SPECTRAL FACTORIZATION OF ND POLYNOMIALS

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Abstract: In this paper recent new approaches to the representation of a positive polynomials as a sum of squares are used in order to compute spectral factorizations of nonnegative multivariable polynomials. In principle this problem is solved due to the positive solution of Hilberts 17th problem by Artin. Unfortunately Artins result is not constructive and the denominator polynomials arising have no special structure which can be used to compute stable factorizations. In this paper a recent result of Demanze will be used to compute spectral factorization with nD Hurwitz stable factors. The approach is computationally feasible and the decomposition is computed using recently developed methods for semidefinite programming. We are able to reduce the factorization problem to a feasibility problem for a linear matrix inequality.

Keywords: multidimensional systems, spectral factorization

1. INTRODUCTION

Spectral factorizations for one variable polynomials and rational functions are a very important tool in system theory in order to solve important problems from automatic control and communications. In this paper we will show for a specific class of multivariable polynomials how a stable spectral factorization can be computed. Given a polynomial $p(\omega)$ depending an the vector $\omega \in \mathbb{R}^n$ which is globally positive then it is known since Hilbert that it can be represented as a sum of squares (sos) only in very specific cases. In 1927 Artin proved Hilberts 17th problem which states that every nonnegative polynomial in n variables is a sum of squares of rational functions. Unfortunately Artins proof is not constructive and the denumerator polynomials can lead to an unstable factorization. In the $n = 2$ case Kummert (Kummert, 1990) showed how a result of Landau can be used to compute spectral factorizations where the spectral factors represent multivariable Hurwitz stable systems. Recent results of Demanze (Demanze, 2000) show that for a specific class of polynomials a very special denominator structure in the representation as a sum of squares of rational functions can be achieved. Especially important is the fact that this denominator structure leads to a factorization with nice properties.

2. NOTATION

In order to have a compact representation for multivariable polynomials we define multiindices

$$\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n \tag{1}$$

with the following further definitions which involve the complex variables $z \in \mathbb{C}^n$

$$z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n} \, , \tag{2}$$

$$\mid \alpha \mid = \alpha_1 + \cdots + \alpha_n \tag{3}$$

we obtain the compact representation

$$p(z) = \sum p_\alpha z^\alpha \tag{4}$$

for a polynomial p with the coefficients $p_\alpha \in \mathbb{C}$. The paraconjugate $p_*(z)$ is defined (Kummert, 1990) as

$$p_*(z) = \left(\sum p_\alpha \left(-z^*\right)^\alpha\right)^* \tag{5}$$

where a superscript asterisk denotes complex conjugation and in the case of matrices hermitian conjugation.

3. STATEMENT OF THE PROBLEM

Let $p(z)$ be a polynomial with the properties $p_*(z) = p(z)$ and $p(j\omega) > 0$ for all $\omega \in \mathbb{R}^n$. Then a factorization of the form

$$p = H_* H \tag{6}$$

is sought where $H(z)$ is a vector which is holomorphic for all $z \in \mathbb{C}$ with $Re z_i > 0$. We call this a stable factorization in the following.

4. REPRESENTATION THEOREM

It is known since Artin that $p(j\omega)$ with the above properties can be represented as a sum of squares of rational functions. Unfortunately the denominators can not be guaranteed to satisfy the required holomorphy condition. For a specific class of polynomials Demanze (Demanze, 2000) showed that with the vector

$$\theta(\omega) = \left(1 + \omega_1^2, \ldots, 1 + \omega_n^2\right) \tag{7}$$

there exist a representation of the form

$$p(j\omega) = \frac{q_1^2(\omega) + \ldots q_l^2(\omega)}{\theta(\omega)^\beta} \tag{8}$$

with real polynomials q_i an appropriate multiindex β and an integer l. Now we follow closely the construction of Kummert from (Kummert, 1990), namely, we define the complex polynomials

$$a_i(z) = q_i\left(\frac{z}{j}\right) \tag{9}$$

which satisfy $a_{i*} = a_i$. The vector

$$\vartheta(z) = \theta\left(\frac{z}{j}\right) , \tag{10}$$

$$= \left(1 - z_1^2, \ldots, 1 - z_n^2\right) \tag{11}$$

is defined accordingly and due to analytic continuation we have

$$p(z) = \frac{a_1^2(z) + \ldots a_l^2(z)}{\vartheta(z)^\beta} \tag{12}$$

which is not far from the factorization we are looking for. In order to finish the construction we use the factorization $1 - z^2 = (1 - z) * (1 + z)$ and define the following vector

$$\phi(z) = (1 + z_1, \ldots, 1 + z_n) \tag{13}$$

and it is easily seen that

$$\left(\phi(z)^\beta\right)_* = \phi(-z)^\beta \tag{14}$$

is valid. Thus, we have

$$\left(\phi(z)^\beta\right)_* \phi(z)^\beta = \vartheta(z)^\beta \tag{15}$$

and finally the vector H is defined as

$$H(z) = \frac{1}{\phi(z)^\beta} \begin{pmatrix} a_1(z) \\ \vdots \\ a_l(z) \end{pmatrix} \tag{16}$$

and is clear from the construction that H is holomorphic for all $z \in \mathbb{C}^n$with $Re z_i > 0$.

5. CONCLUSION AND OUTLOOK

In this short paper a new result from real algebraic geometry has been used to compute spectral factorizations for multivariable polynomials. The presented spectral factorizations have the additional property that the spectral factor has nice holomorphy properties which lead to stable spectral factorizations. Future work will concentrate on the extension of the construction to polynomial matrices. The concrete computation of the spectral factors will also be part of the future work.

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