ADAPTIVE OUTPUT FEEDBACK EXTREMUM SEEKING CONTROL OF LINEAR SYSTEMS

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Abstract: In this paper, we present a control algorithm that incorporates real time optimization and receding horizon control technique to solve an output feedback extremum seeking control problem for a linear unknown plant. The resulting controller is able to drive the system states to the desired unknown optimum by requiring a Lyapunov restriction and a satisfaction of a persistency of excitation condition.

Keywords: Extremum seeking, Input to state stability, Persistence of excitation

1. INTRODUCTION

Optimization has become a key area in control theory due to the increasing need to optimize plant operation in order to reduce operating cost and meet product specifications. As better controllers are developed to adequately control a plant, the focus can be shifted to the solution of controller designs that guarantee optimal plant performance. If, for example, one can generate a reliable estimate of plant profitability, the purpose is shifted to the regulation of the process about conditions that provide maximum profitability. Such a task is usually tackled using a supervisory control technique. One such technique that has received considerable attention in the process industry is real-time optimization (RTO). One of the main challenges involved with the implementation of this technique is the difficulty associated with the integration of RTO with advanced process control (APC) applications. Despite the fact that these technologies are firmly established, their full integration remains troublesome in application.

In this paper, we propose a formal design technique that achieves the integrated task of RTO and APC system where the APC consists of a model predictive controller. The approach is based on the previous work for a class of nonlinear systems with parametric uncertainties (Adetola *et al.*, 2004). The control task is posed as an adaptive output feedback extremum-seeking control problem. Extremum seeking control has been proposed by a number of authors to handle optimization problems in control systems ((Guay and Zhang, 2003) and references therein). The formulation consists of two-phase optimization problems that are solved at every sample time. Assuming that a suitable functional expression for the plant profit is available, which in some application, may depend on unknown plant parameters, the first phase (RTO) uses the current value of the parameter estimates to compute the optimal value which maximizes the economic objective. The second phase (APC) solves the dynamic finite horizon optimal control problem that regulates the output to the desired target value computed by the RTO. The design achieves dynamic tracking of the unknown optimum and ensures both transient and asymptotic performances.

This paper is structured as follows. The problem description is given in section 2 and the design procedure is presented in section 3. The proposed control algorithm and our main result are presented in section 4. Numerical simulation result is shown in section 5 and finally, conclusions are given in section 6.

2. PROBLEM DESCRIPTION

Consider an objective *(profit)* function of the form

$$y_p = p(y, \theta_1) \tag{1}$$

where $\theta_1 \in \mathbb{R}^q$ is a parameter vector that satisfies

$$\theta_1 \in \Omega_{\theta_1} = \left\{ \theta_1 \in \mathbb{R}^q \,\middle|\, \frac{\partial^2 p(y,\theta_1)}{\partial y^2} \le c_0 I < 0,\ y \in \mathbb{R} \right\} \tag{2}$$

The objective function depends on the output of the linear plant

$$y(s) = \frac{b_m s^m + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} u(s) \tag{3}$$

where the a_i 's and b_i 's are unknown constants.

The condition given in (2) ensures that the performance function $p(y, \theta_1)$ is strictly convex, which means that the objective function y_p achieves its maximum at a unique point y^*. This study is carried out under the following basic assumptions.

- The plant is minimum phase, the relative degree ρ, an upper bound for the plant order n and the high frequency gain are known.
- The unknown parameter $\theta_1 \in \mathbb{R}^q$ in (1) consists of some of the constants $a = [a_{n-1} \dots a_0]^T$ and $\bar{b} = [b_{m-1} \dots b_0]^T$.

3. DESIGN PROCEDURE

Let us re-write (3) in the observer canonical form

$$\begin{aligned} \dot{x} &= Ax + F(y,u)^T \Theta \\ y &= e_1^T x \end{aligned} \tag{4}$$

where e_i is a row vector of appropriate dimension with ith entry of one and zero elsewhere.

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix}$$

$$F(y,u)^T = \left[\begin{bmatrix} 0_{(\rho-1)\times(m+1)} \\ I_{m+1} \end{bmatrix} u,\ -Iy_n \right]$$

$$\Theta = \left[b_m,\ \dots,\ b_0,\ a_{n-1},\ \dots,\ a_0 \right]^T$$

Following the procedure in (Krstic *et al.*, 1995), the following state estimation filters are employed.

$$\dot{\zeta} = A_0\zeta + Ly \tag{5}$$

$$\dot{\Omega}^T = A_0\Omega^T + F(y,u)^T \tag{6}$$

where the vector L is chosen so that the matrix $A_0 = A - Le^T{}_1$ is Hurwitz. The state estimate is defined as

$$\hat{x} = \zeta + \Omega^T\Theta \tag{7}$$

and the dynamics of the state estimation error $\varepsilon = x - \hat{x}$ becomes

$$\dot{\varepsilon} = A_0\varepsilon \tag{8}$$

To lower the dynamic of the Ω-filter, the first $m+1$ columns of Ω^T are denoted by $\upsilon_m, \dots, \upsilon_0$ and the remaining n columns are denoted by Ξ, *i.e.*

$$\begin{aligned} \Omega^T &= [\upsilon_m, \dots, \upsilon_1, \upsilon_0, \Xi] \\ \dot{\upsilon}_j &= A_0\upsilon j + e_{n-j}u \quad j = 0, \dots, m \end{aligned}$$

A summary of the implemented filters is as follows

$$\dot{\eta} = A_0\eta + e_n y, \qquad \dot{\lambda} = A_0\lambda + e_n u$$

$$\Xi = -\left[{A_0}^{n-1}\eta, \dots, A_0\eta, \eta \right], \qquad \zeta = -{A_0}^n\eta$$

$$\Omega^T = [\upsilon_m, \dots, \upsilon_0, \Xi], \quad \upsilon_j = -{A_0}^j\lambda \quad j = 0, \dots, m$$

Consider the first equation in (4), *i.e.*

$$\dot{y} = x_2 - a_{n-1}y = x_2 - ye_1^T a \tag{9}$$

If we replace x_2 by its estimate $\hat{x}_2 = \zeta_2 + \Omega_{(2)}^T\Theta + \varepsilon_2$ we have

$$\dot{y} = \zeta_2 + b_m\upsilon_{m,2} + \omega^T\theta + \varepsilon_2 \tag{10}$$

where the regressor vector, ω, and the unknown parameter θ are defined as

$$\omega = \left[\upsilon_{m-1,2}, \dots, \upsilon_{0,2}, \Xi_{(2)} - ye_1^T \right]^T$$

$$\theta = [b_{m-1}, \dots, b_0, a_{n-1} \dots, a_0]^T$$

3.1 ISS Controller design via Backstepping

The controller design is started by choosing $\upsilon_{m,2}$ as the 'virtual control' because both $\upsilon_{m,2}$ and the unmeasured state x_2 are separated by only $\rho - 1$ integrators from the actual control u. Considering (10) and (9) for $j = m$, the design system chosen to replace (4) is

$$\begin{aligned} \dot{y} &= \zeta_2 + b_m\upsilon_{m,2} + \omega^T\theta + \varepsilon_2 \\ \dot{\upsilon}_{m,2} &= \upsilon_{m,3} - k_2\upsilon_{m,1} \\ &\vdots \\ \dot{\upsilon}_{m,\rho} &= \upsilon_{m,\rho+1} - k_\rho\upsilon_{m,1} + u \end{aligned}$$

Given a constant set point, y^r, to be tracked, our goal is to achieve ISS of the tracking error $z_1 = y - y^r$ with respect to the parameter estimation error $\tilde{\theta}$. Note that in the iss-controller design, $\hat{\theta} = \bar{\theta} =$ constant estimate, which implies that $\dot{\hat{\theta}} = 0$.
The dynamic of the tracking error is given as:

$$\dot{z}_1 = b_m\upsilon_{m,2} + \zeta_2 + \omega^T\theta + \varepsilon_2 \tag{11}$$

Let $z_i = b_m\upsilon_{m,i} - \alpha_{i-1}$ for $i = 2, \dots, \rho$ and choose

$$\alpha_1 = -c_1z_1 - g_1z_1 - \zeta_2 - \omega^T\hat{\theta} - k_1\omega^T\omega z_1$$

Then, we have

$$\dot{z}_1 = -c_1z_1 - g_1z_1 + \varepsilon_2 + \omega^T\tilde{\theta} - k_1\omega^T\omega z_1 + z_2$$

which will be asymptotically stabilizing if $\tilde{\theta}$, ε_2 and z_2 were zero. Consider a Lyapunov function $V_1 = \frac{1}{2}z_1^2$. We have

$$\dot{V}_1 = -c_1z_1^2 - g_1z_1^2 + \varepsilon_2z_1 + \omega^T\tilde{\theta}z_1 - k_1\omega^T\omega z_1^2 + z_1z_2$$

From the fact that $-g_1 z_1^2 + \varepsilon_2 z_1 \leq \frac{1}{4g_1}\varepsilon_2^2$ and $\omega^T \tilde{\theta} z_1 - k_1 \omega^T \omega z_1^2 \leq \frac{1}{4k_1}\left\|\tilde{\theta}\right\|^2$, we have

$$\dot{V}_1 \leq -c_1 z_1^2 + \frac{1}{4g_1}\varepsilon_2^2 + \frac{1}{4k_1}\left\|\tilde{\theta}\right\|^2 + z_1 z_2$$

Step $2 \dots \rho - 1$

$$\begin{aligned}
z_i &= b_m \upsilon_{m,i} - \alpha_{i-1} \\
\beta_i &= b_m k_i \upsilon_{m,1} + \frac{\partial \alpha_{i-1}}{\partial y}\left(\omega^T \hat{\theta} + \zeta_2 + b_m \upsilon_{m,2}\right) \\
&\quad + \frac{\partial \alpha_1}{\partial \eta}\dot{\eta} + \sum_{j=1}^{m+i-1} \frac{\partial \alpha_{i-1}}{\partial \lambda_j}\left(-k_j \lambda_1 + \lambda j + 1\right) \\
V_i &= V_{i-1} + \frac{1}{2} z_i^2 \\
\alpha_i &= -c_i z_i - g_i (\frac{\partial \alpha_{i-1}}{\partial y})^2 z_i + \beta_i - z_{i-1} \\
&\quad - k_i (\frac{\partial \alpha_{i-1}}{\partial y})^2 \|\omega\|^2 z_i \\
\dot{V}_i &\leq \sum_{m=1}^{\rho-1} -c_m z_m^2 + \frac{1}{4g_m}\varepsilon_2^2 + \omega^T \tilde{\theta} z_1 - k_1 \omega^T \omega z_1^2 \\
&\quad - \sum_{j=2}^{\rho-1} \frac{\partial \alpha_{j-1}}{\partial y} z_j \omega^T \tilde{\theta} - \sum_{j=2}^{\rho-1}\left[\frac{\partial \alpha_{j-1}}{\partial y} z_j\right]^2 k_j \|\omega\|^2 \\
\Rightarrow \dot{V}_i &\leq \sum_{m=1}^{i} -c_m z_m^2 + \frac{1}{4g_m}\varepsilon_2^2 + \frac{1}{4k_m}\left\|\tilde{\theta}\right\|^2 + z_i z_{i+1}
\end{aligned}$$

Now for $i = \rho$, define z_ρ, $\dot{z}_\rho$, β_ρ, α_ρ as in the previous step, $z_{\rho+1} = 0$ and

$$V_\rho = \sum_{j=1}^{\rho} \frac{1}{2} z_j^2 \tag{12}$$

$$u = \frac{1}{b_m}\left[\alpha_\rho - b_m \upsilon_{m,\rho+1}\right] \quad \textbf{ISS control law}$$

$$\dot{V}_\rho \leq \sum_{m=1}^{\rho} -c_m z_m^2 + \frac{1}{4g_m}\varepsilon_2^2 + \frac{1}{4k_m}\left\|\tilde{\theta}\right\|^2 \tag{13}$$

This implies that the error dynamic z is bounded whenever $\tilde{\theta}$ and ε_2 are bounded. Hence, (12) is an iss-Lyapunov function candidate for the problem under consideration. Since z_1 and y^r are bounded, y is also bounded. This means that η is bounded. Following the argument presented in (Krstic *et al.*, 1995), it can be shown that λ is bounded which implies the boundedness of x.

4. EXTREMUM SEEKING RHC FORMULATION AND ANALYSIS

The formulation consists of two-phase design procedure as described below.

First Phase: At every time step t, the maximum value of the objective *(profit)* function (1) is obtained via an online set-point update law as follows.
Consider a Lyapunov function candidate

$$V_{sp} = \frac{1}{2}\left(\frac{\partial p(r, \hat{\theta}_1)}{\partial r}\right)^2 \tag{14}$$

where r denotes an optimal set point for the output y. Taking the time derivative of V_{sp}, we have

$$\dot{V}_{sp} = \frac{\partial p}{\partial r}\left[\frac{\partial^2 p}{\partial r^2}\dot{r} + \frac{\partial^2 p}{\partial r \partial \hat{\theta}_1}\dot{\hat{\theta}}_1\right].$$

Choosing the update law as

$$\dot{r} = -\left(\frac{\partial^2 p}{\partial r^2}\right)^{-1}\left[\frac{\partial p}{\partial r} - \frac{\partial^2 p}{\partial r \partial \hat{\theta}_1}\dot{\hat{\theta}}_1\right]$$

leads to

$$\dot{V}_{sp} \leq -\left(\frac{\partial p}{\partial r}\right)^2 \tag{15}$$

Equations (14) and (15) imply that r approaches the ($\hat{\theta}_1$-dependent) optimal set-point y^* as $t \to \infty$. To provide some richness condition on the set-point, we append it with a bounded dither signal $d(t)$ and define an approximate set-point $y^r(t) = r(t) + d(t)$. In general, $d(t)$ is chosen to contain at least n (n=no of unknown parameters) distinct frequencies, required for parameter convergence. Other specific details will be given later.

Second Phase: At this step, a finite horizon optimal control problem is solved subject to the system dynamics and terminal state inequality constraints at every time step with the estimated plant states $x(t)$ as initial condition. The goal of this phase is to minimize a given cost while ensuring that the system's output y tracks the reference setpoint y^r dictated by the first phase. Let us re-write (4) as

$$\dot{x} = Ax + e_\rho b_m u + \bar{F}(y, u)^T \theta \tag{16}$$

where $\bar{F}(y, u)^T = \left[\begin{bmatrix} 0_{\rho \times m} \\ I_m \end{bmatrix} u,\ -I y_n\right]$ and define

$$\mathcal{J} = z^T P z + \int_t^{t+T_p} z^T(\tau) C z(\tau) + u(\tau)^T R u(\tau) d\tau$$

The proposed ESRHC scheme is given by:

$$\min_{\mathrm{u}} \mathcal{J} \tag{17}$$

$$\text{s.t} \quad \dot{x} = Ax + e_\rho b_m u + \bar{F}(x_1, u)^T \bar{\theta} \tag{18}$$

$$x(t) = \hat{x}(t), \quad \bar{\theta} = \hat{\theta}(t) \tag{19}$$

$$u_{min} \leq u \leq u_{max} \tag{20}$$

$$V(t + T_p) \leq V^{iss}(t + T_p) \tag{21}$$

$$\begin{aligned}
z_1 &= x_1 - y^r, \quad y^r = r + d(t), \\
\alpha_1 &= -c_1 z_1 - g_1 z_1 + e_{m+1}^T \bar{\theta} x_1 \\
\alpha_{i-1} &= -(c_{i-1} + g_{i-1}) z_{i-1} - z_{i-2} + \dot{\alpha}_{i-2} + e_{m+i}^T \bar{\theta} x_1, \\
z_i &= x_i - \alpha_{i-1}, \quad i = 2 \dots \rho
\end{aligned}$$

where P and R are positive definite weighting matrices, T_p is the length of the prediction horizon, the function V is the value of the CLF resulting from the application of ESRHC and V^{iss} is the value of the CLF that results from the application of the iss controller. Constraint (21) guarantees that the states under the ESRHC are brought within the level set of the iss-controller at the end of the prediction horizon, thereby ensuring that the states under the ESRHC remain bounded. By (19) the optimization problem is initialized by the estimated state, and the unknown parameters in (17) and (18) are replaced by the estimated values. The optimizer computes the required control moves over the horizon. The input $u(t)$ is implemented on the plant at time t. An estimate of the unmeasured state $\hat{x}$ and the unknown parameters $\hat{\theta}(t)$ are obtained via an observer and a parameter update law respectively. The horizon is shifted forward and a new optimization problem is solved at the next time step $t+\delta$ with the new $x = \hat{x}(t+\delta)$ and $\bar{\theta} = \hat{\theta}(t+\delta)$. The control $u(t+\delta)$ is applied at time $t+\delta$ and the process is repeated. In general, it is assumed that the time step length δ can be chosen to be arbitrarily small.

4.1 Main Result

The stability and performance of the proposed scheme is demonstrated in the following. Consider the function

$$W(z(t)) = z^T(t+T_p)Pz(t+T_p) + \frac{1}{2}\int_t^{t+T_p} z(\sigma)^T C z(\sigma) d\sigma \qquad (22)$$

where $P = I$ and $z(.)$ is the error trajectory resulting from the ESRHC. This function is positive definite and it is radially unbounded if V is positive definite and radially unbounded.
For $\tau \in [t, t+\delta]$, eq. (22) becomes

$$W(z(\tau)) = z^T(\tau+T_p)Pz(\tau+T_p) + \frac{1}{2}\left[\int_\tau^{t+T_p} z(\sigma)^T C z(\sigma) d\sigma + \int_{t+T_p}^{\tau+T_p} z(\sigma)^T C z(\sigma) d\sigma\right] \qquad (23)$$

From the iss-controller design section, we have

$$\dot{V}^{iss} \leq -z^T C z + \frac{1}{4G}\varepsilon_2^2 + \omega^T\tilde{\theta} z_1 - \sum_{j=2}^{\rho} \frac{\partial \alpha_{j-1}}{\partial y} z_j \omega^T \tilde{\theta} \qquad (24)$$

where $C = c_i I$ $i = 1 \ldots \rho$ and $G = \sum_{m=1}^{\rho} g_m$
Integrating (24) over $[t+T_p, \tau+T_p]$, we obtain

$$\frac{1}{2}\int_{t+T_p}^{\tau+T_p} z(\sigma)^T C z(\sigma) d\sigma \leq V(t+T_p) - V(\tau+T_p) + \int_{t+T_p}^{\tau+T_p} \Upsilon(\sigma) d\sigma$$

where

$$\Upsilon = -\frac{1}{2} z^T C z + \frac{1}{4G}\varepsilon_2^2 + \omega^T\tilde{\theta} z_1 - \sum_{j=2}^{\rho} \frac{\partial \alpha_{j-1}}{\partial y} z_j \omega^T \tilde{\theta}$$

Hence, equation (23) becomes

$$W(z(\tau)) \leq W(z(t)) + \int_{t+T_p}^{\tau+T_p} \Upsilon(\sigma) d\sigma$$

dividing both sides by $\tau - t$ and taking the lim sup as τ goes to t results in

$$\dot{W}(z(t)) \leq \Upsilon(t+T_p) \qquad (25)$$

Close-loop Analysis: Re-write (16) as

$$\dot{x} = Ax + Bu(x)^{RHC} - ya, \quad x(t) = x(t) \qquad (26)$$

$$\dot{\hat{x}} = A\hat{x} + Bu(\hat{x})^{RHC} - ya, \quad x(t) = \hat{x}(t) \qquad (27)$$

where $B = f_1\bar{b} + e_\rho b_m$, $f_1 = \begin{bmatrix} 0_{\rho\times m} & I_m \end{bmatrix}^T$, $\bar{b} = \begin{bmatrix} \bar{b}_{m-1} & \ldots & \bar{b}_0 \end{bmatrix}^T$ and $a = \begin{bmatrix} a_{n-1} & \ldots & a_0 \end{bmatrix}^T$. The state error dynamic $\tilde{x} = x - \hat{x}$ between (26) and (27) is

$$\tilde{x}(t) = A\tilde{x} + Bu(\tilde{x})^{RHC}, \quad \tilde{x}(t) = x(t) - \hat{x}(t) \qquad (28)$$

and the solution of (28) for $\tau \in [t, t+T_p]$ is

$$\tilde{x}(\tau) = e^{A(\tau-t)}\tilde{x}(t) + \int_t^\tau e^{A(\tau-s)} Bu(\tilde{x}(s))^{RHC} ds \qquad (29)$$

The solution u^{RHC} resulting from the ESRHC scheme has been shown to be piecewise affine (Bemporad *et al.*, 2000). *i.e.*

$$u(t) = k_i x(t) + m_i \qquad (30)$$

for $x(t) \in C_i \triangleq [x : H_i s \leq s_i]$ $i = 1, \ldots, s$ where $\bigcup_{i=1}^{s} C_i$ is the set of states for which a feasible solution to the finite horizon optimal control problem (second phase) exists.
Therefore,

$$u(\tilde{x}(t)) = k_i\tilde{x}(t) + \epsilon_i \leq \bar{k}\tilde{x}(t) + \bar{\epsilon}$$

where $\bar{k} := \max_i |k_i|$, $\bar{\epsilon} := \max_i |\epsilon_i|$. When $\|x\| \geq 1$, $|u| \leq (\bar{k} + \bar{\varepsilon})\|x\|$. Also, when $0 \leq \|x\| < 1$, and $\bar{\epsilon}$ sufficiently small, there exists $\nu > 0$ such that $|u| \leq (\bar{k} + \nu)\|x\|$.
So, without loss of generality, it is assumed that

$$|u(\tilde{x}(t))| \leq L\|\tilde{x}(t)\|, \quad L := \max\left(\bar{k} + \bar{\varepsilon},\ \bar{k} + \nu\right)$$

Then, using Bellman Gronwall Lemma, (29) results in

$$\|\tilde{x}(\tau)\| \leq e^{(\tau-t)}\|\tilde{x}(t)\| + \int_t^\tau e^{(\tau-s)}\varpi\|\tilde{x}(s)\| ds \leq \varrho\|\tilde{x}(t)\| \qquad (31)$$

where $\varrho = \exp\left(-\varpi + \varpi e^{(\tau-t)}\right)$, $\|A\| = 1$ and $L\|B\| = \varpi$.
Parameter Estimation: We define the predicted state, $\hat{x}_a$ as

$$\hat{x}_a = \zeta + \Omega^T\hat{\Theta}$$

Considering (5) and (6), the predicted state dynamic is given as

$$\dot{\hat{x}}_a = A_0\zeta + Ly + A_0\Omega^T\hat{\Theta} + F(y,u)^T\hat{\Theta} \quad (32)$$

Noting that $F(y,u)^T\Theta = \bar{F}(y,u)^T\theta + e_\rho b_m u$ and b_m is assumed known, the prediction error $\varepsilon_a = x - \hat{x}_a$ dynamic results in

$$\dot{\varepsilon}_a = \bar{F}(y,u)^T\tilde{\theta} + A_0\varepsilon_a \quad (33)$$

Consider a Lyapunov function

$$\begin{aligned} V_1(t) =& W(t) + \frac{1}{G}\varepsilon^T(t)P_0\varepsilon(t) + \frac{1}{2}\varepsilon_a^T(t)Q_0\varepsilon_a(t) \\ &+ \frac{1}{2}\tilde{\theta}^T(t)\Gamma^{-1}\tilde{\theta}(t) \end{aligned} \quad (34)$$

where $\Gamma = \Gamma^T > 0$, Q_0 and P_0 are real symmetric positive definite matrices that satisfy $P_0A_0 + A_0^TP_0 = -\varrho^2 I$ and $Q_0A_0 + A_0^TQ_0 = -I$ respectively. Taking the time derivative of V along the solutions of (8) and (33) we have

$$\begin{aligned} \dot{V}_1(t) \leq& \Upsilon(t+T_p) - \frac{\varrho^2}{G}\varepsilon(t)^T\varepsilon(t) - \dot{\hat{\theta}}^T(t)\Gamma^{-1}\tilde{\theta}(t) \\ &- \frac{1}{2}\varepsilon_a^T(t)\varepsilon_a(t) + \varepsilon_a^T(t)Q_0\bar{F}(y,u)^T\tilde{\theta}(t) \end{aligned}$$

Considering the fact that there is no adaptation along the prediction horizon, we have $\tilde{\theta}(t+T_p) = \tilde{\theta}(t)$. Moreover, It is deduced from (31) that $\|\varepsilon(t+Tp)\| \leq \varrho\|\varepsilon(t)\|$, since the plant is initialized by the state estimates obtained via (7) at the beginning of the prediction horizon *i.e.* $\tilde{x}(t) = \varepsilon(t)$. We therefore obtain

$$\begin{aligned} \dot{V}_1(t) \leq& -\frac{1}{2}z(t+T_p)^TCz(t+T_p) - \frac{3\varrho^2}{4G}\varepsilon(t)^T\varepsilon(t) \\ &-\varepsilon_a^T(t)K\varepsilon_a(t) + \left[\psi - \dot{\hat{\theta}}^T\Gamma^{-1}\right]\tilde{\theta} \end{aligned}$$

where $\psi \triangleq \varepsilon_a^T\bar{F}(y,u)^T + z_1(t+T_p)\omega^T - \sum_{j=2}^{\rho}\frac{\partial\alpha_{j-1}}{\partial y}\omega^T z_j(t+T_p).$

The parameter adaptation rule is selected to ensure that $\left[\psi - \dot{\hat{\theta}}^T\Gamma^{-1}\right]\tilde{\theta} \leq 0$ and that the parameter estimates remain in some given set. This is achieved by using standard parameter projection law. Refer to (Krstic *et al.*, 1995) for more details. Defining $\Psi = \Gamma\psi^T$, the update law is given by

$$\dot{\hat{\theta}} = Proj\left\{\hat{\theta}, \Psi\right\} \quad (35)$$

The properties of the projection operator ensures that the parameters are bounded and that

$$\begin{aligned} \dot{V}_1(t) \leq& -\frac{1}{2}z(t+T_p)^TCz(t+T_p) \\ &- \frac{3}{4G}\varrho^2\varepsilon(t)^T\varepsilon(t) - \varepsilon_a^T(t)K\varepsilon_a(t) \end{aligned} \quad (36)$$

From (36), it is concluded that z, ε, ε_a and $\tilde{\theta}$ are uniformly bounded and that z, ε and ε_a converge to the origin asymptotically.

Parameter Convergence: From the previous subsection, it is established that ε_a converges to zero, hence, $\int_0^\infty \dot{\varepsilon}_a(\sigma)d\sigma = -\varepsilon_a(0)$ exists and is finite. Also, from (33), it is known that $\dot{\varepsilon}_a$ is a function of bounded signals $y, u, \tilde{\theta}$ and ε_a which means that $\ddot{\varepsilon}_a$ is bounded. Hence, $\dot{\varepsilon}_a$, is uniformly continuous. By Barbalat's lemma (Krstic *et al.*, 1995), it is concluded that $\dot{\varepsilon}_a \to 0$ as $t \to \infty$. This implies that $\lim_{t\to\infty}\bar{F}(y,u)^T\tilde{\theta} = 0$ or $\lim_{t\to\infty}\tilde{\theta}^T\bar{F}(y,u)\bar{F}(y,u)^T\tilde{\theta} = 0$. From the argument presented in (Adetola *et al.*, 2004), we conclude that if the dither signal $d(t)$ satisfies a richness condition:

$$\frac{1}{T_0}\int_t^{t+T_0}\bar{F}(\tau)\bar{F}(\tau)^Td\tau \geq c_0 I, \quad c_0 > 0 \quad (37)$$

then the parameter error $\tilde{\theta}$ converges to zero asymptotically.

Lemma 1. Consider the adaptive system, eq.(4), with receding horizon controller eqs.(17)-(21), the adaptive laws (35), the state observer (7) and state estimation dynamics eq.(33). If the dither signal $d(t)$ is chosen such that the PE condition (37) is satisfied, then the parameter estimation error $\tilde{\theta}$ converges to zero asymptotically.

The proof is similar to the one presented in ((Adetola *et al.*, 2004), Lemma 1)

Theorem 2. Consider the objective function (1) subject to the system dynamics (4), and satisfying the given assumptions. If the dither signal $d(t)$ satisfies the persistence of excitation condition (37), then the ESRHC (17)-(21), the state observer (7) and the parameter estimation scheme (33) and (35) solves the extremum seeking problem.

Proof: It follows from the stability analysis that $z_1 \triangleq y - y^r \to 0$ as $t \to \infty$. If $d(t)$ is designed to satisfy the PE condition (37), then it is concluded by Lemma 1 that $\lim_{t\to\infty}\hat{\theta} = \theta$. Hence y converges to a neighborhood of the optimal set point r^* whose size depends on $d(t)$. □

Lemma 3. The tracking error z of the closed loop dynamical system is bounded by

$$\|z\|_p \leq 2\sqrt{\gamma V_1(0)}, \;\; p = 2 \text{ or } \infty, \quad \gamma = \frac{1}{\lambda_{\min}(C)}$$

Proof: $\mathcal{L}_2$ Performance:

$$\|z\|_2^2 = \int_0^{T_p} z^T(\tau)z(\tau)d\tau + \int_{T_p}^{\infty} z^T(\tau)z(\tau)d\tau$$

From (36) we know that $z(t+T_p)^Tz(t+T_p) \leq -2\gamma\dot{V}_1$. Since V_1 is non-increasing,

$$\int_{T_p}^{\infty} z^T(\tau)z(\tau)d\tau \leq 2\gamma\left[V_1(T_p) - V_1(\infty)\right] \leq 2\gamma V_1(0) \quad (38)$$

Also from (22), we have

$$\int_0^{T_p} z^T(\tau)z(\tau)d\tau \le 2\gamma \left[W(z(0)) - z^T(T_p)z(T_p)\right] \quad (39)$$

Equations (38) and (39), lead to

$$\|z\|_2^2 \le 2\gamma \left[W(z(0)) + V_1(0)\right] \quad (40)$$

Noting from (34), that $W(0) \le V_1(0)$ concludes the $\mathcal{L}_2$ norm proof. The proof for $\mathcal{L}_\infty$ performance follows from (34) and the fact that V_1 is non-increasing. □

Remark 4. From above, it is clear that the transient performance depends on $\tilde{\theta}$, $\varepsilon(0)$, $\varepsilon_a(0)$, $z(0)$, G, P_0 and Γ. We can set $z_1(0)$ to zero by setting $\hat{y}^r(0) = y(0)$ and use the other tuning functions to systematically reduce the bounds.

5. SIMULATION EXAMPLE

Consider the following linearized model for a non-isothermal CSTR where an exothermic reaction $\mathbf{A} \rightarrow \mathbf{B}$ is carried out. The dynamic of the reactor is given as $\dot{x} = Ax + Bu$, $y = [1,\ 0]x$ where x is a vector of the reactor temperature and concentration, and u is the coolant flow rate. The matrix A and B are as follows:

$$A = \begin{bmatrix} \dfrac{-\Delta H k(T^*)}{\rho C_p} & a_{12} \\ -\dfrac{F}{V} - k(T^*) & -\dfrac{E}{RT^{*2}} k(T^*) C_A^* \end{bmatrix}$$

$$B = \begin{bmatrix} -2.1 \times 10^5 \dfrac{T^* - T_{cin}}{V \rho C_p} \\ 0 \end{bmatrix}$$

The expression for the reaction rate is given by $k(T^*) = K_o e^{-(E/RT^*)}$, where K_o is the kinetic constant of the reaction and E/R is the activation energy.

$$a_{12} = -\frac{F}{V} - \frac{UA}{V\rho C_p} - \Delta H \frac{E/(RT^{*2})}{\rho C_p} k(T^*) C_A^*$$

We assume that $\theta_1 = k(T^*)$ and $\theta_2 = E/RT^*$ are not known. The objective is to adaptively stabilize the system to the unknown set-point $(T^*,\ C_A^*)$ that guarantees 90 percent conversion of reactant A. The specific parameters and operating conditions used for the simulation are F = 1m^3/min, V = 1m^3, T_{cin} = 365K, C_{A0} = 2.0kmole/m^3, C_p = 1cal/(g^{o}C), $\rho = 10^6$g/m^3, ΔH_{rxn} = -130×10^6 cal/(kmole). The coolant flow is restricted to $13 \le u \le 17$ m^3/min. $d(t) = (0.5 \sin 9t + \sin 7t)\, e^{-0.5t} + 0.7 \cos 6t\, e^{-0.3t}$. The true values of the unknown parameters are $\theta_1 = 9\text{min}^{-1}$ and $\theta_2 = 21.07$. The target optimal values are $C_A^* = 0.2$ kmol/m^3 and $T^* = 395.27$K.

The simulation result (Figure 1) shows that the adaptive ESRHC stabilizes the system states to

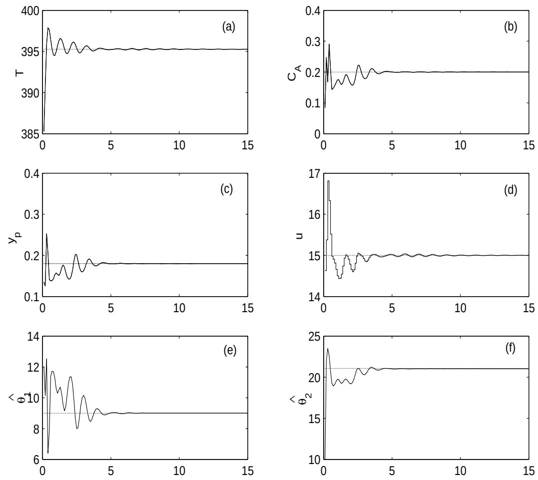

Fig. 1. Reactor Closed loop Trajectories

the required steady-state values and the economic function reaches the optimum in a reasonable time. The parameter estimates converge to the true values. The control input also converges to the appropriate steady-state value and satisfies the given bounds.

6. CONCLUSION

A method is developed to solve an extremum seeking control problem for linear uncertain plant. The technique employs an iss-control Lyapunov function to ensure stability and performance. It is shown that the proposed scheme is able to drive the system states to unknown desired states that optimize the value of an objective function provided an excitation condition is satisfied.

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