PASSIVITY ANALYSIS AND PASSIFICATION OF DISCRETE-TIME HYBRID SYSTEMS

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Abstract: This paper proposes several (sufficient) criteria based on the numerical solution of systems of linear matrix inequalities (LMIs) for proving the passivity of discrete-time hybrid systems in piecewise affine form, and for the synthesis of switched linear control laws that enforce passivity.

Keywords: Passive systems, hybrid systems, piecewise affine systems, discrete-time models.

1. INTRODUCTION

Passivity is a widely adopted tool for analyzing the stability of interconnections of dynamical systems (see Willems (1972); Hill and Moylan (1980); Lozano et al. (2000)). Passivity is used in several domains of engineering sciences, such as in electrical circuit and mechanical system analysis (see Arimoto (1996)), and even in the study of complex phenomena (see Chua (1999)). In particular, passivity is exploited in robotics as a key concept for stability analysis of human/machine interactions involving haptic interfaces (see Miller et al. (2000); Colgate and Schenkel (1997)).

Passivity analysis of interconnected systems hinges upon the ability of characterizing the passivity properties of a single dynamical system. For linear systems a solid theory and analytical/numerical criteria are available, and theoretical characterizations were developed for smooth nonlinear dynamical systems. Most passivity characterizations were proposed for continuous-time models, and recently for sampled-data systems (see Stramigioli et al. (2002)).

In many practical applications, some of the system components exhibit a heterogeneous dynamical discrete and continuous nature that cannot be captured by smooth models because of abrupt mode switches. The study of *hybrid systems*, that has massively emerged in the last few years, has been devoted to analyzing the dynamical interaction between continuous and discrete signals in one common framework (see Antsaklis (2000)). Passivity analysis of hybrid models has received little attention, with the only exception of the contributions of Camlibel et al. (2002), Mahapatra (2003), Zefran et al. (2001), and Pogromski et al. (1998) who formulate a notion of passivity for continuous-time switched systems.

In this paper we characterize the passivity of discrete-time hybrid systems in piecewise affine (PWA) form (Sontag, 1981). Our motivating practical reason for addressing hybrid passivity issues in discrete-time stems from the need of studying haptic problems, where a haptic device interacts with a naturally discrete-time virtual environment (see Colgate and Schenkel (1997)).

After formulating the passivity problem for discrete-time PWA systems and providing a methodology for discretizing linear continuous-time submodels that preserves passivity properties, we propose several (sufficient) criteria for proving the passivity of a given PWA system, and for the synthesis of switched linear control laws that enforce passivity. Such criteria are based on the numerical solution of systems of linear matrix inequalities (LMIs).

2. PASSIVITY ANALYSIS FOR DISCRETE-TIME PWA SYSTEMS

Following the approach of Cuzzola and Morari (2001), in this paper we consider linear discrete-time PWA systems of the form

$$\begin{array}{rl} x_{k+1} &= A_i x_k + B_i u_k + \phi_i \\ y_k &= C_i x_k + D_i u_k + \psi_i \\ & i = 1,\ldots,s \end{array} \quad \text{if } \begin{bmatrix} x_k \\ u_k \end{bmatrix} \in \chi_i, \tag{1}$$

where $k \in \mathbb{T} = \{0, 1, \ldots\}$, $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^m$ is the control input, $y_k \in \mathbb{R}^p$ is the output vector, $A_i, B_i, C_i, D_i, \phi_i, \psi_i$ are constant matrices/vectors of suitable dimensions, and χ_i, $i = 1, \ldots, s$ is a polyhedral partition of a given subset $\mathbb{X}$ of $\mathbb{R}^{n+m}$. The set $\mathbb{X}$ in which the state-input pair is defined is assumed to contain the origin. Each element χ_i of the partition is referred to as a *cell*. Let each cell be a polyhedron of the form

$$\chi_i = \left\{ [x^T \ u^T]^T \in \mathbb{X} \ : \ F_i^x x \geq f_i^x, \ F_i^u u \geq f_i^u \right\} \tag{2}$$

where $F_i^x, f_i^x, F_i^u, f_i^u$, $i = 1, \ldots, s$ are constant matrices/vectors. Furthermore, consider the polyhedra

$$\bar{\chi}_j = \left\{ x \in \mathbb{X} \ : \ F_j^x x \geq f_j^x \right\}, \quad j = 1, \ldots, t \tag{3}$$

where $\bar{\chi}_j \neq \bar{\chi}_h$, $\forall j \neq h$; $j, h = 1, \ldots, t$, and $t \leq s$, and the sets of indices

$$\mathcal{S}_j = \left\{ i \ : \ \exists \ [x^T \ u^T]^T \in \chi_i \ : \ x \in \bar{\chi}_j \right\}, \ j = 1, \ldots, t. \tag{4}$$

Let us denote $\mathcal{I} = \{1, \ldots, s\}$ and $\mathcal{J} = \{1, \ldots, t\}$. Note that $\cup_{j=1}^t \mathcal{S}_j = \mathcal{I}$ and that by (2) the sets $\mathcal{S}_j$ are mutually disjoint.

Let the origin be an equilibrium point of system (1) with zero inputs, that is, ϕ_i, $\psi_i = 0$ for all i such that $0 \in \chi_i$. From the standard dissipativity notion for discrete-time systems (see Lozano et al. (2000)) we have that system (1) is *dissipative* with respect to a given *supply function* $s(u, y) : \mathbb{R}^{m+p} \to \mathbb{R}$ if there exists a constant β such that

$$\sum_{i=0}^{k} s(u_i, y_i) > \beta \quad \forall k \in \mathbb{T}. \tag{5}$$

For $p = m$, system (1) is said to be *passive* if it is dissipative with respect to the supply function $s(u, y) = u^T y$, i.e., if there exists β such that

$$\sum_{i=0}^{k} y_i^T u_i > \beta \quad \forall k \in \mathbb{T}. \tag{6}$$

The following standard result characterizes the dissipativity and passivity conditions (see Lozano et al. (2000)).

Theorem 1. If there exists a positive definite function $V(x) : \mathbb{X} \to \mathbb{R}$ (called the *storage function*) such that along all system trajectories (x_k, u_k, y_k), $[x_k^T \ u_k^T]^T \in \mathbb{X}$, the following inequality holds

$$V(x_{k+1}) - V(x_k) - s(u_k, y_k) < 0 \tag{7}$$

then the system is dissipative with respect to $s(u, y)$. In particular, if

$$V(x_{k+1}) - V(x_k) - u_k^T y_k < 0 \tag{8}$$

then the system is passive.

Note that both the characterizations of dissipativity and passivity are well-posed under the standard assumption that all state-input trajectories of the system satisfy $[x_k^T \ u_k^T]^T \in \mathbb{X}$, $\forall k \in \mathbb{T}$.
For the purpose of this work, the affine terms ϕ_i and ψ_i in the definition of system (1) will be assumed to be zero. Indeed, the proposed results are easily extended to the affine case by performing a suitable augmentation of state vector as done in (Johansson and Rantzer (1998a), Johansson and Rantzer (1998b)).

A practical motivation for addressing the passivity issue for PWA systems in the discrete-time framework lies in the fact that in typical haptic problems the interaction that occurs between the haptic device and the simulated environment is modeled by a (hybrid) discrete-time system. Moreover, the virtual environment is often a simulated discrete-time equivalent (with zero-order hold and samping time T) of a suitable mechanical system (see Colgate and Schenkel (1997)). In order to investigate the properties of the interaction from the passivity point of view in discrete-time, it is desirable that the simulated environment be passive according to (6) when it is designed to be a discrete-time equivalent of a passive system. The usual zero-order hold equivalent does not preserve passivity, in general. For example, the D matrix of the discrete-time equivalent is zero in the absence of input/output feedthrough, and consequently for any initial output value $y(0) \neq 0$ (5) can be easily violated at time $k = 0$ for all β by choosing a suitable $u(0)$.

In order to preserve passivity after time-discretization, it suffices to define the output of the discrete-time system as

$$y_k = \frac{1}{T}\int_{kT}^{(k+1)T} y(\tau)\, d\tau, \quad \forall k \in \mathbb{T} \tag{9}$$

where $y(t)$ is the continuous-time output, rather than setting $y_k = y(kT)$ as usual. Note that this does not affect causality if the input is held constant in each sampling interval. In turn, this implies

$$T\sum_{i=0}^{k} y_i^T u_i = \int_0^{(k+1)T} y^T(\tau)u(\tau)\, d\tau, \quad \forall k \in \mathbb{T} \tag{10}$$

and therefore the discrete-time equivalent is passive according to (6), provided that the continuous-time system is passive according to the standard passivity notion for continuous-time systems. For a linear time-invariant system it is easily shown that this is accomplished once matrices (A, B, C, D) of the discrete-time equivalent are defined as

$$\begin{aligned} A &= e^{\bar{A}T} \\ B &= \int_0^T e^{\bar{A}\tau}\bar{B}\, d\tau \\ C &= \frac{1}{T}\int_0^T \bar{C}e^{\bar{A}\tau}\, d\tau \\ D &= \frac{1}{T}\int_0^T \int_0^\tau \bar{C}e^{\bar{A}(\tau-\sigma)}\bar{B}\, d\sigma d\tau + \bar{D} \end{aligned} \tag{11}$$

where $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ characterize the corresponding continuous-time system. For a PWA system, the above argument can be repeated cell-wise, although this involves some degree of approximation, as switches occurring between sampling instants cannot be captured by any discrete-time model.

2.1 Passivity Analysis via Quadratic Storage Functions

A standard yet conservative approach to investigating passivity of general nonlinear systems is to check the dissipativity inequality (7) against storage functions of prescribed structure (see Lozano et al. (2000)). In particular, quadratic storage functions of the form

$$V(x) = x^T Px, \quad P = P^T > 0 \tag{12}$$

are the most common choice. Such an approach can be successfully applied to the case of PWA systems of the form (1) and the result is an easy-to-check sufficient condition that mimics the one proposed in Mignone et al. (2000) in the context of stability analysis. Indeed, passivity of system (1) with zero affine terms is ensured if there exists a common quadratic storage function satisfying the passivity inequality for all the linear subsystems defined by (A_i, B_i, C_i, D_i), $i \in \mathcal{I}$. Moreover, checking passivity of each subsystem via a quadratic storage function boils down to a standard LMI condition. Hence, the following result is easily obtained.

Theorem 2. Consider system (1) and let $\phi_i, \psi_i = 0$. If the set of LMIs

$$\begin{cases} P = P^T > 0 \\ \begin{bmatrix} A_i^T PA_i - P & A_i^T PB_i - \dfrac{C_i^T}{2} \\ B_i^T PA_i - \dfrac{C_i}{2} & B_i^T PB_i - \dfrac{D_i + D_i^T}{2} \end{bmatrix} < 0, \quad \forall i \in \mathcal{I} \end{cases} \tag{13}$$

has a feasible solution P, then the system is passive with storage function $V(x) = x^T Px$.

2.2 Passivity Analysis via Piecewise Quadratic Storage Functions

A piecewise quadratic (PWQ) candidate storage function for system (1) is a function $V(x) : \mathbb{X} \to \mathbb{R}$ defined as

$$V(x) = x^T P_i x \quad \forall [x^T\ u^T]^T \in \chi_i, \quad i \in \mathcal{I}, \tag{14}$$

where P_i, $i \in \mathcal{I}$, are suitable symmetric positive definite matrices.
According to (7), if matrices $P_i = P_i^T > 0$, $i \in \mathcal{I}$, exist such that

$$V(x_{k+1}) - V(x_k) - u_k^T y_k < 0 \tag{15}$$

for all system trajectories in $\mathbb{X}$, then system (1) is passive. If this is the case, then the system will be termed *PWQ passive*.
Let us define the set of index pairs

$$\begin{aligned} \mathcal{S} = \big\{(i,j) :\ \exists x \in \mathbb{R}^n, u, w \in \mathbb{R}^m :\ [x^T\ u^T]^T \in \chi_i, \\ [(A_i x + B_i u)^T\ w^T]^T \in \chi_j,\ i, j \in \mathcal{I}\big\} \end{aligned} \tag{16}$$

i.e., the set of all ordered pairs of indices corresponding to possibile switches from cell χ_i to cell χ_j. The following result extends the PWQ stability result of Mignone et al. (2000) to passivity analysis.

Theorem 3. Consider system (1) and let $\phi_i, \psi_i = 0$. If matrices P_i, $i \in \mathcal{I}$ exist such that the set of LMIs

$$\begin{cases} P_i = P_i^T > 0 \quad \forall i \in \mathcal{I} \\ \begin{bmatrix} A_i^T P_j A_i - P_i & A_i^T P_j B_i - \dfrac{C_i^T}{2} \\ B_i^T P_j A_i - \dfrac{C_i}{2} & B_i^T P_j B_i - \dfrac{D_i + D_i^T}{2} \end{bmatrix} < 0 \\ \forall (i,j) \in \mathcal{S} \end{cases} \tag{17}$$

is feasible, then system (1) is PWQ passive with storage function (14).

Clearly, the feasibility of the LMIs in (17) ensures that the left-hand term of the dissipation inequality (15) is negative along all possible system trajectories.

Remark 1. The set of all possible switches $\mathcal{S}$ can be computed by means of reachability analysis using linear programming (see Bemporad et al. (2000)). It is worth noting that the computational burden related to such calculation is usually negligible compared to that needed for solving the LMIs (17).

2.3 Relaxed PWQ Passivity Test

By exploiting the same idea as in Johansson and Rantzer (1998a) and Mignone et al. (2000), a certain amount of conservatism can be removed from the PWQ passivity test introduced above. Indeed, the LMIs (17) imply that the passive behaviour that the system exhibits in each cell χ_i or at the switching point between cells χ_i and χ_j would actually be global if the local system dynamics were extended to the whole state space. This is clearly a restriction. For instance, positive definiteness of each P_i is a stronger condition than requiring $x^T P_i x > 0$ for all $x \in \chi_i$, $i \in \mathcal{I}$.

Some conservatism can be removed from conditions (17). For the sake of simplicity, we assume $f_i^x, f_i^u = 0$ and $\phi_i = \psi_i = 0$ for all $i \in \mathcal{I}$, although the result can be easily extended to the affine case by introducing suitable state, input and output augmentations.
Let F_i be symmetric matrices such that

$$x^T F_i x \geq 0, \ \forall [x^T \ u^T]^T \in \chi_i \tag{18}$$

Moreover, for all $(i, j) \in \mathcal{S}$, let us introduce the set

$$\tilde{\chi}_i^j = \left\{ [x^T \ u^T]^T \in \chi_i : \exists w : [(A_i x + B_i u]^T \ w^T]^T \in \chi_j \right\} \tag{19}$$

i.e., the subset of state-input pairs in cell χ_i at time k which are allowed to evolve into cell χ_j at time $k+1$. In view of (2), it turns out that each $\tilde{\chi}_i^j$ is of the form

$$\tilde{\chi}_i^j = \left\{ [x^T \ u^T]^T \ : \ V_{ij} \begin{bmatrix} x \\ u \end{bmatrix} \geq 0 \right\} \tag{20}$$

where V_{ij} is a constant matrix which can be computed explicitly. Let $G_{ij} \in \mathbb{R}^{n+m} \times \mathbb{R}^{n+m}$ be symmetric matrices such that

$$\begin{bmatrix} x^T & u^T \end{bmatrix} G_{ij} \begin{bmatrix} x \\ u \end{bmatrix} \geq 0, \ \ \forall [x^T \ u^T]^T \in \tilde{\chi}_{ij} \tag{21}$$

Then, PWQ passivity is ensured if the following conditions hold

$$\begin{cases} P_i - F_i > 0 \quad \forall i \in \mathcal{I} \\ \begin{bmatrix} A_i^T P_j A_i - P_i & A_i^T P_j B_i - \dfrac{C_i^T}{2} \\ B_i^T P_j A_i - \dfrac{C_i}{2} & B_i^T P_j B_i - \dfrac{D_i + D_i^T}{2} \end{bmatrix} + G_{ij} < 0. \\ \forall (i,j) \in \mathcal{S} \end{cases} \tag{22}$$

Matrices F_i satisfying (18) can be easily computed from the definition (2) of χ_i by applying the same reasoning as in Johansson and Rantzer (1998a) (note that continuity of $V(x)$ is not an issue in the discrete-time case). Matrices F_i can be chosen of the form

$$F_i = (F_i^x)^T U_i F_i^x \tag{23}$$

where U_i is any matrix with positive entries. By the same argument, taking (20) into account, matrices G_{ij} can be chosen of the form

$$G_{ij} = V_{ij}^T Z_{ij} V_{ij} \tag{24}$$

where matrices Z_{ij} have positive entries. A less conservative version of Theorem 3 which still yields a set of LMI conditions for PWQ passivity is then obtained.

Theorem 4. Consider system (1) with $\phi_i = \psi_i = 0$ and let $f_i^x = f_i^u = 0$. If there exist symmetric matrices P_i, $i \in \mathcal{I}$, matrices U_i with positive entries, $i \in \mathcal{I}$, and matrices Z_{ij} with positive entries, $(i,j) \in \mathcal{S}$ such that the set of LMIs (22), with F_i as in (23) and G_{ij} as in (24), is feasible, then the system is PWQ passive with storage function (14).

Note that matrices P_i and G_{ij} need not be positive definite.

3. PASSIVITY ENFORCEMENT VIA PIECEWISE LINEAR STATE FEEDBACK

We now consider the problem of synthesizing a piecewise linear state feedback control law for PWA systems of the form (1) in order to enforce passivity of the resulting closed loop. More specifically, we look for matrices K_i, $i \in \mathcal{I}$ such that system (1) with state feedback

$$u_k = -K_i x_k + v_k, \quad [x_k^T \ u_k^T]^T \in \chi_i \tag{25}$$

is PWQ passive, i.e., there exists a PWQ storage function $V(x)$ such that

$$V(x_{k+1}) - V(x_k) - v_k^T y_k < 0 \tag{26}$$

holds along all trajectories $[x_k^T \ u_k^T]^T \in \mathbb{X}$.
The closed loop system reduces to

$$\begin{matrix} x_{k+1} = \hat{A}_i x_k + B_i v_k \\ y_k = \hat{C}_i x_k + D_i v_k \end{matrix} \quad ; \quad \begin{bmatrix} x_k \\ u_k \end{bmatrix} \in \chi_i, \ \ i \in \mathcal{I} \tag{27}$$

where $\hat{A}_i = A_i - B_i K_i$ and $\hat{C}_i = C_i - D_i K_i$. Clearly, by Theorem 3, the closed loop system with piecewise linear feedback (25) is PWQ passive if there exist matrices K_i, P_i, $i \in \mathcal{I}$ such that the following set of inequalities hold

$$\begin{cases} P_i = P_i^T > 0, \quad \forall i \in \mathcal{I} \\ \begin{bmatrix} \hat{A}_i^T P_j \hat{A}_i - P_i & \hat{A}_i^T P_j B_i - \dfrac{\hat{C}_i^T}{2} \\ B_i^T P_j \hat{A}_i - \dfrac{\hat{C}_i}{2} & B_i^T P_j B_i - \dfrac{D_i + D_i^T}{2} \end{bmatrix} < 0 \\ \forall (i,j) \in \mathcal{S} \end{cases} \tag{28}$$

where $\mathcal{S}$ is the index set (16). Unfortunately, as observed in Cuzzola and Morari (2001), the set $\mathcal{S}$ of all possible system switches under feedback is in general not known until the feedback itself has been designed by solving (28). Moreover, designing a controller K_i for each cell χ_i may not be an easy task. Indeed, since u_k depends on the control gain, at each step k the index i for which the condition $[x_k^T \ u_k^T]^T \in \chi_i$ holds is difficult to compute in advance. A possible way to deal with this problem is to introduce additional conservatism by replacing the piecewise linear feedback in (25) with a set of control gains defined on the cells $\bar{\chi}_j$, $j \in \mathcal{J}$, which are defined by constraints that depend on x_k only, i.e., to consider the piecewise linear feedback

$$u_k = -K_j x_k + v_k, \quad x_k \in \bar{\chi}_j. \tag{29}$$

Note that this choice does not prevent from employing a PWQ storage function defined by matrices P_i, $i \in \mathcal{I}$. The resulting closed loop system is

$$\begin{array}{l} x_{k+1} = \widetilde{A}_{ij} x_k + B_i v_k \\ y_k = \widetilde{C}_{ij} x_k + D_i v_k \end{array} ; \begin{bmatrix} x_k \\ u_k \end{bmatrix} \in \chi_i, \ i \in \mathcal{I}, \ x_k \in \bar{\chi}_j \tag{30}$$

where $\widetilde{A}_{ij} = A_i - B_i K_j$ and $\widetilde{C}_{ij} = C_i - D_i K_j$. With this restriction, a sufficient PWQ passivity condition that replaces (28) is derived.

Lemma 1. Consider system (1) and let $\phi_i, \psi_i = 0$. If there exist matrices P_i, $i \in \mathcal{I}$ and K_j, $j \in \mathcal{J}$ such that the set of inequalities

$$\begin{cases} P_i = P_i^T > 0, \quad \forall i \in \mathcal{I} \\ \begin{bmatrix} \widetilde{A}_{ij}^T P_l \widetilde{A}_{ij} - P_i & \widetilde{A}_{ij}^T P_l B_i - \dfrac{\widetilde{C}_{ij}^T}{2} \\ B_i^T P_l \widetilde{A}_{ij} - \dfrac{\widetilde{C}_{ij}}{2} & B_i^T P_l B_i - \dfrac{D_i + D_i^T}{2} \end{bmatrix} < 0 \\ \forall j \in \mathcal{J}, \quad \forall i \in \mathcal{S}_j, \quad \forall l : (l,i) \in \mathcal{I} \times \mathcal{I} \end{cases} \tag{31}$$

holds, then the closed loop system with piecewise linear feedback (29) is PWQ passive.

Note that, contrary to condition (28), in this case the cell χ_l that contains vector $[x_{k+1}^T \ u_{k+1}^T]^T$ is not accounted for and the only requirement here is that the pair of indices (l, i) belongs to the set $\mathcal{I} \times \mathcal{I}$ of all switches.
The characterization introduced in Lemma 1 is not computationally appealing since the inequalities in (31) are bilinear in K_j and P_i and hence the synthesis problem cannot be approached directly by means of convex optimization techniques. Indeed, by introducing additional conservatism and manipulating inequalities (31), a sufficient LMI condition for PWQ passivity of the closed loop can be obtained as the following result shows.

Theorem 5. Consider system (1) and let $\phi_i, \psi_i = 0$. If there exist matrices $Q_i = Q_i^T$, $i \in \mathcal{I}$ and matrices G_j, Y_j, $j \in \mathcal{J}$ such that the set of LMIs

$$\begin{cases} Q_i = Q_i^T > 0 \quad \forall i \in \mathcal{I} \\ \begin{bmatrix} R_j + R_j^T - Q_i & \frac{1}{2}(R_j^T C_i^T - Y_j^T D_i^T) & R_j^T A_i^T - Y_j^T B_i^T \\ \frac{1}{2}(C_i R_j - D_i Y_j) & \dfrac{D_i + D_i^T}{2} & B_i^T \\ A_i R_j - B_i Y_j & B_i & Q_l \end{bmatrix} > 0 \\ \forall j \in \mathcal{J}, \ \forall i \in \mathcal{S}_j, \ \forall (l,i) \in \mathcal{I} \times \mathcal{I} \end{cases} \tag{32}$$

holds, then the system with piecewise linear state feedback (29) where

$$K_j = Y_j R_j^{-1} \tag{33}$$

is PWQ passive.

Proof. Since $Q_i > 0$ and $R_j + R_j^T > Q_i$ by (32), it turns out that R_j is nonsingular and moreover it is easily shown that $R_j^T Q_i^{-1} R_j \geq R_j + R_j^T - Q_i > 0$. Hence (32) implies

$$\begin{cases} Q_i = Q_i^T > 0 \quad \forall i \in \mathcal{I} \\ \begin{bmatrix} R_j^T Q_i^{-1} R_j & \frac{1}{2}(R_j^T C_i^T - Y_j^T D_i^T) & R_j^T A_i^T - Y_j^T B_i^T \\ \frac{1}{2}(C_i R_j - D_i Y_j) & \dfrac{D_i + D_i^T}{2} & B_i^T \\ A_i R_j - B_i Y_j & B_i & Q_l \end{bmatrix} > 0 \\ \forall j \in \mathcal{J}, \ \forall i \in \mathcal{S}_j, \ \forall (l,i) \in \mathcal{I} \times \mathcal{I} \end{cases} \tag{34}$$

By multiplying (34) from the left by $\text{diag}\{R_j^{-T}, I\}$ and from the right by $\text{diag}\{R_j^{-1}, I\}$ we obtain

$$\begin{cases} Q_i = Q_i^T > 0 \quad \forall i \in \mathcal{I} \\ \begin{bmatrix} Q_i^{-1} & \dfrac{\widetilde{C}_{ij}^T}{2} & \widetilde{A}_{ij}^T \\ \dfrac{\widetilde{C}_{ij}}{2} & \dfrac{D_i + D_i^T}{2} & B_i^T \\ \widetilde{A}_{ij} & B_i & Q_l \end{bmatrix} > 0 \\ \forall j \in \mathcal{J}, \quad \forall i \in \mathcal{S}_j, \quad \forall l : (l,i) \in \mathcal{I} \times \mathcal{I} \end{cases} \tag{35}$$

which is equivalent to (31) by a Schur complement argument, where $Q_i = P_i^{-1}$. The result then follows by Lemma 1. ♢

Clearly, a less conservative version of the previous result can be obtained if the definition of the cells χ_i does not depend the input vector u. In that case, it is indeed possible to define a feedback gain K_i for each cell, thus recovering condition (28) fully.

Corollary 1. Let system (1) be given and let $\phi_i, \psi_i = 0$. If there exist matrices $Q_i = Q_i^T > 0$, $i \in \mathcal{I}$ and Y_i, $i \in \mathcal{I}$ such that the set of LMIs

$$\begin{cases} Q_i = Q_i^T > 0 \quad \forall i \in \mathcal{I} \\ \begin{bmatrix} Q_i & \frac{1}{2}(Q_i C_i^T - Y_i^T D_i^T) & Q_i A_i^T - Y_i^T B_i^T \\ \frac{1}{2}(C_i Q_i - D_i Y_i) & \dfrac{D_i + D_i^T}{2} & B_i^T \\ A_i Q_i - B_i Y_i & B_i & Q_j \end{bmatrix} > 0 \\ \forall (i,j) \in \mathcal{S} \end{cases} \tag{36}$$

holds, then the system with piecewise linear state feedback

$$u_k = -K_i x_k + v_k, \quad x_k \in \chi_i \tag{37}$$

where

$$K_i = Y_i Q_i^{-1} \tag{38}$$

is PWQ passive.

4. CONCLUSION

This paper has proposed a characterization of passivity for discrete-time piecewise affine systems. Based on such characterization, easy-to-check sufficient analysis criteria have been derived in the form of LMI tests by employing quadratic and piecewise quadratic storage functions. The problem of designing a piecewise linear state feedback control law that enforces passivity of the closed loop system has also been addressed by exploiting piecewise quadratic storage functions, and is solved by finding a feasible solution of a certain set of LMIs.

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