility to preserve this dependence is to derive a functional relation between states and parameters at every iteration that will transport the system from the initial state to the present state. Then, two approaches about the assumption of the time-variance of the uncertain parameters are possible:

- the *time-varying approach* which assumes that uncertain parameters are unknown but bounded

in their uncertainty intervals and can vary at each time step since one-step ahead recursion (i.e. region based approach) algorithms are used. This is the approach followed by Barmish (1979), ElGhaoui (1999), Norton (1999) and Puig (2001), among others.

- the ***time-invariant approach*** which assumes that uncertain parameters are unknown but bounded in their uncertainty intervals and guarantee that they can not vary at each time step since a functional relation between parameters and states is used (i.e. trajectory based approach) instead of a one-step ahead recursion. This is the approach followed by Tibken (1993), Bonarini (1994), Horak (1988) and Puig (1999), among others.

Considering one or other assumption about parameter time-invariance different algorithms and results are obtained. Typically the time varying (region based) approach is preferred in the literature because of its lower complexity. However, cooperative systems since they do exhibit the wrapping effect, a time-varying interval simulation based on one-step recursion will provide the same interval simulation than a time-invariant based on vertices algorithm described in *Section 1* (Cugueró, 2002). This result can be interpreted intuitively: states ranges can be computed independently from uncertain parameters and states since isotonic parameters and states are decoupled. In this case, not preserving the relation between parameters and states is not important at all.

3.4 Bounding the discretisation error

In case that the uncertain system is described by a continuous time model as (1), some discretisation should be introduced when computing its numerical interval solution. The usual discretisation is based on one-step ahead recursion that provide the following approximation of the solution at time t starting at time t-h being h de discretisation step:

$$x(t,u,x_{t-h},\theta) = x^a(t,u,x_{t-h},\theta) + e(t,u,x_{t-h},\theta) \qquad (7)$$

where: $x^a(t,u,x_{t-h},\theta)$ is the approximating function and $e(t,u,x_{t-h},\theta)$ is the approximating error due to discretisation. One the most used methods to compute the approximating function $x^a(t,u,x_{t-h},\theta)$ is based on the Taylor series of the state function $\dot{x}(t) = f(x(t),u(t),\theta)$

$$x^a(t,u,x_{t-h},\theta) = x_{t-h} + \sum_{i=1}^{k-1} h^i f^{[i]}(x_{t-h},u_{t-h},\theta) \qquad (8)$$

with: $f^{[1]} = f$ and $f^{[i]} = \dfrac{\partial f^{[i-1]}}{\partial x} f$, $i \geq 2$.

Then, the approximating error is given by

$$e_i(t,u,x_{t-h},\theta) = h^k f_i^{[k]}(x_{\xi_{ii}}, u_{\xi_{ii}}, \theta) \qquad (9)$$

evaluated at a given but unknown $\xi_i \in [t-h,t]$ with $i = 1,...,n$. Then, the interval solution $[x_t]$ at time t using as initial condition the interval solution $[x_{t-h}]$ at time t-h is based on computing the interval extension of the approximating solution (8) and discretisation error (9):

$$[x_t] = [x_t^a] + [e_t] \qquad (10)$$

Interval extensions of (8) are computed by replacing each occurrence of x_{t-h} and θ by its corresponding interval and each standard function by its interval evaluation, while to evaluate interval extension of (9) the interval for x_ξ should be obtained such that $x_\xi \in [\tilde{x}_{t-h}]$ for all $\xi \in [t-h,t]$ using for example the Picard-Lindelöf operator (Moore, 1966). However, as explained in *Section 2.1* using interval extensions by replacing real numbers in a function by intervals often leads to large overestimations what derive in an interval simulation $[x_t]$ that always increases, even if the true solution contracts. A better approach is to apply the mean-value theorem to $f^{[i]}$ at some $\hat{x}_{t-h} \in [x_{t-h}]$, we have:

$$f^{[i]}(x_{t-h}) = f^{[i]}(\hat{x}_{t-h}) + J(f^{[i]}; x_{t-h}, \hat{x}_{t-h})(x_{t-h} - \hat{x}_{t-h}) \qquad (11)$$

where: $J(f^{[i]}; x_{t-h}, \hat{x}_{t-h})$ is the Jacobian of $f^{[i]}$ with its jth row evaluated at $x_{t-h} + \alpha_{il}(\hat{x}_{t-h} - x_{t-h})$ for some $\alpha_{ij} \in [0,1](j = 1,\cdots,n)$. Other classical integration methods such as Runge-Kutta can also be used in discretisation process. However, as the error term contains the Taylor error term, these interval methods do not usually provide better enclosures.

4. APPROACHES COMING FROM ODE VALIDATED SOLUTION AND CONSTRAINT SATISFACTION COMMUNITIES

This group of approaches come from the field of Numerical Analysis and Applied Mathematics, and in particular, from the Interval Analysis community. Interval Analysis was introduced by Moore in the sixties (Moore, 1966). The problem of interval system simulation was first formulated by Moore in the context of the validated solution of ordinary

differential equations (ODE) (Moore, 1966)(Moore,1979). He proposed an algorithm based on Taylor series and interval arithmetic. He discovered the **wrapping effect** and proposed an algorithm to avoid this problem using a transformation of the state space. Since then, many algorithms for ODE validated solution have been proposed. All the approaches presented in this section are region based except the Berz's algorithm.

4.1 Moore's algorithm

Moore's algorithm is based on the Taylor series expansion described in *Section 3.4*. Then, the following one-step ahead formula to the interval solution $[x_t]$ at time t using as initial condition the interval solution $[x_{t-h}]$ at time *t-h* is obtained combining (10) and (11)

$$[x_t] = \hat{x}_t + [S_{t-h}]([x_{t-h}] - \hat{x}_{t-h}) + [e_t] \qquad (12)$$

where[3]:

$$[S_{t-h}] = I + \sum_{i=1}^{k-1} h^i J(f^{[i]}; [x_{t-h}], [\theta])$$

$$\hat{x}_t = \hat{x}_{t-h} + \sum_{i=1}^{k-1} \frac{h^i}{i!} f^{[i]}(\hat{x}_{t-h}, \hat{\theta})$$

$$[e_t] = h^k f^{[k]}([\tilde{x}_{t-h}], \theta)$$

with $\hat{x}_t$, $\hat{x}_{t-h}$ and $\hat{\theta}$ being the mid-points respectively of intervals $[x_t]$, $[x_{t-h}]$ and $[\theta]$.

If the previous formula is evaluated using interval arithmetic, the obtained enclosures are smaller than the obtained applying the formula (10) corresponding directly to the Taylor series expansion, but still this evaluation could be seriously affected by the wrapping effect (Moore,1966).

4.2 Lohner's algorithm

One of the most successful approaches to reduce the problem due to wrapping effect was proposed by Rudolf Lohner (Lohner,1987). This approach is based on transforming the space state following its rotation avoiding the overestimation that produces a crude approximation of the region of system states just computing for each variable the maximum and the minimum. Lohner's algorithm is an evolution of Moore's algorithm presented in (12) according to

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[3] In case that the uncertain system (1) is in discrete-time formula (12) should be modified taking into account that there is no discretisation error, $[S_{t-h}] = J(f^{[i]}; [x_{t-h}], [\theta])$ and $\hat{x}_t = f(\hat{x}_{t-h}, u, \hat{\theta})$

$$[x_t] = \hat{x}_t + [S_{t-h}]A_{t-h}[r_{t-h}] + [e_t] \quad (13)$$

where: $[r_t] = (A_t^{-1}([S_{t-h}]A_{t-h}))[r_{t-h}] + A_t^{-1}([e_t] - \hat{e}_t)$ being $[r_0] = [x_0] - \hat{x}_0$ and $A_0 = I$. A good choice for A_t is the Q-factor from the QR-factorisation of the mid-point of $[S_{t-h}]A_{t-h}$. Lohner's algorithm works very well in many systems but Kühn (1998) has discovered some cases where this approach fails. To avoid the problems of Lohner's algorithm, Kühn has proposed a new algorithm based on approximating the region of system states of the interval system using zonotopes.

4.3. Neumaier's algorithm

Instead of using an interval hull of the reachable set $X(t)$, Neumaier (1993) proposes to use the smallest ellipsoid containing it and a new method for reducing the wrapping effect based on an interval ellipsoid arithmetic. In this paper, only the algorithm for the linear case will be presented. The extension for the non-linear case is a very simple task, since the non-linear system will be linearized around the estimated trajectory (Neumaier, 1993). An ellipsoid is the set of the form:

$$\mathcal{E}(z, L, r) = \left\{ z + L\xi \mid \xi \in \Re^n, \|\xi\| \le r, r > 0 \right\} \quad (14)$$

where $z \in \Re^n$ is the centre, $L \in \Re^{n \times n}$ is the axis matrix and $r \in \Re$ is the radius. This algorithm generalise for an uncertain system described as (3), the property that for a linear certain system given the ellipsoid enclosing the set of possible states at time *t-h* such that $\hat{x}_{t-h} \in \mathcal{E}(z_{t-h}, L_{t-h}, r)$, then the enclosing ellipsoid at time t such that $\hat{x}_t \in \mathcal{E}(z_t, L_t, r)$ can be constructed by propagating separately the centre and axis matrix according to:

$$z_t = Az_{t-h} + Bu_{t-h}$$
$$L_t = AL_{t-h} \qquad (15)$$

being implicitly relative. The interval simulation then can be generated by computing the interval hull of the ellipsoid $\mathcal{E}(z_k, L_k, r)$ at each time instant according to

$$[\hat{x}_t] = z_t + [-r, r] \|L_{ti \bullet}\| \quad (16)$$

where $i \bullet$ represent the ith row of the matrix L_t. The advantage of using ellipsoids instead of parallelepipeds as in Moore's and Lohner's algorithm is that the rotation of the state space of the interval system is implicit being not necessary to make additionally computations. The disadvantage is that the algorithm for computing with ellipsoids is more complicated than those of parallelepipeds and in

general the wrapping effect when uncertain parameters are considered is not avoided (Neumaier, 1993).

4.4 Kühn's algorithm

Kühn's algorithm (1998) is based on approximating the region of system states using zonotopes. A zonotope $\mathcal{Z}$ of order m is the Minkowski sum

$$\mathcal{Z} = \mathcal{P}^1 + \cdots + \mathcal{P}^m \qquad (17)$$

of m parallelepipeds $\mathcal{P}^i$ (Fig. 3). The order m is a measure for the geometrical complexity of the zonotopes. It can be chosen freely and is a performance parameter for the Kuhn's algorithm. Given the zonotope $\mathcal{Z}_{t-h}$ enclosing the reachable set $X(t-h)$, then the reachable set $X(t)$ is enclosed by the following zonotope

$$\mathcal{Z}_t = \mathcal{R}(E_t + T_t \mathcal{Z}_{t-h}) \qquad (18)$$

where T_t are square matrices and E_t are intervals such that

$$f(\mathcal{Z}_{t-h}) \subseteq E_t + T_t \mathcal{Z}_{t-h} \qquad (19)$$

and the reduction operator $\mathcal{R}$ is defined in the following way: let $\mathcal{Z} = \mathcal{P}^0 + \mathcal{P}^1 + \cdots + \mathcal{P}^m$ be a $m+1$ zonotope and $1 \le \ell \le m$ be the largest integer such that:

$$diam(\mathcal{P}^0 + \mathcal{P}^1 + \cdots + \mathcal{P}^{\ell-1}) \ge diam(\mathcal{P}^\ell) \quad (20)$$

or $\ell = 1$ otherwise, then:

$$\mathcal{R}(\mathcal{Z}) := \square(\mathcal{P}^0 + \mathcal{P}^1 + \cdots + \mathcal{P}^\ell) + \mathcal{P}^{\ell+1} + \cdots + \mathcal{P}^m \quad (21)$$

According Adrot (2003), the reachable set $X(t)$ when the state function f is linear and only uncertainty in initial conditions is considered is a zonotope. Then, Kühn's algorithm provides a good solution to the enclosure of $X(t)$. However, considering only uncertainty in parameters, the reachable set $X(t)$ becomes a more complex than a zonotope. In this case even approximating this set using subpavings and algorithms to propagate them (Jaulin, 2001), the wrapping effect could not be avoided at a reasonable computing time.

4.5 Berz's algorithm

Berz's algorithm (1998) for interval simulation proposes formulating the differential equation that describes system dynamics as an integral equation:

$$x(t) = x(0) + \int_0^t f(x(\tau), u(\tau), \theta) d\tau \qquad (22)$$

and solving it using high-order Taylor series expansions in both time and initial conditions. Then, using interval arithmetic validates existence and uniqueness and also computes tight bounds on the solution at each time iteration in one phase. This algorithm has been implemented in COSY INFINITY simulator (Berz,1997). This algorithm avoids the wrapping problem because it computes actual interval hull for the reachable set states based on the initial interval for system states, preserving the functional relation between initial and final state values a not approximating and propagating the state uncertainty using one step ahead iterations.

4.6 Recent improvements

Recently, Nedialkov (1999) has developed a new algorithm based on an interval Hermite-Obreschkoff method. It is an interval Taylor series method that consists of two phases: a predictor and a corrector. The predictor computes an initial enclosure of the solution and from that the corrector computes a tighter enclosure. The corrector applies a Newton-like step to tighten the initial enclosure. Comparing with traditional Taylor series methods based on Lohner's algorithm for the same order and step size, Nedialkov's algorithm provides smaller local error, better stability and fewer Jacobian evaluations. The extra cost introduced by the use of the Newton step is one matrix inversion and a few matrix multiplications. On the other hand, from the Artificial Intelligence field, particularly from the Constraint Satisfaction community, new approaches have appeared that move in the same direction that Nedialkov's algorithm, but using consistency techniques. Deville (1998 has presented, for the first time, the idea of applying consistency techniques for the generation and for the reduction of solution enclosures. The idea of this approach consists in generalising classical methods for validated solution of ODEs into a two-step process: a forward process that computes an initial enclosure and a backward process that reduces this enclosure. Consistency techniques apply naturally to the backward (pruning) step but can also be applied to the forward step.

5. APPROACHES COMING FROM AUTOMATIC CONTROL AND FAULT DETECTION COMMUNITIES

The group of approaches described in this section come from the field of Automatic Control and Fault Detection. These approaches have considered explicitly the problem of parameter uncertainty, while approaches described in previous section do not address directly this problem. They can be classified in two groups depending on the assumption about the temporal variance of uncertainty.

5.1 Time-varying approaches

Time-varying approaches assume, as it has been described in *Section 2*, that uncertain parameters are

only known to belong to their corresponding uncertainty intervals but from iteration to iteration the value can change because a one-step ahead recursion is used. The first time-variant approach was proposed by Barmish (1979) considering linear uncertain systems. This approach is based on propagating recursively the state space region using polytopes. Recently, El Ghaoui (1999) proposed one approach based on approximating the state space region using ellipsoids also considering linear uncertain systems. The ellipsoids are propagated recursively solving convex optimization problems using LMI. This approach has been extended by Calafiore (2003) to quadratic uncertain systems. On the other hand, Puig (2001) has proposed an approach based on approximating the state space region using zonotopes and Kuhn's algorithm presented in *Section 3* of this paper. Uncertain parameters are considered as extra states, as in extended Kalman Filter. Finally, Adrot (2003) has proposed an algorithm based on the use of subpavings (Jaulin, 2001) to propagate an enclosure of the reachable set at each iteration. At the same time, Cherrier (2003) has proposed an algorithm based on propagating independently each bound of the interval enclosing the reachable set by formulating an augmented system where the new state vector is composed by the upper and lower bounds of the state. Additionally conditions of the stability of such scheme based on Lyapunov theory have been introduced.

5.2 Time-invariant approaches

Time-invariant approaches assume, as it has been described in *Section 2*, that uncertain parameters are only known to belong to their corresponding uncertainty intervals and from iteration to iteration the value can not change. Time-invariance in parameters can only be guaranteed if the relation between parameters and states could be preserved at every iteration. One possibility to preserve this dependence is to derive a functional relation between states and parameters at every iteration that will transport the system from the initial state to the present state. In the case of a discrete time-invariant linear system in state space, this relation is the solution of (1) for a particular values of the system matrices A and B:

$$x(k) = A^k x(0) + \sum_{j=0}^{k-1} A^{(k-1-j)} Bu(j) \quad (23)$$

In deriving (5), it has been assumed time-invariant parameters, i.e., $A(k+1) = A(k)$ and $B(k+1) = B(k)$. At the same time that time invariance is preserved the wrapping effect is avoided because state uncertainty is not propagated from step to step but instead always from the initial state as in the case of Berz's algorithm presented in previous section. The first-time invariant approach

was proposed by Horak (1988) in the fault detection community. He proposed to formulate the interval simulation as an optimisation problem using the maximum principle. The optimisation problem is then solved approximately using dynamic optimisation algorithms. Later, Tibken (1993,1995) proposed formulating the time invariant interval simulation for non-linear uncertain systems in discrete-time formulating an optimisation problem that has as objective function the functional relation between the initial state and the present by optimising directly the state trajectory $x(k, u, x_o, \theta)$ subject to the uncertainty on the initial state and parameters. The optimisation problems associated with this approach are solved using *global optimisation* algorithms based on *branch and bound* and *interval arithmetic* proposed by Hansen (Hansen,1992). A global optimisation algorithm is needed as justified in *Section 2* when optimising directly the state trajectory. Danes (1995) using tools from optimal control reformulates the problem of interval simulation as a dynamic optimisation problem. The problem is nicely reformulated but it does not provide a feasible computation algorithm. Finally, another algorithm for simulating an interval linear system coming the fault detection community was proposed by Puig (1999,2003). This algorithm is based on formulating the interval simulation as a global optimisation problem as was proposed by Tibken but exploiting the fact that the system is linear. In this case the objective function is describing the state trajectory is (23). The main drawbacks of this approach and of Tibken's approach, besides its high computational complexity, is that since computations are referred to the initial state the optimization problem grows with time. In case of Puig's approach the objective function is a polynomial with degree increasing by one at every iteration since the uncertain systems considered are linear. Theferore, the amount of computation needed is increasing with time being impossible to operate over a large time interval. Then, some kind of approximation should be introduced to make the approach more tractable. Assuming that the uncertain system is asymptotically stable, any transients in the system settle to negligible values in a finite-time, more precisely in t_s/T samples, being t_s is the system settling time and T the sample time. This assumption implies that the outputs of the system at time k depend only on the inputs that occurred during the last t_s/T samples with an accuracy selectable by the choice of the length L of a **time sliding window**. Typically this length is of the order of the settling time measured in number of samples. Therefore, for any time k, it is possible to approximate (23) using a sliding window, starting at time $k-L$ and ending at k:

$$x(k) = A^L x(k-L) + \sum_{j=k-L}^{k-1} A^{(k-1-j)} Bu(j) \quad (24)$$

Of course, the parameter time-invariance is only guaranteed inside the sliding window. This is why this approach is knows as **almost time-invariant** (Puig, 2003). It also can be easily extended to deal with the problem of interval observation (2002).

In Armengol (2001), it is proposed a similar approach than in Puig (1999.2003). The main difference is that optimisation problem is solved using a global optimisation algorithm based on branch and bound and modal interval arithmetic. This interval arithmetic in some cases can handle multi-incidences very efficiently and provide directly inner and outer solutions.

6. APPROACHES COMING FROM CIRCUIT THEORY COMMUNITY

In the field of circuit theory, interval simulation is related to the problem to the problem of analysing the effect of tolerances in components on the transient response. First studies of this problem are due to Oppenheimer (1988). He proposed an algorithm which enables to generate estimates for solution bounds of initial-value problems described by systems of linear, autonomous, first-order ordinary differential equations with a parameter which belongs to an interval and which enter linearly into the system description. This algorithm is based on a method of constructing augmented partial sums which approximate interval matrix exponential function as closely as desired, using interval arithmetic tools. Later, Kolev (1993) has proposed two methods that provide approximate solutions for interval simulation of linear circuits with constant input. The first method is based on computing only the solutions taking the vertices of the space of uncertainty defined by parameters and states. This method only provide an inner solution. The second method is based on the previous method and the implicit Euler method for integrating ordinary differential equations. It uses some interval analysis techniques and proves to be more computationally efficient than the first method. Recently, Femia (1999; 2000) has proposed the combined use of genetic algorithms and affine arithmetic, a modified interval arithmetic that takes into account dependencies of first order, that provides an inner and an outer solution. Genetic algorithms are used to improve pure Monte Carlo simulation providing the inner solution and affine arithmetic provides an outer solution. However, because the wrapping effect is not correctly handled, in many cases simulation derive in unstable interval simulations.

7. APPROACHES COMING FROM ARTIFICIAL INTELLIGENCE COMMUNITY

The approaches presented in this section come from the field of Artificial Intelligence, in particular, from the Qualitative Reasoning community leaded by Kuipers (Kuipers,1994). In particular these approaches are an evolution of qualitative simulation. A deep survey of such approaches when applied to fault detection can be found in Armengol (2000).

7.1 Qualitative simulation

Qualitative simulation allows the simulation of systems which are incompletely specified by transforming the system into a related system in a more abstract space of qualitative values where model imprecision can be dealt with by rules of qualitative mathematics. A qualitative description of a model (or QDE for Qualitative Differential Equation) is composed of the set of model variables whose domains are defined in terms of quantity spaces, the structural constraints that describes the relationship between variables in terms of arithmetic operators and functions, and shape constraints on the functional relations. The structural level of the model is known as SDE (for Structural Differential Equation). Using QSIM algorithm developed by Kuipers (Kuipers, 1986), it is possible to simulate the behaviour of a qualitative model. QSIM simulates the qualitative model by tracking the magnitude and direction of change of each variable, where the magnitude is defined as being at or between two landmarks in the variable's quantity space and the direction of change is either increasing, steady, or decreasing starting from a user-specified initial state. By applying the rules of qualitative arithmetic, QSIM generates a set of variable descriptions consistent with the model constraints at the initial state. It models time as a sequence of alternating points and intervals in time. QSIM produces a set of alternating time-point and time-interval qualitative descriptions of the system. Such a set of state is called a **qualitative behaviour**. Normally, there will be multiple qualitative behaviours for a given model because the qualitative model represents a family of ODE systems and since qualitative mathematics is inherently ambiguous. Therefore, some behaviours generated by this method will be spurious. However, one property of QSIM is that all real behaviours of the qualitative model are predicted. But, one weakness of QSIM is that some spurious behaviours may also be predicted.

7.2 Semiquantitative simulation

Since QSIM ignores information at the quantitative level, it generates the set all possible qualitative behaviours of ODEs entailed by the qualitative model, but it may also include behaviours of systems that are not part of the family. To bridge the gap between qualitative and quantitative simulation, this qualitative reasoning community has proposed an approach known as **semiquantitative simulation**. This approach is based on semiquantitative models. Semiquantitative models (or SQDE for

Semiquantitative Differential Equation) reduce model imprecision by adding numerical knowledge to the purely qualitative representation, and in some cases refuting qualitative behaviours predicted by a pure qualitative simulation because they are inconsistent with the quantitative information. Predictions from semiquantitative models are more precise (i.e. more tightly bounded), while still retaining the accuracy (i.e. all possible behaviours are found) provided by purely qualitative methods. The Q2 algorithm proposed by Kuipers and Berleant (Kuipers, 1988) produces semiquantitative inferences from a SQDE. At each qualitative time-point, QSIM defines events for each variable. An event is the magnitude of a variable with the magnitude of the time variable at the time-point. Since an event represents the instantaneous snapshot of a particular variable at a particular time, the portion of the structural differential equation of the semiquantitative model can be viewed as a set of equations relating the model variables, and the events must satisfy these equations. Recording the events with the range information described in the semiquantitative model, the process end up with a set of interval equations that must be hold at each time-point. This system of equations can be solved using propagation of intervals through model constraints using interval arithmetic (Moore,1966). Whenever an equation that computes the value of a model variable is found, an intersection of its existing range with the computed one is performed. If their intersection reduces the range, the updated range is asserted and the consequences of this new range are followed. If the intersection is null, the behaviour is refuted since the interval equations are inconsistent with the qualitative behaviour. This process is continued until either the state is refuted or a fixed-point is reached with respect to the ranges of all variables. The propagation algorithm can be extended across time-intervals by suing the mean-value theorem to relate the derivative of each state variable to the bounds of the state variable and the bound on the difference between the two time-points. This type of inference reduces the width of the interval associated with the magnitude of the time value at each time-point. Thus Q2 refines the behaviour description produced by QSIM by generating numeric bounds for both events and time-intervals. While Q2 is a powerful inference method for reducing the ambiguity in the qualitative behaviour description, it may still permit spurious behaviours and overly wide dynamic trajectory envelopes. In Q3 (Berleant,1992), an adaptive interpolation of landmarks to refine step-size is provided. It can be proved that it converges to numerical behavior as uncertainty tends to zero. Kay (1993)(1996) proposes the use of *dynamic envelopes* that more fully exploits the semiquantitative representation that existing methods as Q2 (Kuipers, 1988) or Q3 (Berleant,1992), because they use a simulation time-step determined by qualitative

distinctions. The problem is that the precision of a numerical simulation is directly related to the number and density of the time-points in the simulation and a simulation whose time-step is based solely on the qualitative distinctions can not adequately control these quantities. The approach proposed by Kay works by numerically simulating a set of differential equations whose solutions are guaranteed to bound all behaviours of the semiquantitative model. This approach solves the problem presented before by using a standard numerical method (such as Euler, Runge-Kutta,...) which chooses time-points based on local simulation error estimated, resulting in a much smaller time-step, and hence a more precise simulation. The remaining imprecision more closely reflects the incomplete knowledge in the model itself. To numerically simulate de bounds of the semiquantitative model, Kay proposed to find a set of extremal equations for a system. An extremal equation is a bound on the derivative of a state variable, as opposed to a bound on the value of the state variable. It may be either minimal or maximal. The extremal ODE system is computed using an a lower and upper translation for each equation of the semiquantitative model. Thus, the original imprecisely defined system is replaced with a precise ODE system of twice the order that is guaranteed to bound the original semiquantitative model. Since the new system is an ODE system, it can be simulated using conventional simulation methods. The simulator developed by Kay that uses this idea is called NSIM. While NSIM produces tighter bounds than Q2 in many cases, it can also generate spurious behaviours in the form of overly-wide dynamic envelopes. Another approach born in the QR community is the proposed by Martinez Gasca (1998). It is based on following the uncertainty state region centre and on computing using a transformation matrix the uncertainty region but always relatively to the uncertainty state region centre. The transformation matrix is derived using a linear approximation of the uncertain system around the uncertainty state region.

7.2 Fuzzy Semiquantitative Simulation

In parallel, several researchers have proposed to represent the uncertainty in parameters and states by fuzzy sets instead of intervals. Firsts approaches come from Bonarini and Bontempi (1994). They proposed extending interval simulation to fuzzy simulation by means of α-cuts. But the underlying simulation is interval based. The interval-based simulation algorithm is based on mixing a classical optimisation algorithm with a classical differential equation algorithm formulating the simulation problem always from the initial condition to avoid the wrapping effect (Bontempi, 1996) as in Berz's algorithm (1998). They implemented these algorithms in a series of simulators called QuaSI I, II

and III. Later, Hüllermeier (1997)(1999) proposed a method for fuzzy simulation based on three kind of discretisation: with respect to the fuzzy problem by a finite number of crisp problems using α-cuts, with respect to time and with respect to space approximating reachable sets by geometrical bodies and replacing such bodies by a finite number of points. This method provides an outer approximation or even exact results in the limit. Finally, Keller proposes an algorithm for fuzzy simulation also based in α-cuts that uses vertices algorithm approach (Kolev, 1993) to generate an envelope that it is improved using splines interpolation.

8. CONCLUSIONS

In this paper, simulation of uncertain dynamic systems described by interval models is presented using different group of approaches coming from several research areas: ODE validated solution, constraint satisfaction, automatic control, fault detection, circuit theory and qualitative reasoning. All these approaches can be classified in region or trajectory based. Typically region based approaches present a lower computationally complexity than trajectory based approaches being this their main interest. However, region based approaches when applied to even linear systems but with uncertain parameters still present instability problems because of the wrapping effect. The main problems that each approach should face in order to produce a good interval simulation, namely the range evaluation of interval functions, the wrapping effect, the parameter time-invariance and the discretisation errors (in case of continuous time systems) are presented and discussed. All these approaches have never been presented together establishing a connection between them. In many cases, each research community worked without knowing many about the research developed by the others. The problems described in *Section 2* appear recurrently in all the approaches, but even in some approaches they are not considered. The aim of this paper is to remark that the field of interval simulation is very wide when it is considered from several points of view and when developing new interval simulation algorithms in each field of application all existent results and solutions to the presented problems coming from the other fields should be considered.

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