DESIGN OF ROBUST PARAMETRIC MPC FOR HYBRID SYSTEMS

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Abstract: This paper presents an algorithm for the design of robust model-based predictive controller for hybrid system under uncertainty via parametric programming. A min-max approach is adopted to design robust hybrid parametric predictive controller (RHPPC) where a cost function is minimized for the maximum violation of the uncertainty involved. The proposed hybrid control scheme guarantees stability and feasible operation in the presence of bounded input uncertainty. The governing piecewise affine optimal control policy as a function of states can then be administered on-line as a sequence of simple function evaluations. An example is presented to illustrate the details of the proposed RHPPC design.

Keywords: Hybrid systems, MPC, robust control, parametric programming

1. INTRODUCTION

Hybrid systems are systems involving interactive combination of logic, dynamics and constraints, (Grossman *et al.*, 1993), (Antsaklis, 2000), also called mixed logical dynamical (MLD) systems, (Bemporad and Morari, 1999). Many practical engineering applications are inherently hybrid in nature. Modelling, optimization and control of hybrid systems is one of the most active areas of research in control and systems engineering.

The presence of uncertainty due to inevitable parameter variations and exogenous disturbances may severely affect the performance of a hybrid system, potentially leading to infeasibility. Therefore the key control objective is to achieve robust stability and robust performance using the optimal switching among the family of state and input models, while guaranteeing economics and operational safety. However, the issue of robust controller design for hybrid systems under uncertainty has not been fully addressed in the open literature. Mayne and Raković, (Mayne and Raković, 2003) have proposed first on-line MPC algorithm for hybrid systems by exploiting the special problem structure.

In the current work, the mixed integer parametric programming principles are applied to design explicit MPC under the influence of uncertainty to achieve robust performance. A min-max control formulation is proposed to derive the explicit control policy to safeguard against worst-case un-

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certainty scenario, thus guaranteeing feasibility as well as stability.

The paper is outlined as follows. Section 2 presents the problem formulation of the robust hybrid control. Section 3 describes the theoretical tools used to achieve robust feasibility and stability. In section 4, a min-max problem is proposed for both the open-loop and the closed-loop cases. Finally section 5 presents a design example to illustrate the implementation procedure of the proposed robust hybrid parametric predictive controller (RHPPC).

2. HYBRID SYSTEM MODEL

2.1 System Representation

Consider the following discrete dynamical system in the form of multi-model state space system (Bemporad and Morari, 1999), (Sakizlis *et al.*, 2002).

$$x(k+1) = \begin{cases} A_1x(k) + B_1u(k) + Gw(k) \\ \quad if \ \ S_1x(k) + T_1u(k) \le E_1 \\ A_2x(k) + B_2u(k) + Gw(k) \\ \quad if \ \ S_2x(k) + T_2u(k) \le E_2 \\ \quad \vdots \\ A_sx(k) + B_su(k) + Gw(k) \\ \quad if \ \ S_sx(k) + T_su(k) \le E_s \end{cases} \tag{1}$$

where $x(k) \in \Re^n$ are states, $u(k) \in \Re^m$ are control input, and $w(k) \in \Re^l$ are disturbance variables with $x(0) = x_0$ and corresponding system matrices $A_i \in \Re^{n\times n}$, $B_i \in \Re^{n\times m}$, $G \in \Re^{n\times l}$; $\forall i = 1,\ldots,s$. Furthermore we enforce $x(k)$, $u(k)$ and $w(k)$ to be enclosed inside the bounded polyhedral sets i.e., $\forall k \ge 0$, $x(k) \in \mathcal{X}$, $u(k) \in \mathcal{U}$, and $w(k) \in \mathcal{W}$ representing operating limitations. Constraint matrices S_i, T_i and E_i defines the convex polyhedra in the state+input space.

2.2 Reformulation to Mixed-Integer Form

Consider a binary variables $\delta_i(k) \in \{0,1\}$ corresponding to each of the i^{th} system dynamics, which implies whether that system is active or not by assigning the appropriate switching sequence. Now, by defining the nonlinear terms $z_i(k) \triangleq [A_ix(k) + B_iu(k)]\,\delta_i(k)$ system (1) can be reformulated as (Bemporad and Morari, 1999), (Sakizlis, 2003),

$$\begin{aligned} x(k+1) &= \sum_{i=1}^{s} z_i(k) + Gw(k) \\ z_i(k) &\le M\delta_i(k) \\ z_i(k) &\ge m\delta_i(k) \\ z_i(k) &\le A_ix(k) + B_iu(k) - m\,(1-\delta_i(k)) \\ z_i(k) &\ge A_ix(k) + B_iu(k) - M\,(1-\delta_i(k)) \\ E_i &\ge S_ix(k) + T_iu(k) - M_i^*\,(1-\delta_i(k)) \end{aligned} \tag{2}$$

where $\sum_{i=1}^{s}\delta_i(k) = 1$, while M and M_i^* are appropriately dimensioned large numbers with $m = -M$.

2.3 Problem Formulation

The finite-horizon predictive control problem for the hybrid system is given by,

$$\begin{aligned} \Phi_p(x(0)) = \min_{u(k)} &\left\{ \sum_{k=0}^{N-1} \|Qx(k)\|_p + \|Ru(k)\|_p \right\} \\ &+ \|Px(N)\|_p \end{aligned} \tag{3}$$

$$\begin{aligned} s.t.\ x(k+1) \ &= \ A_ix(k) + B_iu(k) + Gw(k) \\ & x(k) \in \mathcal{X}, u(k) \in \mathcal{U},\ w(k) \in \mathcal{W}, \\ & x(N) \in \mathcal{O}_\infty \subseteq \mathcal{X},\ if\ m_i \le x(k) \le M_i; \\ & \forall k = 0,1,\ldots,(N-1);\ \forall i = 1 \vee 2 \vee \ldots \vee s \end{aligned}$$

where $Q \succeq 0$ and $R \succ 0$ are the weighting matrices for state and control while positive definite P is the stabilizing terminal cost for the prediction horizon N. The objective is defined over $p = 1$, 2 or ∞ based on l_1, l_2 or l_∞ performance criterion and $\vee$ is disjunction denoting logical "or" for $i = 1,\ldots,s$ systems.

Furthermore after the N^{th} time step we enforce the states to lie in the positive invariant set, (Kolmanovsky and Gilbert, 1998), (Blanchini, 1999), containing the origin in its interior by defining the positive invariant set $\mathcal{O}_\infty \subseteq \mathcal{X}$:

$$\mathcal{O}_\infty \equiv \left\{ \begin{array}{l} x \in \Re^n, u \in \Re^m \mid Kx \in \mathcal{U}, \\ (A_i + B_iK)x + Gw \in \mathcal{X}; \forall w \in \mathcal{W} \end{array} \right\} \tag{4}$$

where K is the optimal feedback gain.

Rewriting the system (2) in terms of constraint sets $\mathcal{X}$ and $\mathcal{U}$ and substituting $x(k)$ into the objective function of equation (3), the problem

(3) can be reformulated as following multiparametric mixed integer quadratic program, (Dua and Pistikopoulos, 2000).

$$
\begin{aligned}
\Gamma(X) &= \min_{U,V,Z,D} E_{W\in\mathcal{W}^N}\left[\Phi(U,V,Z,D,X,W)\right] \quad (5)\\
s.t.\ g(z_i(k),v_i(k),w(k),\delta_i(k),u(k),X) &\le 0\\
\sum_{i=1}^{s}\delta_i(k) &= 1\\
v_i(k)\in\mathcal{U},\ w(k)\in\mathcal{W},\ z_i(0)\in\mathcal{X},\\
U\in\mathcal{U}^N,\ z_i(k)\in\mathcal{X},\ z_i(N)\in\mathcal{O}_\infty\\
\delta_i(k)\in\{0,1\};\ \forall i = 1\vee 2\vee\ldots\vee s
\end{aligned}
$$

where the column vector V, Z, D defined as

$$
V \triangleq \begin{bmatrix} [v_1(0),\ldots,v_s(0)]^T,\ldots,\\ [v_1(N-1),\ldots,v_s(N-1)]^T \end{bmatrix};
$$

$$
Z \triangleq \begin{bmatrix} [z_1(1),\ldots,z_s(1)]^T,\ldots,\\ [z_1(N-1),\ldots,z_s(N-1)]^T \end{bmatrix};
$$

$$
D \triangleq \begin{bmatrix} [\delta_1(0),\ldots,\delta_s(0)]^T,\ldots,\\ [\delta_1(N-1),\ldots,\delta_s(N-1)]^T \end{bmatrix}
$$

are the optimization vectors, the column vector W defined as

$$
W \triangleq \begin{bmatrix} [w_1(0),\ldots,w_s(0)]^T,\ldots,\\ [w_1(N-1),\ldots,w_s(N-1)]^T \end{bmatrix}
$$

is the expected disturbance vector, U is a vector of control sequence, $U \triangleq [u(0)^T,\ldots,u(N-1)^T]^T$; and $X \triangleq [z_1(0)^T,\ldots,z_s(0)^T]^T$ are the current states treated as parameters.

3. THEORETICAL DEVELOPMENTS

3.1 Stability and Terminal Cost for l_2 Criterion

Definition 1. Assuming pairs (A_i, B_i) are both stabilizable and detectable, system (A_i, B_i) is asymptotically stable if there exists quadratic Lyapunov function given by $S(\xi) = \xi^T P \xi > 0$.

Using above definition we find the positive definite P from the following theorem.

Theorem 1. (Lyapunov Stability). According to Lyapunov stability theorem, an open-loop system is stable if and only if $\forall i = 1,\ldots,s$; $\exists\ P = P^T \succ 0$ such that

$$
A_i^T P A_i - P < 0 \qquad (6)
$$

and closed-loop system pairs (A_i, B_i) are stable if and only if $\forall i = 1,\ldots,s$; $\exists\ P = P^T \succ 0$ such that

$$
(A_i + B_i K)^T P (A_i + B_i K) - P < 0. \qquad (7)
$$

With $X = P^{-1}$ and $Y = KX$ equation (7) is converted into following set of LMIs, (for proof see, (Boyd *et al.*, 1994)).

$$
\begin{bmatrix} X & (A_i X + B_i Y)^T \\ (A_i X + B_i Y) & X \end{bmatrix} \succ 0. \qquad (8)
$$

Equations (6) and (8) are both LMIs. After N^{th}, time step the control law $u(k) = Kx(k)$, with control gain $K = YX^{-1}$ is implemented.

3.2 Feasibility

Definition 2. The robust polytopic parametric predictive controller steers the plant into the feasible operating region for a specific range of uncertain variations, $\forall k \ge 0$, $\forall w \in \mathcal{W}$.

According to the flexibility analysis theory of (Pistikopoulos and Grossmann, 1988), maximum constraint violation define the feasible operating region. This feasible region is depicted by the feasibility constraints, $\psi(U,V,Z,D,X) \le 0$ and is given by,

$$
\psi(U,V,Z,D,X) \le 0 \quad \Leftrightarrow
$$

$$
\max_{W,j} \left\{ \begin{array}{l} g_j\left(U,V,Z,D,X,W\right)\\ V\in\mathcal{U}^{Ns},\ Z\in\mathcal{X}^{(N-1)s},\\ U\in\mathcal{U}^N,\ W\in\mathcal{W}^N,\\ D\in\{0,1\}^{Ns};\ \forall j = 1,\ldots,J \end{array} \right\}. \qquad (9)
$$

Equation (9) can be solved by identifying critical uncertainty points of each maximization $\forall j = 1,\ldots,J$ and $\forall k = 0,\ldots,(N-1)$ as,

$$
\text{if } \frac{\partial \mathrm{g_j}}{\partial \mathrm{w(k)}} > 0 \ \Rightarrow\ w(k)^{cr} = w(k)^{ub} \qquad (10)
$$

$$
\text{if } \frac{\partial \mathrm{g_j}}{\partial \mathrm{w(k)}} < 0 \ \Rightarrow\ w(k)^{cr} = w(k)^{lb} \qquad (11)
$$

Thus, by substituting the sequence of critical uncertainty, $w(k)^{cr}$ in the constraints set $g(.)$, a multi-parametric linear program (mpLP) is formulated as,

$$\psi(U,V,Z,D,X)$$
$$= \max_j g_j\left(U,V,Z,D,X,W^{cr}\right)$$
$$= \min_{\varepsilon}\left\{\begin{array}{l} \varepsilon \geq g_j\left(V,Z,D,X,W^{cr}\right) \\ V\in\mathcal{U}^{Ns},\ Z\in\mathcal{X}^{(N-1)s}, \\ U\in\mathcal{U}^{N},\ W\in\mathcal{W}^{N}, \\ D\in\{0,1\}^{Ns},\ \forall j=1,\ldots,J \end{array}\right\}. \quad (12)$$

Equation (12) can then be solved using the formal comparison procedure of Acevedo and Pistikopoulos, (Acevedo and Pistikopoulos, 1997).

4. DESIGN OF RHPPC

4.1 Open-Loop RHPPC

The feasibility constraints (9) from section 3.2 are incorporated in problem (5) to obtain the following open-loop robust predictive control problem,

$$\begin{aligned} \Gamma_{ol}(X) &= \min_{U,V,Z,D} E_{W\in\mathcal{W}^N}\left[\Phi(U,V,Z,D,X,W)\right] \quad (13) \\ s.t.\ & g(z_i(k),v_i(k),w(k),\delta_i(k),u(k),X) \leq 0 \\ & \sum_{i=1}^{s}\delta_i(k) = 1 \\ & v_i(k)\in\mathcal{U},\ w(k)\in\mathcal{W},\ z_i(0)\in\mathcal{X}, \\ & z_i(k)\in\mathcal{X},\ z_i(N)\in\mathcal{O}_\infty \\ & \delta_i(k)\in\{0,1\};\ \forall i=1\vee 2\vee\ldots\vee s \\ & \max_{W,j}\left\{\begin{array}{l} g_j\left(U,V,Z,D,X,W\right) \\ W\in\mathcal{W}^N;\ \forall j=1,\ldots,J \end{array}\right\}. \end{aligned}$$

This open-loop robust predictive control problem is a bi-level optimization problem. Note that the inner minimization problem is equivalent to equation (12), which can be solved separately resulting into a set of linear feasibility constraints, $\psi(U,V,Z,D,X)\leq 0$. Substituting it into equation (13) results in the following single-level optimization problem:

$$\begin{aligned} \Gamma_{ol}(X) &= \min_{U,V,Z,D} E_{W\in\mathcal{W}^N}\left[\Phi(V,Z,D,X,W)\right] \quad (14) \\ s.t.\ & g(z_i(k),v_i(k),w(k),\delta_i(k),u(k),X) \leq 0 \\ & \psi(U,V,Z,D,X)\leq 0 \\ & \sum_{i=1}^{s}\delta_i(k) = 1 \\ & v_i(k)\in\mathcal{U},\ w(k)\in\mathcal{W},\ z_i(0)\in\mathcal{X}, \\ & z_i(k)\in\mathcal{X},\ z_i(N)\in\mathcal{O}_\infty \\ & \delta_i(k)\in\{0,1\};\ \forall i=1\vee 2\vee\ldots\vee s. \end{aligned}$$

4.2 Closed-Loop RHPPC

As discussed in Sakizlis et al., (Sakizlis *et al.*, 2004), the drawback of open-loop is that it does not take into account the future measurements which contain the information of past uncertainty. The closed-loop problem which preserves feasibility for all uncertainty realizations is achieved by finding feasibility constraints at every time step is given as follows.

$$\begin{aligned} \Gamma_{cl}(X) &= \min_{U,V,Z,D} E_{W\in\mathcal{W}^N}\left[\Phi(U,V,Z,D,X,W)\right] \quad (15) \\ s.t.\ & g(z_i(k),v_i(k),w(k),\delta_i(k),u(k),X) \leq 0 \\ & \sum_{i=1}^{s}\delta_i(k) = 1 \\ & v_i(k)\in\mathcal{U},\ w(k)\in\mathcal{W},\ z_i(0)\in\mathcal{X}, \\ & z_i(k)\in\mathcal{X},\ z_i(N)\in\mathcal{O}_\infty \\ & \delta_i(k)\in\{0,1\};\ \forall i=1\vee 2\vee\ldots\vee s \\ & \max_{w(0)}\min_{u(1)},\ldots,\max_{w(\mathcal{N}-2)}\min_{u(\mathcal{N})}\max_{w(\mathcal{N}-1),j}\left\{g(.)\right\} \\ & \max_{w(1)}\min_{u(2)},\ldots,\max_{w(\mathcal{N}-2)}\min_{u(\mathcal{N})}\max_{w(\mathcal{N}-1),j}\left\{g(.)\right\} \\ & \vdots \\ & \max_{w(\mathcal{N}-1),j}\left\{g(.)\right\} \end{aligned}$$

where $g(.)$ is defined in equation (9).

Equation (15) is multi-level program which can be solved separately similar to the open-loop solution procedure. Thus for each time step the inner max problem is solved backwards in time for the entire horizon N. The resulting flexibility constraints are then incorporated into problem (15) giving rise to single-level program similar to the open-loop case,

$$\Gamma_{cl}(X) = \min_{U,V,Z,D} E_{W \in \mathcal{W}^{\mathcal{N}}} [\Phi(V,Z,D,X,W)] \quad (16)$$

$$s.t.\ g(z_i(k), v_i(k), w(k), \delta_i(k), u(k), X) \leq 0$$

$$\sum_{i=1}^{s} \delta_i(k) = 1$$

$$v_i(k) \in \mathcal{U},\ w(k) \in \mathcal{W},\ z_i(0) \in \mathcal{X},$$

$$z_i(k) \in \mathcal{X},\ z_i(N) \in \mathcal{O}_\infty$$

$$\delta_i(k) \in \{0,1\};\ \forall i = 1 \vee 2 \vee \ldots \vee s$$

$$\psi_{w(0)}\left(V,X,Z,D,u(0)\right) \leq 0$$

$$\psi_{w(0)}\begin{pmatrix} V,X,Z,D,w(0), \\ \left[u(0)^T, u(1)^T\right] \end{pmatrix} \leq 0$$

$$\vdots$$

$$\psi_{w(0)}\begin{pmatrix} V,X,Z,D, \\ \left[w(0)^T,\ldots,w(N-2)^T\right], \\ \left[u(0)^T,\ldots,u(N-1)^T\right] \end{pmatrix} \leq 0.$$

Remark 3. The solution obtained in sections 4.1 and 4.2 both are obtained as a piecewise affine optimal robust parametric predictive control policy as a function of states $V(X)$ for the critical polyhedral regions in which plant operation is stable and feasible $\forall w(k) \in \mathcal{W}$.

5. DESIGN EXAMPLE

Example 1: Consider the following dynamical system, (Bemporad and Morari, 1999).

$$x(k+1) = 0.8\begin{bmatrix} cos\ \alpha(t) & -sin\ \alpha(t) \\ sin\ \alpha(t) & cos\ \alpha(t) \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(k);$$

$$\alpha(t) = \begin{cases} \pi/3 & if \quad [1\ \ 0]x(t) \geq 0 \\ -\pi/3 & if \quad [1\ \ 0]x(t) < 0 \end{cases}$$

$$x(t) \in [-10, 10] \times [-10, 10]$$

$$u(t) \in [-1, 1]\ ;\ \ w(t) \in [-0.5, 0.5].$$

Theorem 1 gives stabilizing $P = \begin{bmatrix} 1.4346 & 0.2684 \\ 0.2684 & 1.0139 \end{bmatrix}$ and control gain is $K = [-0.2241 \quad -0.9253]$.

For l_∞ performance criterion, Fig. 1 depicts the nominal control policy without disturbances while Fig. 2 depicts the open-loop robust control policy in presence of additive disturbances. Fig. 3 gives the open-loop robust control simulation results with starting condition as $[-8 \quad 8]$. The model switching for the open-loop robust parametric control is shown in Fig. 4.

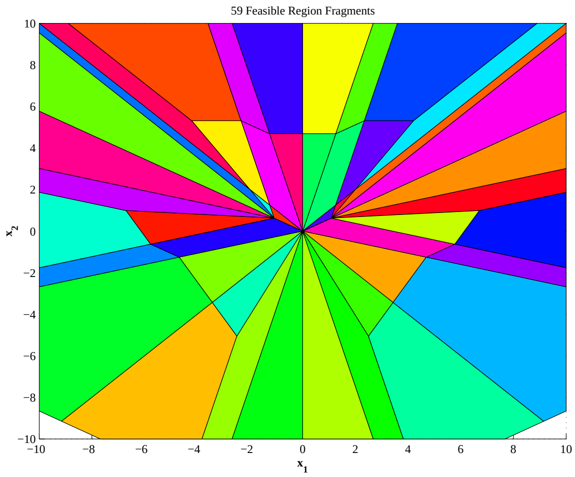

Fig. 1. State-space partition for nominal control

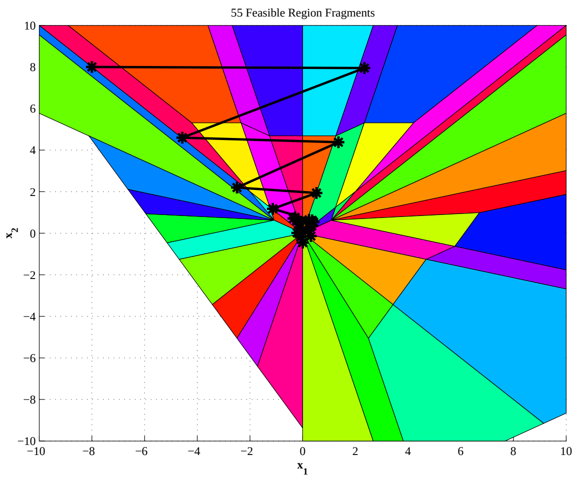

Fig. 2. State-space partition for open-loop robust control

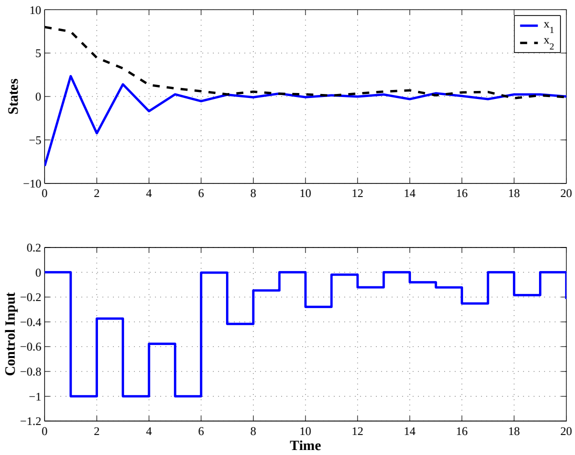

Fig. 3. open-loop robust control simulation

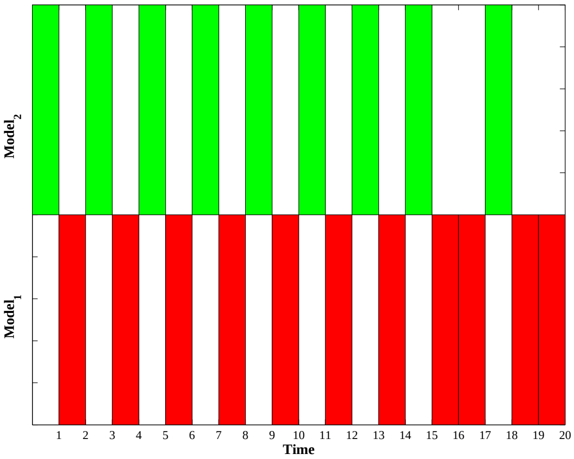

Fig. 4. model switching sequence

6. CONCLUSION

This paper presents an explicit solution to the robust MPC for linear hybrid systems via parametric programming. A min-max based feasibility analysis is described to deal with the worst case uncertainty. The open-loop and closed-loop controller performance guarantees system stability and feasible operation. The resulting controllers yield a piecewise affine control law which can be implemented on-line by simple function evaluations. The proposed framework can be extended to account for the polytopic uncertainty (Manthanwar *et al.*, 2005).

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