INVESTIGATION OF PARAMETER PERTURBATION REGION FOR POSITIVE POLYNOMIALS

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Abstract: In this paper the robust positivity of polynomials under coefficient perturbation is investigated. This robust positivity of polynomials can be used for polynomial systems in order to determine the robust asymptotic stability of the system. It is assumed that the polynomials under investigation depend linearly on some parameters. The aim in the article is to determine the parameter perturbation region as a hypersphere, for which the polynomial is globally positive. The theorem of Ehlich and Zeller is used to achieve this aim. This theorem enables to give conditions in the parameter space for global positivity. These conditions are linear inequalities. By means of these inequalities an inner and an outer approximation are calculated to the relevant perturbation region which is a hypersphere. Two nontrivial examples conclude the paper and show the effectiveness of the presented method.

Keywords: Linear Inequalities, Nonlinear Systems, Optimization, Polynomials, Robustness, Stability

1. INTRODUCTION

In this paper the problem of global positivity of polynomials depending linearly on uncertain parameters (Bose and Guiver, 1980; Djaferis, 1991; Tibken and Dilaver, 2001; Tibken and Dilaver, 2003; Tibken *et al.*, 2003) will be dealt with. Generally a polynomial can be written as

$$p(x) \;=\; \sum_{i=1}^{s} p_{\alpha_i}\, x^{\alpha_i} \;, \quad x \in \mathbb{R}^m \tag{1}$$

where $x^{\alpha_i} = \prod_{j=1}^{m} x_j^{\alpha_{ij}}$ is the i-th monomial of the polynomial $p(x)$, p_{α_i} is the coefficient of the i-th monomial and s is the number of the monomials in the polynomial. The definitions of the degree of the i-th monomial and the degree of the polynomial $p(x)$ are as

$$\mid \alpha_i \mid = \sum_{j=1}^{m} \alpha_{ij} \quad , \tag{2}$$

$$deg\; p(x) = max \mid \alpha_i \mid \;, \quad i = 1,\; \ldots\; , s \tag{3}$$

respectively, where α_{ij} is either a positive entire number or zero. Such a polynomial that depends linearly on some parameters can be written with respect to the uncertainties at the parameters as

$$p(x) \;=\; p_0(x) \;+\; \sum_{i=1}^{r} k_i\; p_i(x) \;, \quad x \in \mathbb{R}^m \tag{4}$$

where k_i, $i = 1, \ldots, r$, represents the uncertainty at the i-th parameter. The investigations in this paper are necessary, for example, when analyzing global asymptotical stability of polynomial dynamical systems (Gahinet *et al.*, 1996; Wu and Mansour, 1995). In this work it will be used the theorem of Ehlich and Zeller (Ehlich and Zeller, 1964) and developed an algorithm in order to compute the maximum domain as a hypersphere in the parameter space for which a given polynomial is globally positive. In the developed algorithm it is assumed that the polynomial $p_0(x)$ in (4) is globally positive in the m dimensional space. Therefore the hyperspherical region of the parameter space for which the considered polynomial is strictly positive, can be defined as the set

$$\Omega := \{\ k \ \in \ \mathbb{R}^r \ \mid \ \|k\| \ \leq \ R\ \} \tag{5}$$

where k is the parameter vector and $\|.\|$ denotes its Euclidean norm. The goal is to compute the maximum R such that $p(x)$ is globally positive for $\forall k \in \Omega$.

First of all the theorem of Ehlich and Zeller is introduced and then polynomial homogenization will be discussed which is necessary for the application of this theorem. By means of homogenization the whole $\mathbb{R}^m$-space (m is the number of the variables in the polynomial) is reduced into a hyperrectangle in the $\mathbb{R}^{m+1}$ space. If the homogenized polynomial is positive on a subset of the boundary of the hyperrectangle, the original polynomial is globally positive. This property will be proved and used in order to determine the parameter region Ω for which the polynomial is globally positive. In the third section the algorithm is presented and is illustrated with two examples in the fourth section. Conclusions and an outlook will finish the paper.

2. THEOREM OF EHLICH AND ZELLER

This section will closely follow the corresponding section in (Ehlich and Zeller, 1964; Tibken and Dilaver, 2003; Tibken and Dilaver, 2004; Tibken *et al.*, 1999). In the following $J = [a, b]$ denotes a nonempty compact real interval with $J \subset \mathbb{R}$. The set of Chebychev points in J is defined for a given natural number $N > 0$ by

$$x(N, J) \ := \ \{\ \mathrm{x}_1\ ,\ \mathrm{x}_2\ ,\ \ldots\ ,\ \mathrm{x}_i\ ,\ \ldots\ ,\ \mathrm{x}_N\ \} \tag{6}$$

where

$$\mathrm{x}_i \ := \ \frac{a+b}{2} \ + \ \frac{b-a}{2}\ \cos\left(\ \frac{(2i-1)\pi}{2N}\ \right). \tag{7}$$

For a continuous function h defined on a set I the norm

$$||h||^I \ := \ \max_{x \in I}\ |h(x)| \tag{8}$$

which is the usual maximum norm, is utilised. Let p_n be the set of polynomials p in one variable with $deg\ p = n$. Then the following inequality

$$\|p\|^J \ \leq \ C\left(\frac{n}{N}\right)\ \|p\|^{x(N,J)} \tag{9}$$

with $N > n$ and

$$C(q) \ := \ \left[\cos(q\frac{\pi}{2})\right]^{-1} \ ,\ \ 0 < q < 1\ \ , \tag{10}$$

is valid for every $p \in p_n$ and every nonempty compact interval J. Inequality (9) is remarkable because the norm $\|p\|^{x(N,J)}$ on the right hand side of (9) depends on the values of p at the Chebychev points only. This result was given by Ehlich and Zeller in (Ehlich and Zeller, 1964). Using (9) the following inequalities

$$p^J_{min} \geq \frac{1}{2}\left\{\left(C\left(\frac{n}{N}\right)+1\right)p^{x(N,J)}_{min} \ - \left(C\left(\frac{n}{N}\right)-1\right)p^{x(N,J)}_{max}\right\}, \tag{11}$$

$$p^J_{max} \leq \frac{1}{2}\left\{\left(C\left(\frac{n}{N}\right)+1\right)p^{x(N,J)}_{max} \ - \left(C\left(\frac{n}{N}\right)-1\right)p^{x(N,J)}_{min}\right\} \tag{12}$$

which are valid for every $p \in p_n$ and $N > n$ are given by Gärtel in (Gärtel, 1987). In the inequalities $p^J_{min} := \min_{x\in J} p(x)$ and $p^J_{max} := \max_{x\in J} p(x)$ are the minimum and maximum of p in the set J respectively. Similarly $p^{x(N,J)}_{min} := \min_{x\in x(N,J)} p(x)$ and $p^{x(N,J)}_{max} := \max_{x\in x(N,J)} p(x)$ are the minimum and maximum of p in the set of Chebychev points respectively. For trigonometric polynomials and for rational functions similar inequalities are given by Gärtel (Gärtel, 1987).

The inequalities (9),(11),(12) are valid for polynomials in one variable. They are extended to polynomials of several variables using the following replacements. The interval J is replaced by

$$\hat{J} \ = \ [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_m, b_m] \tag{13}$$

which represents a hyperrectangle. For the degree of p with respect to the i-th variable x_i the abbreviation n_i is introduced and the set of Chebychev points in $\hat{J}$ is given by

$$x(\hat{N}, \hat{J}) := x(N_1, [a_1, b_1]) \times \cdots \cdots \times x(N_m, [a_m, b_m]) \tag{14}$$

where N_i is the number of Chebychev points for the i-th variable x_i in the interval $[a_i, b_i]$. Then the inequalites

$$p_{min}^{\hat{J}} \geq \frac{1}{2}\Big\{(K+1)\ p_{min}^{x(\hat{N},\hat{J})} \quad - \\ (K-1)\ p_{max}^{x(\hat{N},\hat{J})}\Big\} \qquad (15)$$

$$p_{max}^{\hat{J}} \leq \frac{1}{2}\Big\{(K+1)\ p_{max}^{x(\hat{N},\hat{J})} \quad - \\ (K-1)\ p_{min}^{x(\hat{N},\hat{J})}\Big\} \qquad (16)$$

with

$$K \;=\; \prod_{i=1}^{m} C\left(\frac{n_i}{N_i}\right) \qquad (17)$$

under the conditions $N_i > n_i,\ i = 1, ..., m$, are valid. The theorem of Ehlich and Zeller is used and applied in the next section and then it is showed that if a homogenized polynomial is positive on a subset of the boundary of the hyperrectangle, the original polynomial is globally positive. By means of this property the uncertain parameter region Ω for which the polynomial $p(x)$ (4) is globally positive can be determined.

3. APPROXIMATION METHOD

The theorem of Ehlich and Zeller helps to analyze the positivity of polynomials on finite intervals. Investigating the positivity of a polynomial $p(x)$ on $\mathbb{R}^m$ is the goal and some calculations should be done in order to apply the theorem of Ehlich and Zeller in this case. The main tool is homogenization, i.e. for every polynomial $p(x)$ the polynomial $\tilde{p}(x)$ is introduced which is defined according to the following expression.

$$\tilde{p}(x) := {x_0}^{deg\ p(x)}\ p(\frac{x_1}{x_0},\ \ldots,\ \frac{x_m}{x_0})\ , \qquad (18)$$

$$= \sum_{i=1}^{s} \tilde{p}_{\alpha_i}\ x^{\tilde{\alpha}_i}\ ,\quad x \in \mathbb{R}^{m+1} \qquad (19)$$

Each monomial of the polynomial $\tilde{p}(x)$ has the same degree

$$\mid \tilde{\alpha}_i \mid =\ deg\ p(x)\ ,\quad i = 1,\ \ldots\ , s\ . \qquad (20)$$

Here $\mid \tilde{\alpha}_i \mid$ represents the degree of the i-th monomial in the polynomial $\tilde{p}(x)$. The positivity of $\tilde{p}(x)$ and $p(x)$ is related by the following equivalence.

$$\tilde{p}(x)\ >\ 0\ \ \forall\, x \in \mathbb{R}^{m+1}\ \text{ with }\ x_0\ >\ 0 \quad \Longleftrightarrow \\ p(x)\ >\ 0\ \ \forall\, x \in \mathbb{R}^m \qquad (21)$$

Thus, in order to test $p(x)$ for positivity in $\mathbb{R}^m$ the polynomial $\tilde{p}(x)$ can alternatively be tested for its positivity in $\mathbb{R}^{m+1}$ under the further condition $x_0 > 0$. The following equation

$$\tilde{p}(\lambda\ x) = \lambda^{deg\ p(x)}\ \tilde{p}(x)\ ,\quad \forall\ \lambda \in \mathbb{R} \qquad (22)$$

is valid due to homogeneity of $\tilde{p}(x)$. It follows that if $\lambda > 0$, then $\tilde{p}(\lambda\ x)$ and $\tilde{p}(x)$ have the same sign. If the positive number λ is chosen as

$$\lambda \ :=\ \left(\max_{i=0,\ \ldots,\ m}\ \mid x_i \mid\right)^{-1}\ , \qquad (23)$$

the vector $(\lambda\ x)$ is on the boundary of the hypercube H defined by

$$H\ :=\ H_{00}^{-} \cup H_{00}^{+} \cup \cdots \cup H_{m0}^{-} \cup H_{m0}^{+} \qquad (24)$$

with

$$H_{j0}^{-} := \{\ x \in \mathbb{R}^{m+1} \mid\ x_j = -1\ ,\quad -1 \leq x_i \leq 1\ , \\ i \neq j\ ,\quad i = 0, \ldots, m\ \}\ , j = 0, \ldots, m \qquad (25)$$

$$H_{j0}^{+} := \{\ x \in \mathbb{R}^{m+1} \mid\ x_j = +1\ ,\quad -1 \leq x_i \leq 1\ , \\ i \neq j\ ,\quad i = 0, \ldots, m\ \}\ , j = 0, \ldots, m. \qquad (26)$$

Thus the test of $\tilde{p}(x)$ for positivity in $\mathbb{R}^{m+1}$ under the condition $x_0 > 0$ is reduced to positivity test of $\tilde{p}(x)$ of that part of the boundary of H for which $x_0 > 0$ is fulfilled. This boundary consists of a finite number of hyperrectangles defined by

$$H_0 := \{\ x \in \mathbb{R}^{m+1} \mid\ x_0 = 1\ ,\ \ -1 \leq x_i \leq 1\ , \\ i = 1, \ldots, m\ \} \qquad (27)$$

$$H_j^{\mp} := \{\ x \in \mathbb{R}^{m+1} \mid\ x_j = \mp 1\ ,\ \ 0 < x_0 \leq 1\ , \\ -1 \leq x_i \leq 1\ ,\ \ i \neq j\ ,\quad i = 1, \ldots, m\ \}\ , \\ j = 1, \ldots, m \qquad (28)$$

and thus the theorem of Ehlich and Zeller can be applied to every part of this boundary. Inequality (15) is used to ensure the positivity of $\tilde{p}(x)$ on $H_0, H_1^+, H_1^-,\ \ldots\ , H_m^+, H_m^-$. If for every hyperrectangle $\hat{J} \in \{H_0, H_1^+, H_1^-, \ldots, H_m^+, H_m^-\}$ in this boundary the inequality

$$(K+1)\ \tilde{p}_{min}^{x(\hat{N},\hat{J})} \quad - \quad (K-1)\ \tilde{p}_{max}^{x(\hat{N},\hat{J})}\ >\ 0 \qquad (29)$$

is fulfilled, the polynomial $\tilde{p}(x)$ is positive definite on the boundary of the hyperrectangle in $\mathbb{R}^{m+1}$ for which x_0 is greater than zero. Due to (21) and (22) the polynomial $p(x)$ is global positive in $\mathbb{R}^m$. In case that

$$(K+1)\ \tilde{p}_{min}^{x(\hat{N},\hat{J})} \quad - \quad (K-1)\ \tilde{p}_{max}^{x(\hat{N},\hat{J})}\ >\ 0 \qquad (30)$$

on the set $\hat{J}$, the inequalities

$$(K+1)\ \tilde{p}(\mathrm{x}_i)\ -\ (K-1)\ \tilde{p}(\mathrm{x}_j)\ >\ 0\ \ ,\qquad i,j=1,\ \dots\ ,\hat{N} \quad (31)$$

are valid for all i, j due to fact that

$$\tilde{p}_{min}^{x(\hat{N},\hat{J})}\ \leq\ \tilde{p}(\mathrm{x}_i)\ \leq\ \tilde{p}_{max}^{x(\hat{N},\hat{J})}\ ,\qquad i=1,\dots,j,\dots,\hat{N} \quad (32)$$

where $\mathrm{x}_i,\ \mathrm{x}_j\ \in X(\hat{N},\hat{J})$ are two Chebychev points in the same hyperrectangle. For $\hat{N}$ Chebychev points in one hyperrectangle there are $\hat{N}^2$ inequalities of type (31) which are equivalent to (29). Since $(2m+1)$ hyperrectangles exist to be checked, the total number of the inequalities is $(2m+1)\hat{N}^2$. If the polynomial $p(x)$ depends linearly on some uncertain parameters $k_1,\ \dots,\ k_r$ as in (4), then the polynomial $\tilde{p}(x)$ can be written as

$$\tilde{p}(x)\ \ =\ \ \tilde{p}_0(x)\ \ +\ \ \sum_{i=1}^{r}\ k_i\ \tilde{p}_i(x) \quad (33)$$

and the inequalities (31) can be represented as

$$a_r^{i,j}\ k_r\ +\ \dots\ +\ a_1^{i,j}\ k_1\ +\ a_0^{i,j}\ >\ 0\ \ ,\quad i,j=1,\ \dots\ ,\hat{N} \quad (34)$$

with

$$a_t^{i,j}\ =\ (K+1)\ \tilde{p}_t(\mathrm{x}_i)\ -\ (K-1)\ \tilde{p}_t(\mathrm{x}_j),\qquad t=0\ ,\ \dots\ ,\ r\ ,\ \ i,j=1\ ,\ \dots\ ,\ \hat{N} \quad (35)$$

where k_t, $t=1,\dots,r$, denotes the t-th uncertain parameter in the polynomial $p(x)$ and $a_t^{i,j}$, $t=0,\dots,r,\ i,j\ =\ 1,\dots,\hat{N}$, is constant. For the values of the k_t's that satisfy the $\hat{N}^2$ inequalities in (34) for each hyperrectangle H_0,H_1^+,H_1^-, … ,H_m^+,H_m^-, the polynomial $p(x)$ is globally positive definite. From the inequalities (34) we get an inner approximation to the convex set

$$\Omega=\{\ k\ \in\ \mathbb{R}^r\ \ |\ \ \|k\|\ \leq\ R\ \}\ . \quad (36)$$

Because the inequalities (34) are the sufficient conditions for the strict positivity of the polynomial $p(x)$. An outer approximation to Ω is achieved if only the $\hat{N}$ inequalities

$$\begin{aligned}\tilde{p}(\mathrm{x}_i) &= \tilde{p}_r(\mathrm{x}_i)k_r\ +\ \tilde{p}_{r-1}(\mathrm{x}_i)k_{r-1}\ +\ \dots\\ &\quad +\tilde{p}_1(\mathrm{x}_i)k_1\ +\ \tilde{p}_0(\mathrm{x}_i)\\ &= a_r^i\ k_r\ +\ a_{r-1}^i\ k_{r-1}\ +\ \dots\\ &\quad +\ a_1^i\ k_1\ +\ a_0^i\ >\ 0, i=1,\dots,\hat{N} \quad (37)\end{aligned}$$

for each hyperrectangle H_0,H_1^+,H_1^-, … ,H_m^+,H_m^- are taken into account at the Chebychev points. The total number of the inequalities for the outer approximation is $(2m+1)\hat{N}$. Since the inequalities (37) are the necessary conditions for the strict positivity, by means of the solutions of the inequalities in (37) an outer approximation can be determined to the set Ω. Thus, using the theorem of Ehlich and Zeller it is possible to find an inner and an outer approximation to Ω.

The inequalities in (34) and (37) are in the form

$$a_r\ k_r\ +\ \dots\ +\ a_1\ k_1\ +\ a_0\ >\ 0 \quad (38)$$

where the coefficient a_j, $j=0,\dots,r$, is known and constant. Because it is a function of the Chebychev points in the hyperrectangles H_0,H_1^+,H_1^-, … ,H_m^+,H_m^-.

In the approximation method the following question is firstly answered. "How could the maximal hyperspherical parameter region (5) be determined, if there was only one inequality of type (38)". In order to find a answer to this problem the objective

$$R_*\ \ :=\ \ \min\ \sqrt{k_1^2\ +\ k_2^2\ +\ \dots\ +\ k_r^2} \quad (39)$$

under the constraint

$$a_r\ k_r\ +\ \dots\ +\ a_1\ k_1\ +\ a_0\ =\ 0 \quad (40)$$

must be solved. The solution of this optimization problem is

$$R_*\ \ =\ \ \frac{|a_0|}{\sqrt{a_1^2\ +\ a_2^2\ +\ \dots\ +\ a_r^2}}\ . \quad (41)$$

Thus for inequalities of type (34) which represent the sufficient conditions for the strict positivity of the original polynomial , an inner approximation

$$\Omega_{in}\ :=\ \{\ k\ \in\ \mathbb{R}^r\ \ |\ \ \|k\|\ \leq\ R_{in}\ \} \quad (42)$$

to the set (5) is determined with

$$R_{in}\ =\ \min\ \left(R_*^i\right)\ \ ,\ \ i=1,\dots,(2m+1)\hat{N}^2 \quad (43)$$

where R_*^i indicates the solution of the optimization problem (39) under the constraint

$$a_r^i\ k_r\ +\ \dots\ +\ a_1^i\ k_1\ +\ a_0^i\ =\ 0\ . \quad (44)$$

Similarly an outer approximation

$$\Omega_{out}\ :=\ \{\ k\ \in\ \mathbb{R}^r\ \ |\ \ \|k\|\ \leq\ R_{out}\ \} \quad (45)$$

is found for $(2m+1)\hat{N}$ inequalities which are the necessary conditions for the strict positivity of the polynomial $p(x)$ (4) by means of

$$R_{out}\ =\ \min\ \left(R_*^i\right)\ \ ,\ \ i=1,\dots,(2m+1)\hat{N}. \quad (46)$$

The value of the radius R of the hyperspherical region Ω lies in the interval $[R_{in},R_{out}]$ and the set Ω lies between the sets Ω_{in} and Ω_{out}. This numerical method ensures the global positivity of the polynomial $p(x)$ (4) for the parameter values in the set Ω_{in}. In the next section there are two nontrivial examples that illustrate the presented method.

4. EXAMPLES

4.1 Example 1

In the following example the number of variables of the main polynomial is one and the relevant hyperrectangles are a subset of the edges of the unit square which are given by

$$\begin{aligned} H_0 &= \{x \mid x_0 = 1,\ -1 \le x_1 \le 1\}\ , && (47)\\ H_1^+ &= \{x \mid x_1 = 1,\ 0 < x_0 \le 1\}\ , && (48)\\ H_1^- &= \{x \mid x_1 = -1,\ 0 < x_0 \le 1\}\ . && (49)\end{aligned}$$

The numerical method will be tested with the polynomial

$$p(x) = (1+k_1)x_1^4 \ - \ (2+k_2)x_1^3 \ + \ (1+k_3)x_1^2 + \ (5+k_4) \tag{50}$$

that is taken from (Bose and Guiver, 1980). After the transformation the homogenized polynomial

$$\begin{aligned}\tilde{p}(x) &= (1+k_1)x_1^4 \ - \ (2+k_2)x_0x_1^3 \ + \\ &\quad (1+k_3)x_0^2x_1^2 \ + \ (5+k_4)x_0^4 \\ &= (x_1^4 - 2x_0x_1^3 + x_0^2x_1^2 + 5x_0^4) \ + \ k_1x_1^4 \ - \\ &\quad k_2x_0x_1^3 \ + \ k_3x_0^2x_1^2 \ + \ k_4x_0^4 \end{aligned} \tag{51}$$

is obtained.

Fig.1 shows the inner and outer approximations to the value of R depending on the number of Chebychev points on the sets H_0, H_1^+ and H_1^-. For 100 Chebychev points per variable the following interval

$$R_{in} = 0.4433 \ \le \ R \ \le \ \ R_{out} = 0.4473 \tag{52}$$

was found for the value of the maximum radius R.

4.2 Example 2

In the works of Bose (Bose, 1982) and Parrilo (Parrilo, 2000) it was proven that the polynomial

$$p(x) = x_1^4 \ - \ 2x_1^2x_2x_3 \ - \ x_1^2 \ + \ x_2^2x_3^2 + \ 2x_2x_3 \ + \ 2 \tag{53}$$

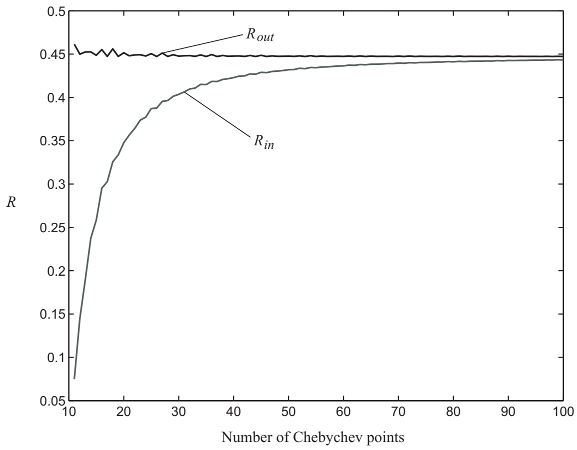

Fig. 1. Inner and outer approximations to the maximum radius R.

is globally positive. Thus the parameter perturbation region for the given polynomial (53) can be determined as a hypersphere. Therefore the numerical method will be tested in this example with the polynomial

$$\begin{aligned} p(x) = (1+k_1)x_1^4 \ &- \ (2+k_2)x_1^2x_2x_3 \\ &- \ (1+k_3)x_1^2 \ + \ (1+k_4)x_2^2x_3^2 \\ &+ \ (2+k_5)x_2x_3 \ + \ (2+k_6)\ . \end{aligned} \tag{54}$$

After the transformation the homogenized polynomial

$$\begin{aligned}\tilde{p}(x) &= (1+k_1)x_1^4 \ - \ (2+k_2)x_1^2x_2x_3 \ - \\ &\quad (1+k_3)x_0^2x_1^2 \ + \ (1+k_4)x_2^2x_3^2 \ + \\ &\quad (2+k_5)x_0^2x_2x_3 \ + \ (2+k_6)x_0^4 \\ &= (x_1^4 - 2x_1^2x_2x_3 - x_0^2x_1^2 + x_2^2x_3^2 + 2x_0^2x_2x_3 \\ &\quad +2x_0^4) \ + \ k_1x_1^4 \ - \ k_2x_1^2x_2x_3 \ - \ k_3x_0^2x_1^2 \\ &\quad + \ k_4x_2^2x_3^2 \ + \ k_5x_0^2x_2x_3 \ + \ k_6x_0^4 \end{aligned} \tag{55}$$

is obtained. According to the presented method the homogenized polynomial $\tilde{p}(x)$ must be positive definite in the relevant hyperrectangles H_0, H_1^+, H_1^-, H_2^+, H_2^-, H_3^+ and H_3^- defined by means of (27) and (28).

Fig.2 shows the inner and outer approximations to the value of R depending on the number of Chebychev points on the sets H_0, H_1^+, H_1^-, H_2^+, H_2^-, H_3^+ and H_3^-. For 100 Chebychev points per variable the following interval

$$R_{in} = 0.3132 \ \le \ R \ \le \ \ R_{out} = 0.3160 \tag{56}$$

was found for the value of the maximum radius R.

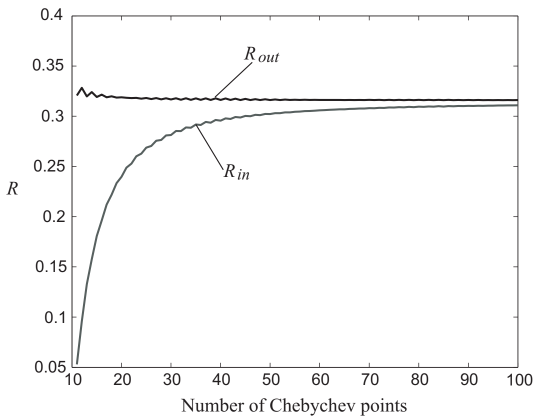

Fig. 2. Inner and outer approximations to the maximum radius R.

5. CONCLUSIONS AND OUTLOOK

In this paper the positivity of polynomials depending on uncertain parameters has been investigated. By means of the theorem of Ehlich and Zeller a new algorithm has been developed that defines the parameter region as a hypersphere where a polynomial $p(x)$ is positive definite. The examples presented in this paper illustrate the result which can be achieved with this new algorithm. In contrast to other methods our method is able to produce inner and outer approximations to the set Ω and relies entirely on linear inequalities. This offers the possibility of using methods from linear programming in the case of higher parameter dimension. This will be the focus of future research.

REFERENCES

Bose, N.K. (1982). *Applied multidimensional systems theory.* Van Nostrand Reinhold Co.. New York.

Bose, N.K. and J.P. Guiver (1980). Multivariate polynomial positivity invariance under coefficient perturbation. *IEEE Trans. on Acoust. Speech Signal Process.* **28**, 660–665.

Djaferis, T.E. (1991). Simple robust tests for polynomials with real parameter uncertainty. *International Journal of Control* **53**, 907–927.

Ehlich, H. and K. Zeller (1964). Schwankung von Polynomen zwischen Gitterpunkten. *Mathematische Zeitschrift* **86**, 41–44.

Gahinet, P., P. Apkarian and M. Chilali (1996). Affine parameter-dependent lyapunov functions and real parametric uncertainty. *IEEE Transactions on Automatic Control* **41**, 436–442.

Gärtel, U. (1987). *Fehlerabschätzungen für vektorwertige Randwertaufgaben zweiter Ordnung, insbesondere für Probleme aus der chemischen Reaktions Diffusions Theorie.* Ph.D. Thesis, University of Cologne. Germany.

Parrilo, P.A. (2000). On a decomposition of multivariable forms via LMI methods. *Proceedings of the American Control Conference, Chicago, USA* pp. 322–326.

Tibken, B. and K.F. Dilaver (2001). Robust positivity of polynomials with uncertain parameters. *International Symposium on Theoretical Electrical Engineering ISTET'01, Linz, Austria.*

Tibken, B. and K.F. Dilaver (2003). New numerical method for the polynomial positivity invariance under coefficient perturbation. *European Control Conference ECC'03, Cambridge, England.*

Tibken, B. and K.F. Dilaver (2004). Investigation of the global positivity of polynomials with uncertain parameters. *49. Internationales Wissenschaftliches Kolloquium, Technische Universität Ilmenau, Germany* **1**, 375–380.

Tibken, B., E.P. Hofer and C. Demir (1999). Guaranteed regions of attraction for dynamical polynomial systems. In: *Dynamics and Control* (G. Leitmann, F. Udwadia and A. V. Kryazhimskii, Eds.). Vol. 9. pp. 119–128. Gordon and Breach Publishers. London.

Tibken, B., K.F. Dilaver and O. Hachicho (2003). New numerical algorithms for the investigation of robust positivity of polynomials with uncertain parameters. *International Journal of Applied Electromagnetics and Mechanics* **17**, 107–120.

Wu, Q.H. and M. Mansour (1995). Robust stability of family of polynomials with 1-norm-bounded parameter uncertainties. In: *Stability Theory* (R. Jeltsch and M. Mansour, Eds.). Vol. 121. pp. 163–171. Birkhäuser Verlag. Zürich.