LINEAR QUADRATIC GAUSSIAN CONTROL OF DISCRETE-TIME MARKOV JUMP LINEAR SYSTEMS WITH HORIZON DEFINED BY STOPPING TIMES

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Abstract: The linear quadratic Gaussian control of discrete-time Markov jump linear systems is addressed in this paper, first for state feedback, and also for dynamic output feedback using state estimation. In the model studied, the problem horizon is defined by a stopping time τ which represents either, the occurrence of a fix number N of failures or repairs (T_N), or the occurrence of a crucial failure event (τ_Δ), after which the system paralyzed. From the constructive method used here a separation principle holds, and the solutions are given in terms of a Kalman filter and a state feedback sequence of controls. The control gains are obtained by recursions from a set of algebraic Riccati equations for the former case or by a coupled set of algebraic Riccati equation for the latter case.

Keywords: Markov models, State and Output feedback control, Stopping times.

1. INTRODUCTION

Markov jump linear systems comprise a class of processes that presents changes of structure or modes according to an underlying Markov chain. The theory of stability and the quadratic optimal control problem for MJLS, also refereed as JLQ control problem, can be found in several papers, under both assumptions, complete and partial state observations. In general, in these studies the performance index associated with the JLQ control problem is related to finite horizon or to purely infinite horizon. A interesting situation from of point of view of applications arises when one considers a stopping time τ of the joint process $\{x_k, \theta_k, k \geq 0\}$ modelled by (1) and (2), as horizon of the functional cost associated with the problem . More specifically, when τ represents the occurrence of a fix number N of failures or repairs of the system ($\tau = T_N$) or a occurrence of a crucial failure event ($\tau = \tau_\Delta$). The stochastic stability analysis for the cases above described has been developed in (do Val *et al.*, 2003) and in (do Val and Nespoli, 2003), respectively. The JLQ control problem for complete observation has been studied in (Nespoli *et al.*, 2004) for the free noise case. The proposal of this paper consists in extending the results in (Nespoli *et al.*, 2004) to the MJLS subject to a stochastic input $\{w_k; k \geq 0\}$. One considers that the jump variable state is perfectly observable to the controller and the linear variable is observed only through an output variable. The results are presented in the following way. The notation to be employed is containing in Section 2 as well as the concept of stability appropriated to the proposed problems. Section 3 provides the solution for the output feedback control

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problem. A numerical example is finally presented in Section 4.

2. NOTATION AND PROBLEM FORMULATION

Throughout this paper, $\mathbb{R}^n$ denotes the n-dimensional real space and $\mathcal{M}^{m\times n}$ ($\mathcal{M}^m$) and the normed linear space of all $m\times n$ ($m\times m$) real matrices. The transpose of matrix U is indicated by U' and a positive semidefinite matrix (positive definite) is represented by $U\geq 0$ ($U>0$). We denote $\mathcal{M}^{m0}=\{U\in\mathcal{M}^m : U=U'\geq 0\}$, and $\mathcal{M}^{m+}=\{U\in\mathcal{M}^m : U=U'>0\}$. The linear space of all sequences of s real matrices in $\mathcal{M}^{m\times n}$ ($\mathcal{M}^m$) is represented by $\mathbb{M}^{m\times n}=\{\mathbf{U}=(U_1,\cdots,U_s) : U_i\in\mathcal{M}^{m\times n}, i=1,\ldots,s\ \}$ ($\mathbb{M}^m$). In addition $\mathbb{M}^{m0}$ ($\mathbb{M}^{m+}$) is written when $U_i\in\mathcal{M}^{m0}$ ($U_i\in\mathcal{M}^{m+}$), for $i=1,\ldots,s$. The standard vector norm in $\mathbb{R}^n$ is indicated by $\|\cdot\|$. In addition, $r_\sigma(U)$ and $\mathcal{N}\{U\}$ indicate the spectral radius and the null space of $U\in\mathcal{M}^m$, respectively, and $a\wedge b$ denotes $\min\{a,b\}$. Let $\mathbb{1}_{\{.\}}$ be the Dirac measure. For $\mathbf{U}\in\mathbb{M}^{m0}$, the following operators are defined

$$\mathcal{E}_i^\Delta(\mathbf{U})=\sum_{j\neq i, j\neq\Delta} p_{ij}U_j \quad \text{and} \quad \mathcal{E}_i(\mathbf{U})=\sum_{j\neq i} p_{ij}U_j.$$

Consider the discrete-time homogeneous Markov chain $\{\theta_k; k\geq 0\}$ with space state $\mathfrak{X}=\{1,\ldots,s\}\cup\{\Delta\}$ ($\mathcal{T}=\{1,\ldots,s\}$ is the collection of transient states and Δ is a absorbent state), initial distribution $\mu=(\mu_1,\ldots,\mu_s,0)$ where $\mu_i=P(\theta_0=i)$, for all $i\in\mathfrak{X}$ and transition probability matrix $\mathbb{P}=[p_{ij}]$ where

$$p_{ij}=P(\theta_{k+1}=j\mid\theta_k=i),\ \forall i,j\in\mathfrak{X},\ k\geq 0. \quad (1)$$

Let the discrete-time **M**arkovian **J**ump **L**inear **S**ystems (MJLS) defined on the fundamental probability space $(\Omega,\mathfrak{F},\{\mathfrak{F}_k\},P)$,

$$\mathcal{S}:\begin{cases} x_{k+1}=A_{\theta_k}x_k+B_{\theta_k}u_k+H_{\theta_k}w_k, & x_0\in\mathbb{R}^n,\\ z_k=C_{\theta_k}x_k+D_{\theta_k}u_k, & \theta_0\sim\mu,\\ y_k=F_{\theta_k}x_k+G_{\theta_k}w_k, & k\geq 0,\end{cases} \quad (2)$$

where $\{x_k,\theta_k; k\geq 0\}$ is the process state taking values in $\mathbb{R}^n\times\mathfrak{X}$; $\{u_k; k\geq 0\}$, $\{z_k; k\geq 0\}$ and $\{y_k; k\geq 0\}$ are the control, the output and the measured output process, respectively. The process $\{w_k; k\geq 0\}$ is a sequence of l-vector independent random vectors normally distributed with mean 0 and covariance I. We assume that $\{w_k; k\geq 0\}$ and $\{\theta_k; k\geq 0\}$ are independent. Whenever $\theta_k=i$, one has that $A_{\theta_k}=A_i\in\mathcal{M}^n, B_{\theta_k}=B_i\in\mathcal{M}^{n\times p}, C_{\theta_k}=C_i\in\mathcal{M}^{q\times n}$, $D_{\theta_k}=D_i\in\mathcal{M}^{q\times p}$, $F_{\theta_k}=F_i\in\mathcal{M}^{m\times n}$, $G_{\theta_k}=G_i\in\mathcal{M}^{m\times l}$ and $H_{\theta_k}=H_i\in\mathcal{M}^{n\times l}$. In this work the horizon of the performance index associated with the system $\mathcal{S}$ is given by a stopping time τ of the joint process $\{x_k,\theta_k, k\geq 0\}$ modelled by (1) and (2), that is

$$J(u)=E\left[\sum_{k=0}^{\tau-1}\|z_k\|^2+x_\tau' S_{\theta_\tau}x_\tau\right], \quad (3)$$

where $\mathbf{S}\in\mathbb{M}^{m0}$ is some terminal cost. In particular we consider the following cases: (i) $\tau=T_N$: τ represents the time of occurrence of a finite number N of failures or repairs. We deal with this by defining the sequence $\mathcal{T}^N=\{T_n; n=0,1,\ldots,N\}$ of $\{\mathfrak{F}_k\}$-stopping times

$$T_0=0,\ T_n=\min\{k>T_{n-1} : \theta_k\neq\theta_{T_{n-1}}\}, n\geq 1.$$

(ii) $\tau=\tau_\Delta$: τ represents the time of the jump into the state Δ, associated with a crucial failure occurrence. Thus τ_Δ is defined as the hitting-time of Δ, i. e.,

$$\tau_\Delta=\min\{k\geq 1 : \theta_k=\Delta\}.$$

We assume that the jump variable is perfectly observable but the linear variable is observed only through an output variable. In these scenario the problem consists in obtaining a τ-stabilizable control action for the cases $\tau=T_N$ and $\tau=\tau_\Delta$ such that minimize the cost criteria as in (3).

Remark 1. The intermediary case $\tau=\tau_\Delta\wedge T_N$ is used here as strategy for studying the case $\tau=\tau_\Delta$ since considering $\tau_\Delta=\lim_{N\to\infty}\{\tau_\Delta\wedge T_N\}$.

Stochastic Stability: Consider the autonomous discrete-time MJLS $\mathcal{S}_0$ ($\mathcal{S}$ with $u\equiv 0$). We adopt the stochastic τ-stability concept introduced in (do Val *et al.*, 2003) that is tailored to the announced problems.

Definition 2. Consider a stopping time τ with respect to $\{\mathfrak{F}_k\}$. Then, the MJLS $\mathcal{S}_0$ is **S**tochastically τ-stable (τ-SS) if for each initial condition x_0 and initial distribution μ

$$E\left[\sum_{k\geq 0}\|x_k\|^2\mathbb{1}_{\{\tau\geq k\}}\right]<\infty. \quad (4)$$

The results below provide necessary and sufficient conditions to ensure the stochastic τ-stability in the cases previously described, see (do Val *et al.*, 2003) and (do Val and Nespoli, 2003).

Theorem 3. Let $\tau\in\mathcal{T}^N$ or $\tau=\tau_\Delta\wedge T_n$, $n\leq N$. The following assertions are equivalent:

i) The MJLS $\mathcal{S}_0$ is τ-SS.
ii) For any given set of matrices $\mathbf{Q}\in\mathbb{M}^{n+}$, there exists a unique set of matrices $\mathbf{L}\in\mathbb{M}^{n+}$, satisfying the Lyapunov equations

$$p_{ii}A_i'L_iA_i-L_i+Q_i=0, \quad i=1,\ldots,s. \quad (5)$$

Theorem 4. Let $\tau=\tau_\Delta$. The following conditions are equivalent:

i) The MJLS $\mathcal{S}_0$ is τ-SS.
ii) For any given set of matrices $\mathbf{Q}\in\mathbb{M}^{n+}$, there exists a unique set of matrices $\mathbf{L}\in\mathbb{M}^{n+}$, satisfying the Lyapunov equations

$$\sum_{j=1}^{s}p_{ij}A_i'L_jA_i-L_i+Q_i=0, \quad i=1,\ldots,s. \quad (6)$$

Remark 5. Applying Theorem 3, notice that the T_n-stabilizability problem is equivalent to determine the stabilizability of the pair $(p_{ii}^{1/2}A_i, p_{ii}^{1/2}B_i)$ for each $i = 1,\ldots,s$ in the deterministic sense. In turn, we can announce from Theorem 4 that the τ_Δ-stabilizability of the pair $(\mathbf{A},\mathbf{B})$ is equivalent to the existence of a set of matrices $\mathbf{M} \in \mathbb{M}^{n+}$ for some $\mathbf{Q} \in \mathbb{M}^{n+}$ such that

$$[A_i + B_iK_i]'[p_{ii}M_i + \mathcal{E}_i^\Delta(\mathbf{M})][A_i + B_iK_i] - M_i + Q_i = 0,$$

holds for each $i = 1,\ldots,s$, and some $\mathbf{K} = (K_1,\cdots,K_s)$. Note that if $\mathbf{K}$ τ_Δ-stabilizes the closed-loop system, then K_i stabilizes $(p_{ii}^{1/2}A_i, p_{ii}^{1/2}B_i)$ for each $i = 1,\cdots,s$.

3. THE CONTROL PROBLEM

3.1 The JLQ problem with additive noise

Assume that at each instant k the linear state x_k and the jump state state θ_k are precisely known to controller, i.e., the system $\mathcal{S}$ with $F_i \equiv I$ and $G_i \equiv 0$ for all $i \in \mathfrak{X}$. Write $E_{T_k}[\cdot]$ and $E_0[\cdot]$ to represent $E[\cdot \mid x_{T_k}, \theta_{T_k}]$ and $E[\cdot \mid x_0, \theta_0]$, respectively, and suppose $\mu_i = 1$. The cost for $x_0 = x$ and $\theta_0 = i$ is denoted by

$$J(x,i,u) := E_0\left[\sum_{k=0}^{\tau-1} \|z_k\|^2 + x_\tau' S_{\theta_\tau} x_\tau\right]. \tag{7}$$

Firstly we solve the one jump problem related to horizon $\tau = T_1 \wedge m$, $m > 1$, which model is

$$\begin{cases} x_{k+1} = A_ix_k + B_iu_k + H_iw_k, & \mu_i = 1, \\ z_k = C_ix_k + D_iu_{k,}, & 0 \le k < T_1 \wedge m. \end{cases} \tag{8}$$

Note that $\theta_k = \theta_0 = i$ para $0 \le k < T_1 \wedge m$. Denote $J_{T_1}^m(x,i,u)$ the functional in (7) when $\tau = T_1 \wedge m$. Consider $u_k = K_i^k x_k$ then $J_{T_1}^m(x,i,u)$ is given by

$$J_{T_1}^m(x,i,u) = \sum_{k=0}^{m-1} p_{ii}^k x_k' \hat{Q}_i^k x_k + p_{ii}^m x_m' S_i x_m + \sum_{k=0}^{m-1} p_{ii}^k tr\{H_i' \mathcal{E}_i^\Delta(\mathbf{S}) H_i\}, \tag{9}$$

with $\hat{Q}_i^k = \hat{C}_i^{k\prime}\hat{C}_i^k + \hat{A}_i^{k\prime}\mathcal{E}_i^\Delta(\mathbf{S})\hat{A}_i^k$, $\hat{A}_i^k = A_i + B_iK_i^k$ and $\hat{C}_i^k = C_i + D_iK_i^k$. Note that in (9) the last term is a constant which do not depend on the choice of u_k, then can be eliminated in the optimization process. As consequence, the problem of minimizing (9) subject to (8) is a standard problem found in the literature, see (Davis and Vinter, 1985). In order to determinate the optimal control law, we define $\hat{A}_i = p_{ii}^{1/2}A_i$, $\hat{B}_i = p_{ii}^{1/2}B_i$, $\tilde{A}_i = A_i - B_i[D_i'D_i]^{-1}D_i'C_i$, $\tilde{C}_i = [I - D_i[D_i'D_i]^{-1}D_i']C_i$, and consider $(\hat{A}_i, \hat{B}_i)$ estabilizable and $(\tilde{C}_i, p_{ii}^{1/2}\tilde{A}_i)$ detectable. Then, there exist an unique matrix $L_i^{k,m}$ which is the solution for the set of algebraic Riccati equations (ARE) given by

$$\begin{aligned} L_i^{k,m} &= A_i'[p_{ii}L_i^{k+1,m} + \mathcal{E}_i^\Delta(\mathbf{S})]A_i - [A_i'[p_{ii}L_i^{k+1,m} + \\ &\mathcal{E}_i^\Delta(\mathbf{S})]B_i + C_i'D_i][B_i'[p_{ii}L_i^{k+1,m} + \mathcal{E}_i^\Delta(\mathbf{S})]B_i + D_i'D_i]^{-1} \\ &[B_i'[p_{ii}L_i^{k+1,m} + \mathcal{E}_i^\Delta(\mathbf{S})]A_i + D_i'C_i] + C_i'C_i, \end{aligned} \tag{10}$$

for $k = m-1,\ldots,0$ with $\mathbf{L}^{m,m} = \mathbf{S}$.

The optimal control law is $u_k = K_i^k x_k$ where

$$K_i^k = [B_i'[p_{ii}L_i^{k+1,m} + \mathcal{E}_i^\Delta(\mathbf{S})]B_i + D_iD_i']^{-1} [B_i'[p_{ii}L_i^{k+1,m} + \mathcal{E}_i^\Delta(\mathbf{S})]A_i + D_i'C_i]. \tag{11}$$

In the next proposition the case $\tau = T_1$ is recovered taking $\lim_{m\to\infty} J_{T_1}^m(x,i,u)$.

Proposition 6. Let $\tau = T_1$. Assume $(\hat{A}_i, \hat{B}_i)$ stabilizable and $(\tilde{C}_i, p_{ii}^{1/2}\tilde{A}_i)$ detectable. Then, the matrix $L_i = \lim_{m\to\infty} L_i^{k,m}$, with $L_i^{k,m}$ obtained as in (10), is the unique solution for the ARE

$$\begin{aligned} L_i = A_i'[p_{ii}L_i + \mathcal{E}_i^\Delta(\mathbf{S})]A_i - [A_i'[p_{ii}L_i + \mathcal{E}_i^\Delta(\mathbf{S})]B_i + C_i'D_i] \\ [B_i'[p_{ii}L_i + \mathcal{E}_i^\Delta(\mathbf{S})]B_i + D_i'D_i]^{-1} \\ [B_i'[p_{ii}L_i + \mathcal{E}_i^\Delta(\mathbf{S})]A_i + D_i'C_i] + C_i'C_i, \end{aligned} \tag{12}$$

The optimal control is $u_k = K_ix_k$, where

$$K_i = [B_i'[p_{ii}L_i + \mathcal{E}_i^\Delta(\mathbf{S})]B_i + D_i'D_i]^{-1} [B_i'[p_{ii}L_i + \mathcal{E}_i^\Delta(\mathbf{S})]A_i + D_i'C_i]. \tag{13}$$

Moreover, the minimal cost is $x'L_ix + l_i$ where l_i is a constant.

Case $\tau = T_N$: Consider the value function $V(z_n,\phi_n)$, defined as the minimal cost starting at the jump instant T_n, namely,

$$V(z_n,\phi_n) = \min_{\mathbf{K}^n,\ldots,\mathbf{K}^{N-1}} E_{T_n}\left[\sum_{k=T_n}^{T_N-1} \|z_k\|^2 + z_N' S_{\phi_N} z_N\right],$$

for $n = N-1,\ldots,0$, with $z_n = x_{T_n}$ and $\phi_n = \theta_{T_n}$. Using the strong Markov property and the optimality principle we have

$$V(z_n,\phi_n = i) = \min_{K_i^n} E_{T_n}\left[\sum_{k=T_n}^{T_{n+1}-1} \|z_k\|^2 + V(z_{n+1},\phi_{n+1})\right],$$

Based on one jump problem results, it can be show that $V(z_n,\phi_n = i) = z_n'L_i^n z_n + l_i^n$ where L_i^n is the unique solution for the ARE

$$\begin{aligned} L_i^n = A_i'[p_{ii}L_i^n + \mathcal{E}_i^\Delta(\mathbf{L}^{n+1})]A_i - [A_i'[p_{ii}L_i^n + \mathcal{E}_i^\Delta(\mathbf{L}^{n+1})] \\ B_i + C_i'D_i][B_i'[p_{ii}L_i^n + \mathcal{E}_i^\Delta(\mathbf{L}^{n+1})]B_i + D_i'D_i]^{-1} \\ [B_i'[p_{ii}L_i^n + \mathcal{E}_i^\Delta(\mathbf{L}^{n+1})]A_i + D_i'C_i] + C_i'C_i. \end{aligned} \tag{14}$$

and l_i^n is a constant.

Theorem 7. Assume $(\hat{A}_i, \hat{B}_i)$ stabilizable and $(\tilde{C}_i, p_{ii}^{1/2}\tilde{A}_i)$ detectable, for each $i = 1,\ldots,s$. Then, the set of matrices $\{\mathbf{L}^0,\ldots,\mathbf{L}^{N-1}\}$ is the unique solution of the backward recursive ARE (14) for each $i = 1,\ldots,s$ with $\mathbf{L}^N = \mathbf{S}$. The optimal control is given by the piecewise linear feedback law

$$u_k = \sum_{n=0}^{N-1} K_{\theta_k}^n x_k \mathbb{1}_{\{T_n \le k < T_{n+1}\}}, \quad k \ge 0, \tag{15}$$

where the optimal gains sequence $\{\mathbf{K}^0,\ldots,\mathbf{K}^{N-1}\}$ is given by

$$K_i^n = [B_i'[p_{ii}L_i^n + \mathcal{E}_i^\Delta(\mathbf{L}^{n+1})]B_i + D_i'D_i]^{-1} [B_i'[p_{ii}L_i^n + \mathcal{E}_i^\Delta(\mathbf{L}^{n+1})]A_i + D_i'C_i], \tag{16}$$

for each $i = 1,\dots,s$ and $n = N-1,\dots,0$.

Remark 8. The optimal gain as in (16) coincides with the optimal gain for the free noisy case, see (Nespoli *et al.*, 2004).

Remark 9. For the mixed case $\tau_\Delta \wedge T_N$, under the same assumptions above, the set of matrices $\{\mathbf{L}^0,\dots,\mathbf{L}^{N-1}\}$ is the unique solution of the backward recursive ARE

$$\begin{aligned}L_i^n = {} & C_i'C_i + A_i'[p_{ii}L_i{}^n + \mathcal{E}_i^\Delta(\mathbf{L}^{n+1}) + p_{i\Delta}S_\Delta]A_i \\ & - [A_i'[p_{ii}L_i{}^n + \mathcal{E}_i^\Delta(\mathbf{L}^{n+1}) + p_{i\Delta}S_\Delta]B_i + C_i'D_i] \\ & [B_i'[p_{ii}L_i{}^n + \mathcal{E}_i^\Delta(\mathbf{L}^{n+1}) + p_{i\Delta}S_\Delta]B_i + D_i'D_i]^{-1} \\ & [B_i'[p_{ii}L_i{}^n + \mathcal{E}_i^\Delta(\mathbf{L}^{n+1}) + p_{i\Delta}S_\Delta]A_i + D_i'C_i], \quad (17)\end{aligned}$$

for each $i = 1,\dots,s$ and $n = N-1,\dots,0$ with $\mathbf{L}^N = \mathbf{S}$. The optimal control is given by the piecewise linear feedback law (15) where $\{\mathbf{K}^0,\dots,\mathbf{K}^{N-1}\}$ is given by (16) for each $i = 1,\dots,s$ and $n = N-1,\dots,0$.

Case $\tau = \tau_\Delta$: The strategy to study this case consists in seeking the limit situation for $\tau_\Delta \wedge T_N$, when $N \to \infty$. In this sense, we use the Weak-detectability concept as introduced in (Costa and do Val, 2002), for ensure the convergence of the solutions L_i^n in (17). Firstly, consider the set of observability matrices $\mathbf{O} \in \mathbb{M}^{n(n^2s)\times n}$, where each of the matrices $O_i \in \mathcal{M}^{n0}$ is defined as

$$O_i := [W_i(0) \vdots W_i(1) \vdots \cdots \vdots W_i(n^2s-1)]'$$

for $i \in \{1,\dots,s\}$, where $W_i(k)$ is defined recursively as $W_i(k) := A_i'[p_{ii}W_i(k-1) + \mathcal{E}_i(\mathbf{W}(k-1))]A_i$, with $W_i(0) := \tilde{C}_i'\tilde{C}_i$.

Definition 10. The pair $(\tilde{\mathbf{C}},\tilde{\mathbf{A}})$ is Weak-detectable iff $\lim_{k\to\infty} E[\|x_k\|^2] = 0$ whenever $x_0 \in \mathcal{N}(O_{\theta_0})$.

The following proposition is a straightforward modification of a result proven in (Costa and do Val, 2002).

Proposition 11. Assume that $(\tilde{\mathbf{C}},\tilde{\mathbf{A}})$ is Weak-detectable. There exists a unique solution $\mathbf{P} \in \mathbb{M}^{n0}$ for the coupled algebraic Riccati equation (CARE)

$$\begin{aligned}P_i = {} & A_i'[p_{ii}P_i + \mathcal{E}_i(\mathbf{P})]A_i - [A_i'[p_{ii}P_i + \mathcal{E}_i(\mathbf{P})]B_i + C_i'D_i] \\ & [B_i'[p_{ii}P_i + \mathcal{E}_i(\mathbf{P})]B_i + D_i'D_i]^{-1} \\ & [B_i'[p_{ii}P_i + \mathcal{E}_i(\mathbf{P})]A_i + D_i'C_i] + C_i'C_i, \quad (18)\end{aligned}$$

iff $(\mathbf{A},\mathbf{B})$ is τ_Δ-stabilizable

Substituting in (17) $p_{ii}L_i^n$ by the approximation $p_{ii}L_i^n = \kappa p_{ii}L_i^n + (1-\kappa)p_{ii}L_i^{n+1}$ with $0<\kappa<1$, we can use the result bellow, adapted from (Costa and do Val, 2002).

Proposition 12. Assume $(\tilde{\mathbf{C}},\tilde{\mathbf{A}})$ Weak-detectable and consider the solutions of the ARE's

$$\begin{aligned}L_i^n = {} & C_i'C_i + A_i'[\kappa p_{ii}L_i{}^n + \mathcal{L}_i(\mathbf{L}^{n+1})]A_i - [A_i'[\kappa p_{ii}L_i{}^n + \\ & \mathcal{L}_i(\mathbf{L}^{n+1})]B_i + C_i'D_i][B_i'[\kappa p_{ii}L_i{}^n + \mathcal{L}_i(\mathbf{L}^{n+1})]B_i + \\ & D_i'D_i]^{-1}[B_i'[\kappa p_{ii}L_i{}^n + \mathcal{L}_i(\mathbf{L}^{n+1})]A_i + D_i'C_i], \quad (19)\end{aligned}$$

where $\mathcal{L}_i(\mathbf{L}^{n+1}) = \mathcal{E}_i^\Delta(\mathbf{L}^{n+1}) + (1-\kappa)p_{ii}L_i^{n+1} + p_{i\Delta}S_\Delta$, for $n = 0,-1,\dots$ and $i = 1,\dots,s$, L_i^0 arbitrary. Then $(\mathbf{A},\mathbf{B})$ is means square stabilizable iff the sequence $\mathbf{L}^n$ converges to $\mathbf{P} \in \mathbb{M}^{n+}$ when $n \to -\infty$, where $\mathbf{P}$ is the solution of the CARE (18).

Finally, the next theorem allow us to find the gain $\mathbf{K}$ as required.

Theorem 13. Suppose $(\mathbf{A},\mathbf{B})$ τ_Δ-stabilizable and $(\tilde{\mathbf{C}},\tilde{\mathbf{A}})$ Weak-detectable. Consider the solutions $L_i^n \in \mathcal{M}^{n0}$ of the ARE's (19). Then $L_i^n \to L_i$ when $n \to -\infty$, where $\mathbf{L} \in \mathbb{M}^{n0}$ is the solution of the CARE (18). Moreover, $u_k = K_i x_k$ where the optimal gain K_i is given by

$$\begin{aligned}K_i = {} & [B_i'[p_{ii}L_i + \mathcal{E}_i^\Delta(\mathbf{L}) + p_{i\Delta}S_\Delta]B_i + D_i'D_i]^{-1} \\ & [B_i'[p_{ii}L_i + \mathcal{E}_i^\Delta(\mathbf{L}) + p_{i\Delta}S_\Delta]A_i + D_i'C_i], \quad (20)\end{aligned}$$

for each $i = 1,\dots,s$.

Remark 14. Hence, adding noise to the case τ_Δ makes no difference to the optimal gain which coincides with the optimal gains to the free noise, see (Nespoli *et al.*, 2004).

3.2 *Output Feedback Control*

We now consider control problem associated with the system $\mathcal{S}$ defined in (2). Suppose additionally that x_0 is normal with mean m_0 and covariance P_0. The state x_k cannot be measured directly, but "noisy observations" $y^{k-1} = (y_0,y_1,\dots,y_{k-1})$ are available at time $k-1$, so that $\hat{x}_{k|k-1} = E[x_k \mid y^{k-1}]$ denotes the linear estimator of x_k given y^{k-1}. Also, the control $u_k = u_k(y^{k-1})$. Analogously to previous section, one studies the one jump problem associated with the model

$$\begin{cases} x_{k+1} = A_i x_k + B_i u_k + H_i w_k & \mu_i = 1, \\ y_k = F_i x_k + G_i w_k & 0 \le k < T_1 \wedge m, \end{cases} \quad (21)$$

We deal this problem by replacing (21) by the corresponding *innovations representation*, see (Davis and Vinter, 1985), which provides an equivalent model in the form

$$\hat{x}_{k+1|k} = A_i\hat{x}_{k|k-1} + B_i u_k + Z_i^k \nu_k, \quad \hat{x}_{0|-1} = m_0, \quad (22)$$

for $0 \le k < T_1 \wedge m$. The innovation process $\nu_k := y_k - F_i\hat{x}_{k+1|k}$ is a white-noise process with mean 0 and covariance function $E[\nu_k,\nu_k'] = F_iP_i^kF_i' + G_kG_k'$, where $P_i^k = E[(x_k - \hat{x}_{k|k-1})(x_k - \hat{x}_{k|k-1})']$ satisfies the recursive ARE

$$\begin{aligned}P_i^{k+1} = {} & A_iP_i^kA_i' + H_iH_i' - [A_iP_i^kF_i' + H_iG_i'] \\ & [F_iP_i^kF_i' + G_iG_i']^{-1}[F_i'P_i^kA_i' + G_iH_i'], \quad (23)\end{aligned}$$

with $P^0 = P_0$. The Kalman gain Z_i^k is given by

$$Z_i^k = [A_iP_i^kF_i' + H_iG_i'][F_iP_i^kF_i' + G_iG_i']^{-1}. \quad (24)$$

Note that, y_k satisfies

$$y_k = F_i\hat{x}_{k|k-1} + \nu_k. \quad (25)$$

Under the conditions above, the new state $\hat{x}_{k|k-1}$ is the best linear estimator of x_k given y^{k-1}. The functional $J^m_{T_1}(i,u)$ bellow, re-express (9) in a way which involves $\hat{x}_{k|k-1}$,

$$J^m_{T_1}(i,u) = E\left[\sum_{k=0}^{m-1} p^k_{ii}\hat{x}'_{k|k-1}\hat{Q}^k_i\hat{x}_{k|k-1} + p^m_{ii}\hat{x}_{m|m-1}S_i\hat{x}_{m|m-1}\right] + \sum_{k=0}^{m-1} p^k_{ii}tr\{H_i\mathcal{E}^\Delta_i(S)H'_i + P^k_i\hat{Q}^k_i\} + p^m_{ii}tr\{P^m_i S_i\}. \quad (26)$$

Notice about the above expressions that the first term is the cost (9) in the variable $\hat{x}_{k|k-1}$, and the remaining two terms are constants which do not depend on the choice of u_k and then can be eliminated in the optimization process.

In this section one intends to apply the results for the completely observable case to the new system (22) and (25). In this sense, define $\tilde{\nu}_k = [F_iP^k_iF'_i + G_iG'_i]^{-1/2}\nu_k$. Consequently (22) can be written

$$\hat{x}_{k+1|k} = A_i\hat{x}_{k|k-1} + B_iu_k + Z^k_i[F_iP^k_iF'_i + G_iG'_i]^{1/2}\tilde{\nu}_k. \quad (27)$$

Since $E[\tilde{\nu}_k\tilde{\nu}'_k] = I$, $\tilde{\nu}_k$ is a normalized white-noise process and then the equation (27) is in the standard form of (21) with H_i replaced by $Z^k_i[F_iP^k_iF'_i + G_iG'_i]^{1/2}$. Defining $\breve{A}_i = A_i - H_iG'_i[G_iG'_i]^{-1}F_i$ and $\breve{H}_i = H_i[I - G'_i(G_iG'_i)^{-1}G_i]$ the solution for the output feedback control with horizon T_1 is obtained analogously to Proposition 6.

Proposition 15. Let $\tau = T_1$. Suppose $(\hat{A}_i,\hat{B}_i)$ and $(\breve{A}_i,\breve{H}_i)$ stabilizable; $(\tilde{C}_i, p^{1/2}_{ii}\tilde{A}_i)$ and $(F_i,\hat{A}_i)$ detectable for each $i = 1,\dots,s$. Then, the matrices $L_i = L^{k,m}_i$ when $m \to \infty$ and P^k_i are the unique solution for the following ARE (12) and (23), respectively. The optimal control is given by $\hat{u}_k = K_i\hat{x}_{k|k-1}$, with K_i obtained as in (13). The filter dynamics is given by

$$\hat{x}_{k+1|k} = A_{\theta_k}\hat{x}_{k|k-1} + B_{\theta_k}u_k + Z^k_{\theta_k}[y_k - F_{\theta_k}\hat{x}_{k|k-1}], \quad (28)$$

where the Kalmam gain Z^k_i for each $i = 1,\dots,s$ is obtained as in (23) and (24).

Observe that the gains K_i are the same as in the complete observation case, so that the named *certainty-equivalence principle* is verified for T_1.

Case $\tau = T_N$: The function value $V(\hat{z}_n,\phi_n)$ associated to the partial observation problem, now is defined as

$$V(\hat{z}_n,\phi_n) = \min_{\mathbf{K}^n,\dots,\mathbf{K}^{N-1}} E\left[\sum_{k=T_n}^{T_N-1} \|z_k\|^2 + z'_N S_{\phi_N} z_N \mid \phi_n\right],$$

$n = N-1,\dots,0$, with $\phi_n = \theta_{T_n}$ and $\hat{z}_n := \hat{x}_{T_n} = E[x_{T_n} \mid y_0,\dots,y_{T_n-1}, T_n = k]$. It can be show

$$V(\hat{z}_n,\phi_n = i) = \min_{K^n_i} E\left[\sum_{k=T_n}^{T_{n+1}-1} \|z_k\|^2 + V(\hat{z}_{n+1},\phi_{n+1}) \mid \phi_n\right], = \hat{z}'_n L^n_i \hat{z}_n + tr\{L^n_i\hat{P}_{T^n}\} + \hat{l}^n_i,$$

where L^n_i is the unique solution of the ARE (14) and the sequence $\{\hat{l}^n_i\}$ is bounded. Considering that the certainty-equivalence principle holds for T_1, the same procedure for complete observation case can be employed here to determine the sequence $\{\mathbf{K}^0,\dots,\mathbf{K}^{N-1}\}$.

Theorem 16. Assume $(\hat{A}_i,\hat{B}_i)$ and $(\breve{A}_i,\breve{H}_i)$ stabilizable; $(\tilde{C}_i, p^{1/2}_{ii}\tilde{A}_i)$ and $(F_i,\hat{A}_i)$ detectable for each $i = 1,\dots,s$. Then, the set of matrices $\{\mathbf{L}^0,\dots,\mathbf{L}^{N-1}\}$ is the unique solution of the backward recursive ARE (14) for each $i = 1,\dots,s$ with $\mathbf{L}^N = \mathbf{S}$. The optimal control is given by the piecewise linear feedback law

$$u_k = \sum_{n=0}^{N-1} K^n_{\theta_k}\hat{x}_{k|k-1}\mathbb{1}_{\{T_n\le k<T_{n+1}\}}, \quad k \ge 0, \quad (29)$$

where the optimal gains sequence $\{\mathbf{K}^0,\dots,\mathbf{K}^{N-1}\}$ is given by (16) for each $i = 1,\dots,s$ and $n = N-1,\dots,0$. The filter dynamics is given by (28) where the Kalmam gain Z^k_i for each $i = 1,\dots,s$ is obtained as in (23) and (24).

Case $\tau = \tau_\Delta$: $\mathbf{K}$ is obtained as in the observable case.

Theorem 17. Suppose $(\mathbf{A},\mathbf{B})$ τ_Δ-stabilizable, $(\tilde{\mathbf{C}},\tilde{\mathbf{A}})$ Weak-detectable, $(\breve{A}_i,\breve{H}_i)$ stabilizable and $(F_i,\hat{A}_i)$ detectable for each $i = 1,\dots,s$. Then there exist a set of matrices $\mathbf{L}$ which is an solution of the CARE (18). The optimal gain $\mathbf{K}$ is determinate by (20). The filter dynamics is given by (28) where the Kalmam gain Z^k_i for each $i = 1,\dots,s$ is obtained as in (23) and (24).

4. ILLUSTRATIVE EXAMPLE

The linear model of one joint of "European Robot Arm" (*ERA*), see (Yang and Blanke, 1999), is utilized as example. The parameters are given in Table 1. The space-state model of the system is given by

Table 1. *ERA* model parameters.

Symbol	Description	Unit
gearbox ratio	$N = -260.6$	-
motor torque constant	$g_m = 0.6$	N/%
the damping coefficient	$\beta = 0.4$	N/%
inertia of the input axis	$I_m = 0.0011$	Kg m^2
inertia of the output axis	$I_{son} = 400$	Kg m^2
motor current	i_c	Am
spring constant	$c = 130000$	N/%

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Hw(t), \quad t \ge 0, \\ z(t) = Cx(t) + Du(t), \\ y(t) = Fx(t) + Gw(t), \end{cases}$$

where $x = \begin{bmatrix}\Omega & \dot{\Omega} & \varepsilon & \dot{\varepsilon}\end{bmatrix}$, $y = \begin{bmatrix}\Omega + \varepsilon & N\dot{\Omega}\end{bmatrix}'$, $u = i_c$,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{c}{N^2 I_m} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-\beta}{I_{son}} & (\frac{-c}{N^2 I_m} + \frac{-c}{I_{son}}) & \frac{-\beta}{I_{son}} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ \frac{g_m}{2NI_m} & \frac{g_m}{2NI_m} \\ \frac{-g_m}{2NI_m} & \frac{-g_m}{2NI_m} \\ 0 & 0 \end{bmatrix},$$

$$H = 0.01I, \quad C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & N & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix},$$

$F = 0.01\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}$ and $G = F$.

Table 2. Optimal control gains for $\tau = T_3$

Intervals	Control gains
$[0, T_1)$	$K_1^0 = \begin{bmatrix} -0.0050 & -0.1193 & -10.8276 & 3.0058 \\ -0.0051 & -0.1454 & -10.8277 & 3.0058 \end{bmatrix}$
	$K_2^0 = \begin{bmatrix} -0.0057 & -0.1326 & 62.1058 & 1. \\ -0.0058 & -0.1586 & 62.1057 & 1.7550 \end{bmatrix}$
	$K_3^0 = \begin{bmatrix} -0.0026 & -0.0692 & -63.1985 & 1.6320 \\ -0.0027 & -0.0953 & -63.1986 & 1.6320 \end{bmatrix}$
$[T_1, T_2)$	$K_1^1 = \begin{bmatrix} -0.0145 & -0.2086 & 0.8923 & 0.0214 \\ -0.0146 & -0.2347 & 0.8922 & 0.0214 \end{bmatrix}$
	$K_2^1 = \begin{bmatrix} -0.0085 & -0.1794 & 148.3138 & 0.7306 \\ -0.0086 & -0.2054 & 148.3137 & 0.7306 \end{bmatrix}$
	$K_3^1 = \begin{bmatrix} -0.0053 & -0.1100 & -14.1831 & 1.7818 \\ -0.0054 & -0.1360 & -14.1832 & 1.7818 \end{bmatrix}$
$[T_2, T_3)$	$K_1^2 = \begin{bmatrix} -0.0254 & -0.2651 & -0.1348 & -0.2260 \\ -0.0255 & -0.2912 & -0.1349 & -0.2260 \end{bmatrix}$
	$K_2^2 = \begin{bmatrix} -0.0235 & -0.2438 & 0.0863 & -0.2320 \\ -0.0236 & -0.2698 & 0.0862 & -0.2320 \end{bmatrix}$
	$K_3^2 = \begin{bmatrix} -0.0123 & -0.1735 & 0.4648 & -0.0494 \\ -0.0124 & -0.1995 & 0.4647 & -0.0494 \end{bmatrix}$

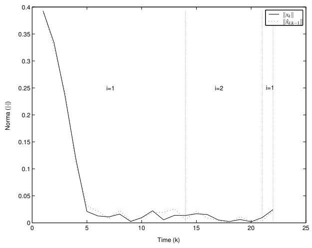
Table 3. Optimal control gains for $\tau = \tau_\Delta$

Control gains
$K_1 = \begin{bmatrix} 0.4758 & 130.0280 & 0.2659 & -0.2178 \\ -0.5242 & -130.5720 & -0.7341 & -0.2178 \end{bmatrix}$
$K_2 = \begin{bmatrix} 0.4779 & 130.0505 & 0.6317 & -0.2234 \\ -0.5221 & -130.5495 & -0.3683 & -0.2234 \end{bmatrix}$
$K_3 = \begin{bmatrix} 0.4781 & 130.0649 & 0.2511 & -0.1937 \\ -0.5219 & -130.5351 & -0.7489 & -0.1937 \end{bmatrix}$

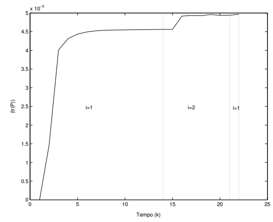
We consider two kinds of possible faults of the system, namely, $g_m^f =: F_{g_m} g_m$ and $I_m^f =: F_{I_m} I_m$ where the parameters F_{g_m} and F_{I_m} represent the fault levels of corresponding system parameters, respectively, assuming the values$F_{g_m} = 1$, $F_{g_m} = 1.2$, $F_{g_m} = 0.12$, $F_{I_m} = 1$ and $F_{I_m} = 0.5$. Since this work deals with discrete time MJLS, the set of systems defined above it was discretesized with sampling interval of 10. The Markov chain with state space $S = \{\Delta, 1, 2, 3\}$ represent the faults that occur according the next values of (F_{g_m}, F_{I_m}), namely, (0.12,1), (0.12,0.5), (1,1), (1,0.5) and (1.2,1), in this order. Notice that $\Delta = \{(0.12, 1), (0.12, 0.5)\}$. When $F_{g_m} = 0.12$ one has a actuator fault which explain the states $(0.12, .)$ be considered absorbent states. The following matrix of probability is adopted

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.05 & 0.90 & 0.05 & 0 \\ 0.05 & 0.05 & 0.85 & 0.05 \\ 0.20 & 0.05 & 0.05 & 0.70 \end{bmatrix}.$$

Assuming $m_0 = [\pi/8 \; 0 \; 0 \; 0]$ and $S_i = 100I$ for $i = \Delta, 1, 2, 3$ we present the control gains from the results, for the cases: (i) $\tau = T_N$ for $N = 3$ in Table 2, and (ii) $\tau = \tau_\Delta$ in Table 3. Figures 1 and 2 present the corresponding trajectories for $\|x_k\|$ and $\|\hat{x}_{k|k-1}\|$ as well as for trace of P^k ($tr\{P^k\}$).

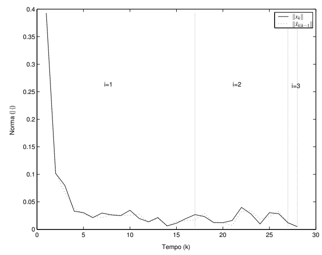


(a) Trajectories of $\|x_k\|$ and $\|\hat{x}_{k|k-1}\|$.

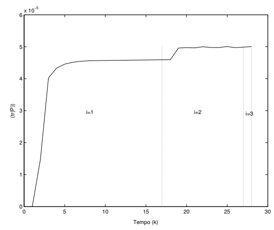


(b) Evolution of $tr\{P^k\}$.

Fig. 1. Typical trajectories for $\tau = T_3$.



(a) Trajectories of $\|x_k\|$ and $\|\hat{x}_{k|k-1}\|$.



(b) Evolution of $tr\{P^k\}$.

Fig. 2. Typical trajectories for $\tau = \tau_\Delta$.

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