OPTIMAL FILTERING FOR LINEAR SYSTEMS WITH MULTIPLICATIVE AND ADDITIVE WIENER NOISES

Huichai Zhang ** **Michael Basin** * **Mikhail Skliar** **

* *Department of Physical and Mathematical Sciences*
Autonomous University of Nuevo Leon, Mexico
** *Department of Chemical and Fuels Engineering*
University of Utah, USA

Abstract The problem of the optimal state estimation is solved for the system described by the continuous, linear, n-dimensional ordinary differential equation with multiplicative and additive Wiener noises. The obtained solution essentially relies on the recently developed optimal filtering theory for Itô-Volterra systems.

Keywords: Filtering, stochastic system, multiplicative and additive Wiener noises

1. INTRODUCTION

The state estimation problem is considered for the continuous stochastic systems subject to additive and multiplicative Wiener processes. The classical example of a system with additive Wiener noises is the Brownian motion (Åström, 1970). The example when a Wiener disturbance is multiplicative is given by the model of a re-entry vehicle with the intensity of the stochastic drag coefficient increasing linearly with time as the vehicle descends into a denser atmosphere.

It seems that the state estimation problem for the systems described by a high-dimensional ordinary differential model with additive and multiplicative Wiener noises has not been previously studied. The case of additive and multiplicative white noises, studied by the authors simultaneously, allows the reformulation of the problem in the state space form with state dependent noise intensities. For the case of multiplicative and additive Wiener noises, the state space reformulation is not possible, and the result of the paper substantially depends on the recently developed optimal filtering theory for the Itô-Volterra systems (Zhang *et al.*, 2004).

[1] This work was supported under the US NSF Grant CTS-0117300 and the Mexican National Science and Technology Council (CONACyT) Grant 39388-A.

The next section summarizes the results on the optimal filtering for the Itô-Volterra systems, which are used to derive the main results of the paper.

2. OPTIMAL FILTER FOR ITO-VOLTERRA SYSTEMS

Let (Ω, F, P) be a complete probability space with an increasing right-continuous family of σ-algebras $F_t, t \geq 0$, and let $(W_1(t), F_t, t \geq 0)$ and $(W_2(t), F_t, t \geq 0)$ be independent Wiener processes with the unit variance intensities. Here, Ω is the sample space, F is a set of subsets on which the probability measure (or, simply, probability) is defined, and P is the probability defined on F. All subsets of F form a σ-algebra, and F_t denotes a family of subsets (σ-algebra) for every t such that for $t_1 < t_2$, $F_{t_1} \subset F_{t_2}$. The partly observed F_t-measurable random process $(x(t), z(t))$ is described using the Itô-Volterra equations:

$$x(t) = \int_0^t (A(t,s)x(s) + B(t,s)u(s))ds + \int_0^t G(x,t,s)dW_1(s), \quad (1)$$

$$z(t) = \int_0^t C(t,s)x(s)ds + \int_0^t H(t,s)dW_2(s), \quad (2)$$

where $x(t) \in R^n$ is the state vector, and $z(t) \in R^m$ is a vector of measurements *integrated* over the time interval $[0,t]$. The vector-valued function $B(t,s)u(s)$ describes the effect of known system inputs. Functions $A(t,s)$, $B(t,s)$ are smooth in t uniformly in s. Functions $C(t,s)$, $H(t,s)$ are of appropriate dimensions and continuous in t and s. $C(t,s)$ is a nonzero matrix and $G(x,t,s)G^T(x(s),t,s) \geq 0$, $H(t,s)H^T(t,s) > 0$. To simplify notation, denote

$$(H(t,s)H^T(t,s))^{-1} = \Upsilon(t,s) \tag{3}$$

throughout the paper. Except for state-dependent function G, all coefficients in the equations (1) and (2) are deterministic functions of t and s, both of which are independent (time) variables with $t \geq s \geq 0$. Without loss of generality, zero initial conditions are assumed.

The estimation problem is to find the estimate $\hat{x}(t)$ of the system state $x(t)$ described by the Itô–Volterra model (1) based on the observation process $Z(t) = \{z(s), 0 \leq s \leq t\}$, such that the Euclidean 2-norm

$$J = \mathrm{E}[(x(t) - \hat{x}(t))^T (x(t) - \hat{x}(t)) \mid F_t^Z] \tag{4}$$

is minimized at every time moment t. Here, $E[\xi(t) \mid F_t^Z]$ means the conditional expectation of a stochastic process $\xi(t) = (x(t) - \hat{x}(t))^T (x(t) - \hat{x}(t))$ with respect to the σ - algebra F_t^Z generated by the observation process $z(t)$ in the interval $[t_0,t]$. In an alternative formulation, the objective is to find the conditional expectation $m(t) = \hat{x}(t) = E(x(t) \mid F_t^Z)$. As usual, $P(t) = \mathrm{E}[(x(t) - m(t))(x(t) - m(t))^T \mid F_t^Z]$ is the estimation error covariance matrix.

This formulation is, in fact, the Kalman filtering problem for the integral Itô-Volterra system. This formulation is more general than the problem considered in (Kleptsina and Veretennikov, 1985; Shaikhet, 1987), since the Itô-Volterra measurements model, equation (2), cannot be reduced to the standard differential form. The standard state space formulation can be recovered by making all functional parameters in (1) and (2) dependent on s only.

The solution of the optimal filtering problem for the system (1)–(2) was first reported in (Basin, 2000), which generalized results (Kleptsina and Veretennikov, 1985; Shaikhet, 1987) obtained for systems with Itô-Volterra dynamics and standard differential measurements. It is shown in (Kleptsina and Veretennikov, 1985; Shaikhet, 1987) that the variance matrix $P(t)$ alone is not sufficient to completely characterize the state estimation process and to obtain a closed form of filtering equations. Instead, for the systems with the Itô-Volterra dynamic model, equation (1), the explicit solution can be obtained in terms of the integral cross-correlation function $f(t,s)$, which characterizes the deviation of the optimal estimate $m(t)$ from unknown true state $x(t)$ and is defined as:

$$f(t,s) = \mathrm{E}[(x_s^t - m_s^t)(x(s) - m(s))^T \mid F_s^Z], \tag{5}$$

where x_s^t can be viewed as a state with independent (time) variable s and parameter t:

$$\begin{aligned} x_s^t = &\int_0^s [A(t,r)x(r) + B(t,r)u(r)]dr \\ &+ \int_0^s G(x(r),t,r)dW_1(r). \end{aligned} \tag{6}$$

The governing equation for x_s^t can be differentiated with respect to s to yield the state space form of equation (6). The conditional mean $m_s^t = \mathrm{E}[x_s^t \mid F_s^Z]$ is the estimate of x_s^t. Note that function f is a generalization of the variance P, since $f(t,t) = P(t)$. Furthermore, for $s = t$, $x_s^t = x(t)$ and $z_s^t = z(t)$.

Theorem 1. (Basin, 2000; Zhang *et al.*, 2004) The optimal in the Kalman sense estimate $m(t)$ of the states of system (1) with measurements (2) satisfies the following optimal filter equation

$$\begin{aligned} m(t) = &\int_0^t \big[A(t,s)m(s) + B(t,s)u(s)\big]ds \\ &+ \int_0^t K_{tttt}(s)\big[dz(s) - C(t,s)m(s)ds\big], \end{aligned} \tag{7}$$

where the componentwise multiplication by the m-dimensional measure $\mu(t)$ is used, and the filter gain is given by

$$K_{abcd}(e) = f(a,e)C^T(b,e)(H(c,e)H^T(d,e))^{-1}. \tag{8}$$

The function $f(t,s)$ is found from the following Riccati-like equation

$$\begin{aligned} f(t,s) = &\int_0^s \big[A(t,r)f^T(s,r) + f(t,r)A^T(s,r) + \Psi\big]dr \\ &- \int_0^s \Big[K_{tsss}(r)C(s,r)f^T(s,r) + K_{tttt}(r)C(t,r)f^T(s,r) \\ &- \frac{1}{2}K_{tttt}(r)H(t,r)H^T(s,r)\Upsilon(s,r)C(s,r)f^T(s,r) \\ &- \frac{1}{2}K_{ssss}(r)H(s,r)H^T(t,r)\Upsilon(t,r)C(t,r)f^T(t,r)\Big]dr, \end{aligned} \tag{9}$$

where $\Psi = \mathrm{E}[G(x(r),t,r)G^T(x(r),s,r) \mid F_r^Z]$.

3. MAIN RESULTS

The problem is to find an optimal estimation of $\xi^{(n-1)}(t)$, $\xi^{(n-2)}(t)$, …, $\xi^{(0)}(t)$ given the ODE model for ξ with additive and multiplicative Wiener noises, and measurements of any linear combination of $\xi^{(n-1)}(t), \cdots, \xi(t)$, or a measurement vector of different linear combinations of the derivatives of ξ up to the order $n-1$. The case of time-invariant deterministic coefficients of the n-dimensional ordinary differential model is first considered.

3.1 *Case 1: Time-Invariant Coefficients*

Consider a general linear ODE with time-invariant deterministic coefficients, and multiplicative and additive Wiener noises:

$$\xi^{(n)}(t)+(a_1+W_1^1(t))\xi^{(n-1)}(t)+\cdots+(a_i+W_1^i(t))\xi^{(i)}(t) + \cdots + (a_n+W_1^n(t))\xi(t)=\lambda(t)+W_1^0(t), \qquad (10)$$

where $W_1^i(t),\ i=0,\ 1,\ldots,n$ are the independent Wiener processes. Assuming zero initial conditions $\xi(0)=\xi'(0)=\ldots=\xi^{(n-1)}(0)=0$ and zero forcing before the initial time, $\lambda(0)=0$, the integration of the ODE model yields

$$\begin{aligned}\xi^{(n-1)}(t) = & \\ -\int_0^t a_1\xi^{(n-1)}(s)ds & - \int_0^t W_1^1(s)\xi^{(n-1)}(s)ds - \cdots \\ -\int_0^t a_i\xi^{(n-i)}(s)ds & - \int_0^t W_1^i(s)\xi^{(n-i)}(s)ds - \cdots \\ -\int_0^t a_n\xi(s)ds & - \int_0^t W_1^n(s)\xi(s)ds \\ +\int_0^t \lambda(s)ds & + \int_0^t W_1^0(s)ds. \qquad (11)\end{aligned}$$

The following lemma (Cakmak *et al.*, 1987) is used in subsequent derivations.

Lemma 1: For $i=2,\ \cdots,\ n$

$$\xi^{(n-i)}(v)=\int_0^v \frac{(v-s)^{(i-2)}}{(i-2)!}\xi^{(n-1)}(s)ds. \qquad (12)$$

Integration of equation (11) with $v\in[0,t]$ gives

$$\begin{aligned}\int_0^t \xi^{(n-i)}(v)dv &= \int_0^t \Big[\int_0^v \frac{(v-s)^{(i-2)}}{(i-2)!}\xi^{(n-1)}(s)ds\Big]dv \\ &= \int_0^t \frac{(t-s)^{i-1}}{(i-1)!}\xi^{(n-1)}(s)ds, \qquad (13)\end{aligned}$$

where the result of Lemma 1 was used.

In equation (11), consider terms of the form $W_1^i(v)\xi^{(n-i)}(v)$, for $i=1,\ldots,n$. The following chain of equalities is obtained using the interchange of the integration variables, followed by integration by parts:

$$\begin{aligned}&\int_0^t W_1^i(v)\xi^{(n-i)}(v)dv = \int_0^t W_1^i(v)d\xi^{(n-i-1)}(v) \\ &= W_1^i(t)\xi^{(n-i-1)}(t) - \int_0^t \xi^{(n-i-1)}(v)dW_1^i(v) \\ &= \int_0^t \xi^{(n-i-1)}(t)dW_1^i(v) - \int_0^t \xi^{(n-i-1)}(v)dW_1^i(v) \\ &= \int_0^t \big(\xi^{(n-i-1)}(t)-\xi^{(n-i-1)}(v)\big)dW_1^i(v) \\ &= \int_0^t \Big[\int_0^t \frac{(t-r)^{i-1}}{(i-1)!}\xi^{(n-1)}(r)dr \\ &-\int_0^v \frac{(v-r)^{i-1}}{(i-1)!}\xi^{(n-1)}(r)dr\Big]dW_1^i(v). \qquad (14)\end{aligned}$$

Therefore, each term of (11), which includes the Wiener process as a part of the integrand, can be transformed into the integral with respect to the differential of the Wiener process $W_1^i(v)$.

Using the results (13) and (14) in equation (11), obtain:

$$\begin{aligned}&\xi^{(n-1)}(t) = -a_1\int_0^t \xi^{(n-1)}(v)dv \\ &-\int_0^t \Big[\int_0^t \xi^{(n-1)}(r)dr - \int_0^v \xi^{(n-1)}(r)dr\Big]dW_1^1(v) \\ &-\cdots - a_i\int_0^t \frac{(t-s)^{i-1}}{(i-1)!}\xi^{(n-1)}(s)ds \\ &-\int_0^t \Big[\int_0^t \frac{(t-r)^{i-1}}{(i-1)!}\xi^{(n-1)}(r)dr \\ &-\int_0^v \frac{(v-r)^{i-1}}{(i-1)!}\xi^{(n-1)}(r)dr\Big]dW_1^i(v) - \cdots \\ &-a_n\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}\xi^{(n-1)}(s)ds \\ &-\int_0^t \Big[\int_0^t \frac{(t-r)^{n-1}}{(n-1)!}\xi^{(n-1)}(r)dr \\ &-\int_0^v \frac{(v-r)^{n-1}}{(n-1)!}\xi^{(n-1)}(r)dr\Big]dW_1^n(v) \\ &+\int_0^t \lambda(v)dv + \int_0^t (t-v)dW_1^0(v), \qquad (15)\end{aligned}$$

where the last term of (11) is transformed into the last term of the above equation following the same procedure as used to obtain equation (14). Setting for consistency $s=r$ and $v=s$, obtain the description of the system with additive and multiplicative Wiener noises in the standard Itô-Volterra form (1) with $\xi^{(n-1)}$ as a state:

$$\begin{aligned}\xi^{(n-1)}(t) &= \int_0^t \Big[\Big(-\sum_{i=1}^n a_i\frac{(t-s)^{i-1}}{(i-1)!}\Big)\xi^{(n-1)}(s)+\lambda(s)\Big]ds \\ &+\int_0^t G(\xi^{(n-1)},t,s)dW_1(s), \qquad (16)\end{aligned}$$

where

$$\begin{aligned}&G(\xi^{(n-1)},t,s) = \big[G_0\ G_1\ \cdots\ G_i\ \cdots\ G_n\big] \qquad (17)\\ &G_0(t,s) = (t-s)\\ &G_1(\xi^{(n-1)},t,s) = -\int_0^t \xi^{(n-1)}(r)dr + \int_0^s \xi^{(n-1)}(r)dr\\ &\cdots\\ &G_i(\xi^{(n-1)},t,s) = -\int_0^t \frac{(t-r)^{i-1}}{(i-1)!}\xi^{(n-1)}(r)dr\\ &+\int_0^s \frac{(s-r)^{i-1}}{(i-1)!}\xi^{(n-1)}(r)dr\\ &\cdots\\ &G_n(\xi^{(n-1)},t,s) = -\int_0^t \frac{(t-r)^{n-1}}{(n-1)!}\xi^{(n-1)}(r)dr\\ &+\int_0^s \frac{(s-r)^{n-1}}{(n-1)!}\xi^{(n-1)}(r)dr,\\ &W_1^T(s) = \big[W_1^0(s)\ W_1^1(s)\ W_1^2(s)\ \cdots\ W_1^{n-1}(s)\ W_1^n(s)\big],\end{aligned}$$

and $W_1(t)$ is the $(n+1)$-dimension column vector of independent Wiener noises with $\mathrm{E}[W_1(t)] = 0$, $cov[W_1(t)] = tI$.

Now, the measurement model should be cast in the Itô-Volterra form. Suppose $\xi^{(n-j)}(t),\ j = 1,2,\cdots,n$, is measured, i.e.,

$$y(t) = \xi^{(n-j)}(t) + H(t)\omega_2(t). \tag{18}$$

Then the *integral* measurement

$$z(t) = \int_0^t y(s)ds$$

can be written in the form of equation (2) with $\xi^{(n-1)}(t)$ as the state variable:

$$z(t) = \int_0^t \frac{(t-s)^{j-1}}{(j-1)!}\xi^{(n-1)}(s)ds + \int_0^t H(s)dW_2(s). \tag{19}$$

The optimal filter for the system (10) with measurements (18) represented in the Itô-Volterra form is obtained applying Theorem 1 to the dynamic system (16), (19):

$$\begin{aligned} m(t) &= \int_0^t \Big[\big(-\sum_{i=1}^n a_i \frac{(t-s)^{i-1}}{(i-1)!}\big)m(s) + \lambda(s)\Big]ds \\ &+ \int_0^t f(t,s)\frac{(t-s)^{j-1}}{(j-1)!}\Upsilon(s) \\ &\times \Big[dz(s) - \frac{(t-s)^{j-1}}{(j-1)!}m(s)ds\Big], \end{aligned} \tag{20}$$

$$\begin{aligned} f(t,s) &= \int_0^s \Big[\big(-\sum_{i=1}^n a_i \frac{(t-r)^{i-1}}{(i-1)!}\big)f(s,r) \\ &+ f(t,r)\big(-\sum_{i=1}^n a_i \frac{(s-r)^{i-1}}{(i-1)!}\big) + \Psi\Big]dr \\ &- \int_0^s f(t,r)\Upsilon(r)\Big[\big(\frac{(s-r)^{j-1}}{(j-1)!}\big)^2 + \big(\frac{(t-r)^{j-1}}{(j-1)!}\big)^2 \\ &- \frac{(s-r)^{j-1}(t-r)^{j-1}}{((j-1)!)^2}\Big]f(s,r)dr, \end{aligned} \tag{21}$$

where $\Psi = E[G(x(r),t,r)G^T(x(r),s,r) \mid F_r^Z]$.

Each component G_i, $i = \overline{0,n}$ can be written as a sum of two terms: the first one is a function of t only, and the second one is the function of s only. Therefore, the vector G can be represented as

$$G(t,s) = f_1(t) + f_2(s) \tag{22}$$

and, in simplified notation,

$$\begin{aligned} \Psi &= \mathrm{E}[(f_1(t)+f_2(r))(f_1(s)+f_2(r)) \mid F_r^Z] \\ &= \mathrm{E}[f_1(t)f_1(s) + f_1(t)f_2(r) + f_1(s)f_2(r) \\ &+ f_2(r)f_2(r) \mid F_r^Z]. \end{aligned} \tag{23}$$

The constructive expression for Ψ is obtained by calculating the expectation operator for four additive terms in the last equation. Consider the estimation of one of the expectation operators. Take $\mathrm{E}[f_1(t)f_1(s) \mid F_r^Z]$ as an example. Since for any random $x_1(t)$ and $x_2(t)$

$$\begin{aligned} &\mathrm{E}[(x_1(t)x_2(t) \mid F_t^Z] \\ &\quad = \mathrm{E}[(x_1(t)-\hat{x}_1(t))(x_2(t)-\hat{x}_2(t)) \mid F_t^Z] \\ &\quad + \hat{x}_1(t)\hat{x}_2(t), \end{aligned} \tag{24}$$

obtain that

$$\begin{aligned} &\mathrm{E}[f_1(t)f_1(s) \mid F_r^Z] \\ &= \sum_{i=1}^n \mathrm{E}\Big[\int_0^r \frac{(t-r_1)^{i-2}}{(i-2)!}\xi^{(n-1)}(r_1)dr_1 \\ &\times \int_0^r \frac{(s-r_2)^{i-2}}{(i-2)!}\xi^{(n-1)}(r_2)dr_2 \mid F_r^Z\Big] \\ &= \sum_{i=1}^n \int_0^t\int_0^s \frac{(t-r_1)^{i-2}}{(i-2)!}\mathrm{E}\Big[\xi^{(n-1)}(r_1)\xi^{(n-1)}(r_2) \mid F_r^Z\Big] \\ &\times \frac{(s-r_2)^{i-2}}{(i-2)!}dr_1dr_2 \\ &= \sum_{i=1}^n \int_0^t\int_0^s \frac{(t-r_1)^{i-2}(s-r_2)^{i-2}}{((i-2)!)^2} \\ &\times [K(r_1,r_2) + m(r_1)m(r_2)]dr_2dr_2. \end{aligned} \tag{25}$$

The expressions for the other three expectation operators can be obtained similarly to yield the following expression for Ψ:

$$\begin{aligned} \Psi = (t-r)(s-r) \\ &+ \sum_{i=1}^n \Big\{\int_0^t\int_0^s \frac{(t-r_1)^{i-1}}{(i-1)!}[K(r_1,r_2) \\ &+ m(r_1)m(r_2)]\frac{(s-r_2)^{i-1}}{(i-1)!}dr_2dr_1 \\ &+ \int_0^t\int_0^r \frac{(t-r_1)^{i-1}}{(i-1)!}[K(r_1,r_2) \\ &+ m(r_1)m(r_2)]\frac{(r-r_2)^{i-1}}{(i-1)!}dr_1dr_2 \\ &+ \int_0^s\int_0^r \frac{(s-r_1)^{i-1}}{(i-1)!}[K(r_1,r_2) \\ &+ m(r_1)m(r_2)]\frac{(r-r_2)^{i-1}}{(i-1)!}dr_1dr_2 \\ &+ \int_0^r\int_0^r \frac{(r-r_1)^{i-1}}{(i-1)!}[K(r_1,r_2) \\ &+ m(r_1)m(r_2)]\frac{(r-r_2)^{i-1}}{(i-1)!}dr_1dr_2\Big\}. \end{aligned} \tag{26}$$

Assuming $r_1 \geq r_2$, the cross-covariance matrix

$$\begin{aligned} &K(r_1,r_2) = \\ &= \mathrm{E}[(\xi^{(n-1)}(r_1) - m(r_1))(\xi^{(n-1)}(r_2) - m(r_2)) \mid F_r^t] \end{aligned} \tag{27}$$

can be found from Theorem 2 in (Kleptsina and Veretennikov, 1985):

$$K(r_1,r_2) = f(r_1,r_2) + \int_{r_2}^{r_1} \Big[-\sum_{i=1}^{n} a_i \frac{(r_1-r_3)^{i-1}}{(i-1)!} - \frac{(r_1-r_3)^{2(j-1)}}{((j-1)!)^2} \Upsilon(r_3) f(r_1,r_3) \Big] K(r_3,r_2) dr_3. \quad (28)$$

The obtained filter, equations (20) and (21), determines the optimal estimate of $m(t) = \hat{\xi}^{(n-1)}(t)$. If the filtering problem consists only in estimating a derivative of any order from 0 to $(n-2)$, the optimal estimate can be found using equation (12), without the need for multiple integrations.

3.2 *Case 2: Time-Varying Coefficients*

The system is described by the following ODE with time-varying deterministic coefficients:

$$\xi^{(n)}(t) + (a_1(t) + W_1^1(t))\xi^{(n-1)}(t) + \cdots + (a_i(t) + W_1^i(t))\xi^{(i)}(t) + \cdots + (a_n(t) + W_1^n(t))\xi'(t) = \lambda(t) + W_1^0(t), \quad (29)$$

where $W_1^i(t)$ are independent Wiener processes. As in the case of time invariant coefficients, assuming $\xi(0) = \xi'(0) = \cdots = \xi^{(n-1)}(0) = \lambda(0) = 0$, integration of the model gives

$$\begin{aligned} \xi^{(n-1)}(t) = & \\ & - \int_0^t a_1(v)\xi^{(n-1)}(v)dv \\ & - \int_0^t W_1^1(v)\xi^{(n-1)}(v)d(v) - \\ \cdots & \\ & - \int_0^t a_i(v)\xi^{(n-i)}(v)dv \\ & - \int_0^t W_1^i(v)\xi^{(n-i)}(v)d(v) - \\ \cdots & \\ & - \int_0^t a_n(v)\xi(v)dv - \int_0^t W_1^n(v)\xi(v)dv \\ & + \int_0^t \lambda(v)dv + \int_0^t W_1^0(v)dv. \end{aligned} \quad (30)$$

Using Lemma 1, obtain that

$$\begin{aligned} I_a = & \int_0^t a_i(v)\xi^{(n-i)}(v)dv \\ = & \int_0^t a_i(v) \Big[\int_0^v \frac{(v-s)^{i-2}}{(i-2)!} \xi^{(n-1)}(s)ds \Big] dv \\ = & \int_0^t \Big[\int_s^t a_i(v) \frac{(v-s)^{i-2}}{(i-2)!} dv \Big] \xi^{(n-1)}(s)ds. \end{aligned} \quad (31)$$

Introducing new coefficients $\tilde{a}_i(t,s)$, defined by the following equation

$$\int_s^t a_i(v) \frac{(v-s)^{i-2}}{(i-2)!} dv = \tilde{a}_i(t,s) \quad (32)$$

for $i = 2,3,\cdots,n$, obtain that

$$I_a = \int_0^t \tilde{a}_i(t,s)\xi^{(n-1)}(s)ds. \quad (33)$$

The standard Itô-Volterra form of equation (30) can now be written as

$$\begin{aligned} \xi^{(n-1)}(t) = & \int_0^t \Big[\big(-\sum_{i=1}^{n} \tilde{a}_i(t,s)\xi^{(n-1)}(s) + \lambda(s) \Big] ds \\ & + \int_0^t G(\xi^{(n-1)},t,s)dW_1(s), \end{aligned} \quad (34)$$

where $G(\xi^{(n-1)},t,s)$ and $W_1(s)$ are defined by equations (17).

Limiting the consideration to the simplest case of the measurement model (19), the optimal filter for (34) is obtained applying the general optimal state estimation result of Theorem 1:

$$\begin{aligned} m(t) = & \int_0^t \Big[\big(-\sum_{i=1}^{n} \tilde{a}_i(t,s)m(s) + \lambda(s) \Big] ds \\ & + \int_0^t f(t,s) \frac{(t-s)^{j-1}}{(j-1)!} \Upsilon(s) \\ & \times \big[dz(s) - \frac{(t-s)^{j-1}}{(j-1)!} m(s)ds, \big] \end{aligned} \quad (35)$$

$$\begin{aligned} f(t,s) = & \int_0^s \Big[\big(-\sum_{i=1}^{n} \tilde{a}_i(t,s) f(s,r) \\ & + f(t,r) \big(-\sum_{i=1}^{n} \tilde{a}_i(t,s) + \Psi \Big] dr \\ & - \int_0^s f(t,r)\Upsilon(r) \Big[\big(\frac{(s-r)^{j-1}}{(j-1)!} \big)^2 + \big(\frac{(t-r)^{j-1}}{(j-1)!} \big)^2 \\ & - \frac{(s-r)^{j-1}(t-r)^{j-1}}{((j-1)!)^2} \Big] f(s,r)dr, \end{aligned} \quad (36)$$

where Ψ is given by (26).

4. CONCLUSIONS

In this paper, an optimal, in the Kalman sense, filter for linear n-dimensional ODE system with multiplicative and additive Wiener disturbances is developed. Though only a simple measurement model is considered in the paper, the extension for the case of vector measurements with the components in the form of an arbitrary linear combination of derivatives of the state $\xi(t)$ of any order between 0 and $n-1$ can be easily obtained. The developed filter provides the optimal estimate of the $(n-1)$-th order derivative of the state of the ODE model. It is shown that the estimation of a state derivative of any order between 0 to $(n-2)$ can be obtained with a single integration of $\hat{\xi}^{(n-1)}(t)$.

Unlike the optimal state estimation for the general linear ODE with multiplicative and additive white Gaussian noises, studied by the authors simultaneously, the case considered in this paper cannot be reduced to the equivalent state space form. Therefore, the obtained result substantially relies on the recently developed optimal filtering theory for the Itô-Volterra systems.

5. REFERENCES

Åström, Karl J. (1970). *Introduction to Stochastic Control Theory*. Vol. 70 of *Mathematics in Science and Engineering*. Academic Press. New York.

Basin, M. V. (2000). On filtering over Itô-Volterra observations. *Journal of Applied Mathematics and Stochastic Analysis* **13**, 347–365.

Cakmak, A. S., J. F. Botha and W. G. Gray (1987). *Computational and Applied Mathematics for Engineering Analysis*. Springer.

Kleptsina, M. L. and A. Yu. Veretennikov (1985). On filtering and properties of conditional laws of Itô-Volterra processes. In: *Statistics and Control of Stochastic Processes*. pp. 179–196. New York.

Shaikhet, L. E. (1987). On an optimal control problem of partly observable stochastic Volterra processes. *Problems of Control and Inform. Theory* **16**, 439–448.

Zhang, Huichai, Michael V. Basin and Mikhail Skliar (2004). An Itô-Volterra optimal state estimation with continuous, multirate, randomly sampled and delayed measurements. *IEEE Trans. Automat. Contr., submited, manuscript at http://implicit.che.utah.edu/∼skliar/TAC2004.pdf*.