ROBUSTNESS OF BOUNDARY CONTROL OF DAMPED WAVE EQUATIONS WITH LARGE DELAYS AT BOUNDARY MEASUREMENT

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Abstract: The Smith predictor is applied to the boundary control of damped wave equations with large delays at the boundary measurement. The instability problem due to large delays is solved and the scheme is proved to be robust against a small difference between the assumed delay and the actual delay.

Keywords: Robust boundary control; Smith predictor; damped wave equation; delay.

1. INTRODUCTION

In recent years, boundary control of flexible systems has become an active research area, due to the increasing demand on high precision control of many mechanical systems, such as spacecraft with flexible attachment or robots with flexible links, which are governed by PDEs (partial differential equations) rather than ODEs (ordinary differential equations), see (Morgül, 2002*a*; Morgül, 2001; Conrad and Morgül, 1998; Morgül, 1998; Guo, 2001; Guo, 2002; Chen, 1979; Chen *et al.*, 1987; Morgül, 2002*b*). In this research area, the robustness of controllers against delays is an important topic and has been studied by many researchers, see (Datko *et al.*, 1986; Datko, 1993; Logemann and Rebarber, 1998; Logemann *et al.*, 1996; Morgül, 1995), due to the fact that delays are unavoidable in practical engineering. All the available publications focus on the analysis of systems against a small delay, *i.e.*, under what conditions a very small delay will not cause instability problem and can be therefore neglected? An equally important and very practical issue is, how to synthesize a boundary controller when the delay is large and makes the systems unstable? To the best of our knowledge, publications studying this problem are very few. In this paper, we solve the instability problem caused by large delays by applying the Smith predictor to the boundary control of the damped wave equation. The control scheme is shown to be stable and robust against a small difference between the actual delay and the assumed delay.

The paper is organized as follows. In Sec. 2, the Smith predictor is introduced briefly. Section 3 formulates the boundary control of the damped wave equation and the control algorithm proposed in this paper. In Sec. 4, the stability and robustness issues of the control algorithm are discussed. Finally, Sec. 4 concludes this paper.

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2. A BRIEF INTRODUCTION TO THE SMITH PREDICTOR

The Smith predictor was proposed by Smith in (Smith, 1957) and is probably the most famous method for control of systems with time delays, see (Levine, 1996) and (Wang *et al.*, 1999). Consider a typical feedback control system with a time delay in Fig. 1, where $C(s)$ is the controller; $P(s)e^{-\theta s}$ is the plant with a time delay θ.

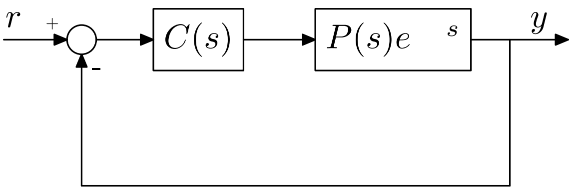

Fig. 1. A feedback control system with a time delay

With the presence of the time delay, the transfer function of the closed-loop system relating the output $y(s)$ to the reference $r(s)$ becomes

$$\frac{y(s)}{r(s)} = \frac{C(s)P(s)e^{-\theta s}}{1 + C(s)P(s)e^{-\theta s}}. \tag{1}$$

Obviously, the time delay θ directly changes the closed-loop poles. Usually, the time delay reduces the stability margin of the control system, or more seriously, destabilizes the system.

The classical configuration of a system containing a Smith predictor is depicted in Fig. 2, where $\hat{P}_0(s)$ is the assumed model of $P_0(s)$ and $\hat{\theta}$ is the assumed delay. The block $C(s)$ combined with the block $\hat{P}(s) - \hat{P}(s)e^{-\hat{\theta}s}$ is called "the Smith predictor". If we assume the perfect model matching, *i.e.*, $\hat{P}_0(s) = P_0(s)$ and $\theta = \hat{\theta}$, the closed-loop transfer function becomes

$$\frac{y(s)}{r(s)} = \frac{C(s)P(s)e^{-\theta s}}{1 + C(s)P(s)}. \tag{2}$$

Now, it is clear what the underlying idea of the Smith predictor is. With the perfect model matching, the time delay can be removed from the denominator of the transfer function, making the closed-loop stability irrelevant to the time delay.

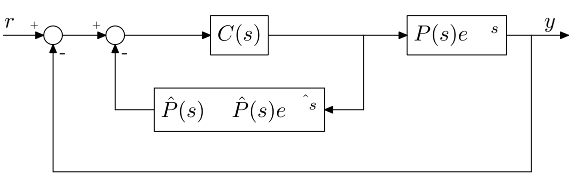

Fig. 2. The Smith predictor

3. BOUNDARY CONTROL OF THE DAMPED WAVE EQUATION WITH LARGE DELAYS

Consider a string clamped at one end and is free at the other end. We denote the displacement of the string by $u(x,t)$, where $x \in [0,1]$ and $t \geq 0$. The string is controlled by a boundary control force at the free end. The governing equations are given as

$$u_{tt}(x,t) - u_{xx}(x,t) + 2au_t(x,t) + a^2u(x,t) = 0, \tag{3}$$

$$u(0,t) = 0, \tag{4}$$

$$u_x(1,t) = f(t), \tag{5}$$

where $a > 0$ is the damping constant and $f(t)$ is the boundary control force applied at the free end of the string.

It is known that the following boundary feedback controller stabilizes the system, see (Chen, 1979),

$$f(t) = -ku_t(1,t), \tag{6}$$

where $k > 0$ is the constant boundary control gain.

Now, we consider the presence of a time delay in the feedback loop, which is shown as follows.

$$f(t) = -ku_t(1, t - \theta), \tag{7}$$

where θ is the time delay.

In (Datko *et al.*, 1986) and (Logemann *et al.*, 1996), it was shown that if k and a satisfy

$$k\frac{e^{2a}+1}{e^{2a}-1} < 1, \tag{8}$$

then the delayed feedback systems is stable for all sufficiently small delays.

In this paper, we will solve the following problem: what if the time delay θ is large enough to make the system unstable? We will apply the Smith predictor to solve this problem.

Comparing the equation (7) with Fig. 2, we can see that in our case, the plant output y is the tip end displacement $u(1,t)$; the controller $C(s)$ is a derivative controller with the transfer function ks; and $P(s)$ is the transfer function from the control force $f(t)$ to the un-delayed displacement of the tip end. If we assume $\hat{P}(s) = P(s)$ and the time delay θ is known, the remaining problem is how to get $P(s)$, which is shown as follows.

Assuming zero initial conditions of $u(x,0)$ and $u_t(x,0)$, take the Laplace transform of (3), (4), and (5) with respect to t, the original PDE of $u(x,t)$ with initial and boundary conditions can be transformed into the following ODE of $U(x,s)$ with boundary conditions.

$$\frac{d^2U(x,s)}{dx^2} - (s+a)^2U(x,s) = 0, \tag{9}$$

$$U(0,s) = 0, \tag{10}$$

$$U_x(1, s) = F(s), \tag{11}$$

where $U(x, s)$ is the Laplace transform of $u(x, t)$ and $F(s)$ is the Laplace transform of $f(t)$.

Solving the ODE (9), we have the following solution of $U(x, s)$ with two arbitrary constants C_1 and C_2 (s can be treated as a constant in this step).

$$U(x, s) = C_1 e^{-(s+a)x} + C_2 e^{(s+a)x}. \tag{12}$$

Substitute (12) into (10) and (11), we have the following two equations.

$$C_1 + C_2 = 0, \tag{13}$$

$$(-C_1 e^{-(s+a)} + C_2 e^{s+a})(s + a) = F(s). \tag{14}$$

Solving (13) and (14) simultaneously, we can obtain the exact value of C_1 and C_2

$$C_1 = \frac{-F(s)}{(s + a)(e^{-(s+a)} + e^{s+a})}, \tag{15}$$

$$C_2 = \frac{F(s)}{(s + a)(e^{-(s+a)} + e^{s+a})}. \tag{16}$$

Now we have obtained the solution of $U(x, s)$. Substituting $x = 1$ into $U(x, s)$, we obtain the following Laplace transform of the tip end displacement.

$$U(1, s) = \frac{F(s)(1 - e^{-2(s+a)})}{(s + a)\left(1 + e^{-2(s+a)}\right)}. \tag{17}$$

So the transfer function of the plant, which is $P(s)$ in Fig. 2, is obtained as

$$P(s) = \frac{U(1, s)}{F(s)} = \frac{1 - e^{-2(s+a)}}{(s + a)\left(1 + e^{-2(s+a)}\right)}. \tag{18}$$

Finally, we have the following expression of the boundary controller (the Smith predictor), denoted as $C_{sp}(s)$:

$$C_{sp}(s) = \frac{ks}{1 + ksP(s)(1 - e^{-\hat{\theta}s})} \tag{19}$$

Notice that the controller (19) is physically implementable.

4. STABILITY AND ROBUSTNESS ANALYSIS

In (Chen, 1979), the stability of the controller (6) was proved for the boundary control of the damped wave equation without delays. If the assumed delay is equal to the actual delay, the Smith predictor removes the delay term completely from the denominator of the closed-loop transfer function. This means the stability of the controller (19) is already proved.

Since the actual delay θ and the assumed delay $\hat{\theta}$ can not be exactly the same, another important issue is the robustness of the controller (19), *i.e.*, what if an unknown small difference ϵ between the assumed delay and the actual delay is introduced to the system, as shown in Fig. 3.

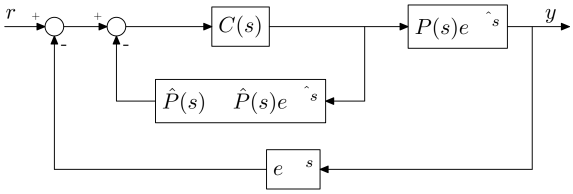

Fig. 3. System with mis-matched delays

To study the robustness of the controller (19), we will first introduce a theorem presented in (Logemann *et al.*, 1996) and (Logemann and Rebarber, 1998).

Theorem 1. Let $H(s)$ be the open-loop transfer function as illustrated in Fig. 4 and $\mathfrak{D}_H$ the set of all its poles. Define two closed-loop transfer functions $G_0(s)$ and $G_\epsilon(s)$ as

$$G_0(s) = \frac{H(s)}{1 + H(s)},$$

and

$$G_\epsilon(s) = \frac{H(s)}{1 + e^{-\epsilon s}H(s)}.$$

Define again

$$\mathbb{C}_0 = \{s \in \mathbb{C} | \Re(s) > 0\},$$

and

$$\gamma(H(s)) = \limsup_{|s|\to\infty, s\in\mathbb{C}_0\setminus\mathfrak{D}_H} |H(s)|.$$

Suppose G_0 is L^2-stable. If γ(H) < 1, then there exists ϵ^* such that G_ϵ is L^2-stable for all $\epsilon \in (0, \epsilon^*)$.

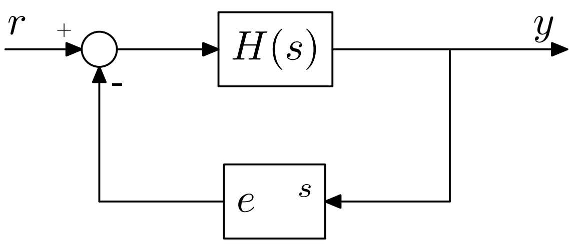

Fig. 4. Feedback system with delay

The underlying idea of the above theorem is that the robustness of the closed-loop transfer function $G_0(s)$ against a small unknown delay can be determined by studying the open-loop transfer function $H(s)$. Now we can prove the robustness of the controller (19).

CLAIM:

If $\hat{\theta}$ is chosen as the minimum value of the possible delay and k is chosen to satisfy

$$k\frac{e^{2a}+1}{e^{2a}-1} \leq \frac{1}{3}, \tag{20}$$

then the controller (19) is robust against a small difference ϵ between the assumed delay $\hat{\theta}$ and the actual delay $\theta = \hat{\theta} + \epsilon$.

Proof:

$$\begin{aligned} H(s) &= C_{sp}(s)P(s)e^{-\hat{\theta}s} \\ &= \frac{ksP(s)e^{-\theta s}}{1 + ksP(s)(1 - e^{-\theta s})} \end{aligned}$$

Let $T(s) = ksP(s)$, then

$$|H(s)| = \frac{1}{|(\frac{1}{T(s)}+1)e^{\hat{\theta}s} - 1|} \tag{21}$$

Let $Q(s) = (\frac{1}{T(s)}+1)e^{\hat{\theta}s} - 1$, then

$$\begin{aligned} |Q(s)| &= \left|(\frac{1}{T(s)}+1)e^{\hat{\theta}s} - 1\right| \\ &\geq \left|\left|(\frac{1}{T(s)}+1)e^{\hat{\theta}s}\right| - 1\right| \\ &\geq \left|\left|\frac{1}{T(s)}+1\right|\left|e^{\hat{\theta}s}\right| - 1\right| \end{aligned} \tag{22}$$

In (Logemann *et al.*, 1996), it was proved that

$$\limsup_{|s|\to\infty, s\in\mathbb{C}_0} |T(s)| = k\frac{e^{2a}+1}{e^{2a}-1}$$

So if $k\frac{e^{2a}+1}{e^{2a}-1} \leq \frac{1}{3}$, for $|s|$ large enough,

$$\begin{aligned} \left|\frac{1}{T(s)}+1\right| &\geq \left|\left|\frac{1}{T(s)}\right| - 1\right| \\ &\geq 2 \end{aligned} \tag{23}$$

Considering $|e^{\hat{\theta}s}| > 1$, we have

$$|Q(s)| > 1 \tag{24}$$

So

$$\limsup_{|s|\to\infty, s\in\mathbb{C}_0} |H(s)| < 1. \tag{25}$$

Remarks:

- In *Theorem 1*, ϵ is positive. To satisfy this condition, $\hat{\theta}$ should be chosen as the minimal value of the possible delay.
- The damping constant a plays a key role in making the controllers (both the original derivative controller ks and the Smith predictor) robust. if $a = 0$, the damped wave equation becomes the conservative wave equation, the transfer function of which becomes

$$P(s) = \frac{1 - e^{-2s}}{s(1 + e^{-2s})}. \tag{26}$$

We can see that $P(s)$ has infinite number of poles on the imaginary axis. In order to make $\gamma(H(s)) < 1$, controllers must cancel these poles completely, which is impossible due to the uncertainty of the plant parameters. This means both the original derivative controller ks and the Smith predictor are not robust when applied to the boundary control of the conservative wave equation.

5. CONCLUDING REMARKS

With the introduction of the Smith predictor, the instability problem caused by large delays in the boundary control of the damped equation is solved. The control algorithm is also robust against a small difference between the actual delay and the assumed delay. Future work includes studying the robustness of the controller against the plant modelling errors and the controller performance of the Smith predictor.

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