ON CONTROLLABILITY OF LINEAR SYSTEMS WITH POSITIVE CONTROL

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Abstract: Necessary conditions for the controllability on linear systems with positive control are presented. The real case is analyzed and the Jordan form representation is employed. The minimum number of necessary controls to obtain controllability is remarked.

Keywords: Controllability, positive control, blocks of Jordan.

1. INTRODUCTION

In the last years we can observe a great interest in the study of controllability of linear systems with positive control (PCS). The restriction in sign for the input is natural in many applications. In (Saperstone and Yorke, 1971), appears the following mechanical problem: can the pendulum be take to the stable equilibrium point by means of the applications of a finite continuous force in a single direction? One demonstrates that the system is controllable with a control to climb positive. In (Leyva and Carrillo, 2004) appears an annotated and positive feedback that stabilizes this problem. (Brammer, 1972) gives a characterization of the controllability on linear systems with positive control in terms of the pair (A, B). (Frias, *et al.*, 2004) give a characterization of the controllability on a class of linear systems with positive control, only in terms of the matrix B.

The stabilization problem is similar to the one of controllability. In (Smirnov, 1999) the nonlinear systems are characterized that are locally stabilizable by means of positive controls while in (Korobov, 1979) necessary and sufficient conditions are establish to determine the local controllability with positive control in linear systems where the control input is modeled in general form. (Leyva and Carrillo, 2004) consider the case of linear PCS with complex eigenvalues, to design positive a global stabilizer.

In this work, the results obtained in (Frias, *et al.*, 2004) are applied to more general linear systems.

2. PROBLEM STATEMENT

Consider the linear system

$$\dot{x} = Ax + Bu \tag{1}$$

with $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and the control parameter u restricted to take values in the cone $U = \mathbb{R}^m_+$. Due to this restriction of no-negativity, we will say that the control is positive.

In this paper we are interested in finding necessary conditions to ensure the controllability of the system (1) with positive control, for the case when the matrix A has only real eigenvalues. Firstly, we will find necessary and sufficient conditions to controllability with positive control, for the next three cases:

(i) a repeated real eigenvalue in a diagonal form

$$A = \begin{pmatrix} \lambda & & & \\ & \lambda & & \\ & & \ddots & \\ & & & \lambda \end{pmatrix} \tag{2}$$

(ii) a repeated real eigenvalue

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$$A = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix} \qquad (3)$$

(iii) different real eigenvalues

$$A = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \qquad (4)$$

For each case, the minimum number of control necessary to ensure the controllability is remarked. Next, we suppose the matrix A in Jordan form,

$$A = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{pmatrix},$$

where each block J_i is giving by (2),(3) or (4), and then we establish our main result putting together the three cases analyzed.

3. PRELIMINARY RESULTS

We present the results obtained in (Frias *et al.*, 2004) which give necessary and sufficient conditions on the matrix B, to assure the controllability with positive control, for the cases in that the matrix A is of the form (2), (3) y (4). Besides, a special emphasis is remarked in those cases where the controllability is possible with a minimum of controls.

Definition 1. The system (1) is called controllable if, for each $x_1, x_2 \in \mathbb{R}^n$ there exists a bounded admissible control $u(t) \in U$, definded in some interval $0 \leq t \leq t_1$, which steers x_1 to x_2.

In each one of the demonstrations was used the next result due to (Brammer, 1972).

Theorem 1. The system (1) is PCS if and only if

(a) The controllability matrix

$$\mathcal{C} = (B\, AB\, \cdots\, A^{n-1}B)$$

has rank n,

(b) There exists not real eigenvector v of A^T satisfying the inequality

$$v \cdot Bu \leq 0$$

for all $u \in \mathbb{R}^m_+$.

An equivalent way to express (b) is: (b') For each real eigenvalue v of A^T, there exist $u_1, u_2 \in U$ such that $(v \cdot Bu_1)\,(v \cdot Bu_2) < 0$.

The next three propositions were proved in (Frias, *et al*,, 2004).

Proposition 1. (i) The control system $\dot{x} = Ax + Bu$, with A of the form (2), is positively controllable with $n+1$ controls if and only if B has rank n and exists a column b_k of B such that

$$b_k = \sum_{j=1 j\neq k}^{n+1} c_j b_j, \quad \text{with } c_j < 0$$

(ii) The control system $\dot{x} = Ax + Bu$, with A of the form (2), is not positively controllable with n or less controls.

Proposition 2. The control system $\dot{x} = Ax+Bu$, with A of the form (3), is controllable with positive control if and only if in the last row of B are two entries of opposite signs.

In this case two controls are only necessary to obtain the controllability of the system (1)

Proposition 3. The control system $\dot{x} = Ax+Bu$, with A of the form (4), is controllable with positive control if and only if in each row of B there are two entries of opposite signs.

There exist in this case systems that can be controlled from two controls, until $2n$ controls.

The next proposition characterize a PCS for the case in that the matrix A is in the form (2), but with any restriction on the number of controls.

Proposition 4. The control system $\dot{x} = Ax+Bu$, with A of the form (2), is positively controllable if and only if each vector $v \in \mathbb{R}^n$ can be written as a positive linear combination of column vectors of B

Proof:
If $A = \lambda I$, where $I \in \mathbb{R}^{n\times n}$ is the identity matrix, observe that

$$A^k B = \lambda^k B,$$

then,

$$\mathcal{C} = (B\, AB\, \cdots\, A^{n-1}B) = (B\, \lambda B\, \cdots\, \lambda^{n-1}B)$$

therefore $rank(\mathcal{C}) = rank(B)$. Besides, in this case, each vector $v \in \mathbb{R}^n$ is an eigenvalue of A^T.
We suppose that the system is positive controllable. Then, $rank(B) = n$, and each vector $v \in \mathbb{R}^n$ can be written as a linear combination of column vectors of B. We must to show that the mentioned linear combination is positive. Consider

$$W = \{\, w \in \mathbb{R}^n \mid w = \sum_{i=1}^{m} c_i b_i,\ c_i \geq 0,\ \text{and } ||w|| = 1 \,\}$$

and, for $v \in \mathbb{R}^n$, with $v \neq 0$, we define $f_v : W \to \mathcal{R}$, giving by $f_v(w) = \frac{v \cdot w}{||v||}$. It is not difficult to show that there exists $w_0 \in W$ such that $f_v(w_0)$ is maximum. If $f_v(w_0) = 1$ we have finished, because in this case, $v = \alpha w_0$ where $\alpha > 0$, and then $\frac{v}{||v||} \in W$. Suppose that $f_v(w_0) < 1$. Let us define

$$v_1 = v - Proy_{w_0} v,$$

where $Proy_{w_0} v$ is the vector that is the projection of v on the vector w_0. Observe that $v_1 \cdot w_0 = 0$. It follows from the theorem 1, that there exists $u \in U$ such that $v_1 \cdot (Bu) > 0$, then, there exists a column of B, b_0, such that $v_1 \cdot b_0 > 0$; if such b_0 there not exists, then $v_1 \cdot (Bu)$ could not be positive. Now, we define

$$v_2 = c_1 b_0 + c_2 w_0, \tag{5}$$

such that

$$(v - v_2) \cdot b_0 = 0 \tag{6}$$
$$(v - v_2) \cdot w_0 = 0 \tag{7}$$

That is, the vector v_2 is the projection of v on the plane generated by b_0 and w_0. Such plane exists, because b_0 and w_0 are not parallel (remember that $v_1 \cdot w_0 = 0$ and $v_1 \cdot b_0 > 0$). Now then, substituting (5) in (6) and (7), we obtain the next system with unknown variables c_1, c_2:

$$\begin{aligned} ||b_0||^2 c_1 + (b_0 \cdot w_0) c_2 &= v \cdot b_0 \\ (b_0 \cdot w_0) c_1 + c_2 &= v \cdot w_0 \end{aligned}$$

which solution is giving by

$$\begin{aligned} c_1 &= \frac{v \cdot b_0 - (v \cdot w_0)(b_0 \cdot w_0)}{||b_0||^2 - (b_0 \cdot w_0)^2} \\ c_2 &= \frac{||b_0||^2 (v \cdot w_0) - (v \cdot b_0)(b_0 \cdot w_0)}{||b_0||^2 - (b_0 \cdot w_0)^2} \end{aligned}$$

Observe that

$$\begin{aligned} ||b_0||^2 - (b_0 \cdot w_0)^2 &= ||b_0||^2 - ||b_0||^2 \cos^2 \theta_{b_0 w_0} \\ &= ||b_0||^2 \left(1 - \cos^2 \theta_{b_0 w_0}\right) \\ &= ||b_0||^2 \sin^2 \theta_{b_0 w_0}, \end{aligned}$$

where $\theta_{b_0 w_0}$ is the angle between b_0 and w_0, which is different of zero, then $||b_0||^2 - (b_0 \cdot w_0)^2 > 0$. Now then, observe that $v \cdot w_0 = ||Proy_{w_0} v||$ and that $(Proy_{w_0} v) \cdot b_0 = ||Proy_{w_0} v||(b_0 \cdot w_0)$, then substituting $v = v_1 + Proy_{w_0} v$ in the numerator of c_1, we obtain

$$\begin{aligned} v \cdot b_0 - (v \cdot w_0)(b_0 \cdot w_0) &= v_1 \cdot b_0 + (Proy_{w_0} v) \cdot b_0 - \\ &\quad ||Proy_{w_0} v||(b_0 \cdot w_0) \\ &= v_1 \cdot b_0 > 0, \end{aligned}$$

therefore, $c_1 = \frac{v_1 \cdot b_0}{||b_0||^2 \sin^2 \theta_{b_0 w_0}} > 0$. The sign of c_2 depends of the vector b_0. If $c_2 > 0$, we define

$$\tilde{w} = \frac{v_2}{||v_2||} \in W,$$

and, by construction, $f_v(\tilde{w}) > f_v(w_0)$, which is a contradiction, because we have suppose that $f_v(w_0)$ is maximum. If $c_2 \leq 0$, we define

$$\tilde{w} = \frac{b_0}{||b_0||} \in W.$$

Observe that $w_0 \cdot \left(\frac{b_0}{||b_0||}\right) < 1$, then, $c_2 \leq 0$ imply that

$$\begin{aligned} &||b_0||^2 (v \cdot w_0) \leq (v \cdot b_0)\,(b_0 \cdot w_0) \quad \Leftrightarrow \\ &\quad (\frac{v}{||v||} \cdot w_0) \leq \left(\frac{v}{||v||} \cdot \frac{b_0}{||b_0||}\right) \left(w_0 \cdot \frac{b_0}{||b_0||}\right) \quad \Leftrightarrow \\ &\quad (\frac{v}{||v||} \cdot w_0) < \left(\frac{v}{||v||} \cdot \frac{b_0}{||b_0||}\right) \\ &\qquad f_v(w_0) < f_v(\tilde{w}), \end{aligned}$$

which is a contradiction.

Now, we assume that each vector $v \in \mathbb{R}^n$ can be written as a positive linear combination of column vectors of B. Then, $rank(B) = n$, therefore, $rank(\mathcal{C}) = n$. Let $v \in \mathbb{R}^n$ be an eigenvector of A^T, we must to prove that there exist $u_1, u_2 \in U$, such that $(v \cdot (Bu_1))\,(v \cdot (Bu_2)) < 0$. Now then, by hypothesis, there exist $c_i \geq 0$ and $d_i \geq 0$, for $i = 1, \ldots, m$, such that $v = \sum_{i=1}^m c_i b_i$ and $-v = \sum_{i=1}^m d_i b_i$, where b_i are column vectors of B. If we made $u_1 = (c_1, \ldots, c_m)^T$ and $u_2 = (d_1, \ldots, d_m)^T$, then $(v \cdot (Bu_1)) = ||v||^2$ and $(v \cdot (Bu_2)) = -||v||^2$.

Corollary 1. The control system $\dot{x} = Ax + Bu$, with A of the form (2), is positively controllable if and only if B has rank n and each column vector of B can be written as a negative linear combination of column vectors of B.

Proof:
Suppose that the system is positively controllable. $Rank(\mathcal{C}) = n$ imply that $rank(B) = n$. Let b_i be a column vector of B, then, by the proposition 4, $-b_i$ can be written as a positive linear combination of column vectors of B.
Now, let us suppose that B has rank n, and that each column vector of B can be written as a negative linear combination of column vectors of B. It follows that $Rank(\mathcal{C}) = n$. Consider $v \in \mathbb{R}^n$, then, by hypothesis, v can be written as a linear combination of column vectors of B. Change those columns of B, that correspond to terms with negative coefficients, by their respective negative linear combination. In this way, v can be written as a positive linear combination of column vectors of B, then, by the proposition (4), we conclude that the system is positively controllable.

4. MAIN RESULT

Consider the linear system

$$\dot{x} = Ax + Bu$$

where the matrix

$$A = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{pmatrix},$$

is in Jordan form, where each block $J_i \in \mathbb{R}^{n_i \times n_i}$ is in the form (2),(3) or (4), and

$$B = \begin{pmatrix} B_1 \\ \vdots \\ B_k \end{pmatrix},$$

where $B_i \in \mathbb{R}^{n_i \times m}$. In a natural way, we can write $x = (x_1, \ldots, x_k)^T$, where $x_i \in \mathbb{R}^{n_i}$, and $\sum_{i=1}^k n_i = n$. Then

$$\dot{x} = Ax + Bu \;\;\Leftrightarrow\;\; \begin{cases} \dot{x}_1 = J_1 x_1 + B_1 u \\ \vdots \\ \dot{x}_k = J_k x_k + B_k u \end{cases}$$

Lemma 1. Consider the control system $\dot{x} = Jx + Bu$ with J of the form (4) and $x \in \mathbb{R}^s$. If the matrix $B = \{b_1, \ldots, b_m\}$ is such that b_j has r entries different of zero, with $0 < r \leq s$, then the collection $\{b_j, Ab_j, \ldots, A^{r-1}b_j\}$ is linearly independent.

Proof:
Let us suppose that

$$c_1 b_j + c_2 A b_j + \cdots + c_r A^{r-1} b_j = 0.$$

We must to prove that $c_i = 0$ for each $i = 1, \ldots, r$. Consider the polynomial

$$f(x) = c_1 + c_2 x + \cdots + c_r x^{r-1},$$

which is of degree $r-1$. Without loss of generality, suppose that $b_j = (b_{1j}, \ldots, b_{rj}, 0, \ldots, 0)^T$. Note that

$$A^k b_j = (\lambda_1^k b_{1j}, \ldots, \lambda_r^k b_{rj}, 0, \ldots, 0)^T,$$

so then

$$\begin{aligned} & \sum_{k=1}^{r} c_k A^{k-1} b_j \;=\; 0 \\ \Leftrightarrow \quad & \sum_{k=1}^{r} c_k \lambda_i^{k-1} b_{ij} \;=\; 0, \text{ for } i = 1, \ldots, r \\ \Leftrightarrow \quad & b_{ij} \left(\sum_{k=1}^{n} c_k \lambda_i^{k-1} \right) \;=\; 0, \text{ for } i = 1, \ldots, r \\ \Leftrightarrow \quad & b_{ij} f(\lambda_i) \;=\; 0, \text{ for } i = 1, \ldots, r, \end{aligned}$$

what implies that $f(x)$ has r different solutions, but $f(x)$ is a $r-1$ degree polynomial, therefore, in virtue of the fundamental theorem of algebra, $f(x) \equiv 0$, consequently,

$$c_1 = c_2 = \cdots = c_n = 0.$$

Theorem 2. If the system $\dot{x} = Ax + Bu$ is positively controllable then the system $\dot{x}_i = J_i x_i + B_i u$ is positively controllable, for $i = 1, \ldots, k$.

Proof:
Observe that

$$\begin{aligned} A^r B &= \begin{pmatrix} J_1^r & & \\ & \ddots & \\ & & J_k^r \end{pmatrix} \begin{pmatrix} B_1 \\ \vdots \\ B_k \end{pmatrix} \\ &= \begin{pmatrix} J_1^r B_1 \\ \vdots \\ J_k^r B_k \end{pmatrix}, \end{aligned}$$

then,

$$\begin{aligned} \mathcal{C} &= (B \; AB \; \cdots \; A^{n-1}B) \\ &= \begin{pmatrix} B_1 & J_1 B_1 & \cdots & J_1^{n_1-1} B_1 & \cdots & J_1^{n-1} B_1 \\ & & \vdots & & & \\ B_i & J_i B_i & \cdots & J_i^{n_i-1} B_i & \cdots & J_i^{n-1} B_i \\ & & \vdots & & & \\ B_k & J_k B_k & \cdots & J_k^{n_k-1} B_k & \cdots & J_k^{n-1} B_k \end{pmatrix}. \end{aligned}$$

We define

$$\mathcal{C}_i = (B_i \; J_i B_i \; \cdots \; J_i^{n_i-1} B_i \; \cdots \; J_i^{n-1} B_i),$$

then, $rank(\mathcal{C}_i) \leq n_i$, for each $i = 1, \ldots, k$, but,

$$rank(\mathcal{C}) = \sum_{i=1}^{k} rank(\mathcal{C}_i) = n,$$

therefore, $rank(\mathcal{C}_i) = n_i$, for each $i = 1, \ldots, k$. Now, let us define

$$\tilde{\mathcal{C}}_i = (B_i \; J_i B_i \; \cdots \; J_i^{n_i-1} B_i).$$

We are going to prove that $rank(\tilde{\mathcal{C}}_i) = n_i$ by cases.
(i). Suppose that $J_i = \lambda_i I$, then

$$n_i = rank(\mathcal{C}_i) = rank(B_i) = rank(\tilde{\mathcal{C}}_i).$$

(ii). Consider $J_i = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix} = \lambda I + N$, where N is a nilpotent matrix such that $N^r = 0$ if $r \geq n_i$. Observe that

$$J_i^r B_i = (\lambda I + N)^r B_i$$

$$= \sum_{j=0}^{r} \mathrm{C}_j^r \lambda^{r-j} N^j B_i$$
$$= \sum_{j=0}^{r} c_j N^j B_i,$$

where $c_j = \mathrm{C}_j^r \lambda^{r-j} \neq 0$, that is, each column vector in $J_i^r B_i$ can be written as a linear combination of column vectors of $(B_i\, NB_i\, \cdots\, N^r B_i)$. If $r \geq n_i$, then each column vector in $J_i^r B_i$ can be written as a linear combination of column vectors of $(B_i\, NB_i\, \cdots\, N^{n_i-1} B_i)$. We know that $rank(\mathcal{C}_i) = n_i$, then there exist n_i linearly independent column vectors in the matrix $(B_i\, NB_i\, \cdots\, N^{n_i-1} B_i)$, that is, $rank(\tilde{\mathcal{C}}_i) = n_i$.
(iii) Finally, consider the case when J_i has n_i different real eigenvalues. Let $v \in \mathbb{R}^{n_i}$ be a column vector of B_i with r entries different of zero, with $0 < r \leq n_i$. By the lemma 1, $\{v, J_i v, \ldots, J_i^{r-1} v\}$ are linearly independent, then each vector $J_i^s v$ can be written as a linear combination of them, for $s \geq n_i$. But we know that $rank(\mathcal{C}_i) = rank(B_i\, J_i B_i\, \cdots\, J_i^{n_i-1} B_i\, \cdots\, J_i^{n-1} B_i) = n_i$, then each column vector in $(J_i^{n_i} B_i\, \cdots\, J_i^{n-1} B_i)$ is a linear combination of column vectors in $\tilde{\mathcal{C}}_i = (B_i\, J_i B_i\, \cdots\, J_i^{n_i-1} B_i)$, it follows then that $rank(\tilde{\mathcal{C}}_i) = n_i$. This finish the proof.

The conditions are not sufficient. Consider the next system, that is positively controllable by blocks, but that not is PCS.

$$\dot{x} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} x + \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} u$$

We have the blocks,

$$\dot{x}_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x_1 + \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix} u,$$

which is positively controllable by the proposition 2, and

$$\dot{x}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x_2 + \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix} u,$$

which is positively controllable by the proposition 1, but for the whole system the controllability matrix has rank 3.

5. CONCLUSIONS

Necessary conditions for positively controllable systems have been found. Brammer has establish an important characterization in terms of the pair (A, B), while in this document have been found necessary conditions only in terms of the matrix B. The principal result can be used as a very simple negative criteria for positively controllable systems. In addition, the characterization of the particular Jordan blocks, allows to establish the number minimum of necessary controls to obtain the controllability of the system.

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