STABILIZATION OF DYNAMIC VEHICLE FORMATION CONFIGURATIONS USING GRAPH LAPLACIANS

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Abstract: In this paper, stability of a formation of vehicles coupled through a communication pattern which changes over time is examined. This analysis uses the graph theoretic techniques developed by Fax and Murray (2002*a*) in their analysis of vehicle formations with static communication patterns. The graph Laplacian stability properties defined by Fax and Murray are extended to include dynamic sensing and communication graphs. This method is applied to a periodic communication pattern and stability is evaluated. Finally, these ideas are considered relative to optimal communication schemes, adding robustness to these schemes, and evaluating general periodic formation patterns.

Keywords: graph theory, communication networks, control system analysis

1. INTRODUCTION

Controlling a formation of vehicles has become a topic garnering much attention in the field of control theory due to both the complexity of the problem and its wide applicability. Formations of vehicles are being used throughout various commercial and military applications due to the potential benefits of having multiple vehicles available to complete a task. The stability of these formations with information coupling has been examined using various methods. These approaches have included, among other concepts, the use of potential functions (Loizou and Kyriakopoulos (2003)) and the use of nearest-neighbor rules (Jadbabaie *et al.* (2003)). Several of these methods involve representing the flow of information between vehicles as a graph. In practice, this flow of information can represent either sensing (vehicle *i* detects vehicle *j* and thus gains knowledge of its position/velocity) or communication (vehicle *j* transmits to vehicle *i* its state information). The goal is to use this information to create a local, or decentralized, controller which stabilizes the formation. This approach has proven to be a rich source of material for analysis as can be seen in Mesbahi and Hadaegh (2001), Fax and Murray (2002*a*), Fax and Murray (2002*b*) and Olfati-Saber and Murray (2002), among others.

In particular, Fax and Murray (2002*a*) found that the stability of a controller which uses information from other vehicles can be evaluated by examining a particular property of the graph representing the flow of information. This property is the graph's Laplacian, which corresponds to a measure of the graph's connectedness and will be shown to appear naturally in the controller equations.

In practical applications, the flow of information between vehicles can change drastically over time: communication windows may appear and disappear, vehicles may move in or out of each other's sensor range, communication equipment may fail, etc. Therefore, evaluating the stability of the formation subject to these changes in the information flow between vehicles becomes critical. Here, the results of Fax and

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Murray are considered and extended to this dynamic environment where the graph representing information flow will change with time. Sec. 2 contains a review of the static graph results derived by Fax and Murray. In Sec 3, these results are extended to generic dynamic communication, and in Sec. 4 these results are constrained to periodic communication sequences. An example is given in Sec. 5, and the paper concludes with a discussion of ongoing and future work.

2. STATIC COMMUNICATION PATTERNS

This section comprises a brief review of the results from Fax and Murray (2002*a*), where stability of formations is evaluated using graph Laplacians. These results will be extended to a dynamic graph in the next section. In both cases, consider a system of N vehicles, each with identical individual dynamics

$$\begin{aligned}\dot{x}_i = Ax_i + Bu_i,\ x_i \in \mathbb{R}^n,\ u_i \in \mathbb{R}^m \\ y_i = C_1 x_i,\ y_i \in \mathbb{R}^p.\end{aligned} \quad (1)$$

A distributed control algorithm will be used with the local controller for each vehicle (u_i) as follows:

$$\begin{aligned}\dot{v}_i &= Fv_i + G_1 y_i + G_2 z_i \\ u_i &= Hv_i + J_1 y_i + J_2 z_i.\end{aligned} \quad (2)$$

Here v_i is a dummy state which represents the dynamics associated with the controller. The z_i represent the information available to each vehicle regarding other vehicles and is defined as:

$$z_i = \frac{1}{|J_i|} \sum_{j \in J_i} C_2 (x_i - x_j) \quad (3)$$

where J_i is the set of vehicles for which vehicle i has information. In a graph representation, J_i is the set of nodes which point to vehicle i. A sample communications graph is shown in Fig. 1. This graph represents vehicles 2 and 3 transmitting their information to vehicle 1 (or being sensed by vehicle 1). In this example, $J_1 = \{2, 3\}$. This means that the control law for vehicle 1 can incorporate feedback of the state information from vehicles 2 and 3. The definition of the normalized graph Laplacian is given by

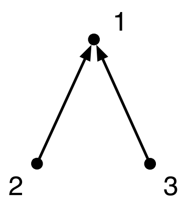

Fig. 1. Static communication/sensing graph.

$$L = I - D^{-1} A \quad (4)$$

where D is a diagonal matrix with d_{ii} equal to the in-degree of node i and A is the standard adjacency matrix of the graph. By using this definition, the Laplacian can be rewritten as:

$$L_{ij} = \begin{cases} 1 & i = j \\ -\dfrac{1}{|J_i|} & i \text{ receives information from } j \\ 0 & \text{all other } i, j. \end{cases} \quad (5)$$

Defining the total state vector for the system as the combination of the states for the i vehicles by writing $x = \left[x_1^T\ x_2^T\ \ldots\ x_N^T\right]^T$, the following relation holds:

$$z = C_2 L x. \quad (6)$$

Using these definitions, Fax and Murray went on to prove that stabilizing a formation of vehicles is equivalent to stabilizing the set of systems

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= C_1 x \\ z &= \lambda_i C_2 x\end{aligned} \quad (7)$$

where the λ_i are the eigenvalues of the Laplacian matrix.

3. DYNAMIC COMMUNICATION PATTERNS

Consider again a system of N vehicles with dynamics as defined in the previous section. However, the communications/sensing patterns will now be assumed dynamic rather than static, as shown in Fig. 2. In the previous section, J_i was defined as the set of vehicles about which vehicle i had information either from communication or from sensing. This same definition can be applied in the time-varying case, with an added dimension coming from traversing discrete time steps. Information is assumed to travel instantaneously at the beginning of each time step. For example in Fig. 2, vehicle 1 at time step 2 has information from vehicle 2 at time step 1 and from vehicle 3 at time step 2. Therefore, J_1 at time step 2 can be defined as

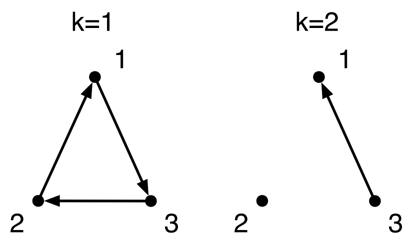

Fig. 2. Dynamic communication/sensing graph.

$$J_1(2) = \{x_2(1), x_3(2)\}$$

where $x_i(k)$ is the state of node i at time step k. Nodes at different time steps may then be treated as having independent states for the purpose of evaluating the behavior of the system. This approach leads to the creation of a “meta-graph” which describes the evolution of the information flow of the system. This concept is shown graphically in Fig. 3. This meta-graph represents the communication pattern from Fig. 2. In this

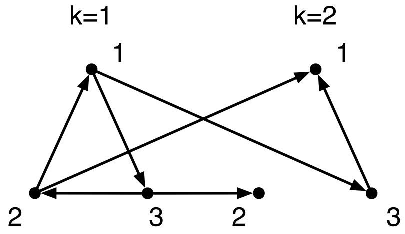

Fig. 3. Meta-graph combining graphs at $k = 1$ and $k = 2$.

example, there are 2 "copies" of each vehicle—one for each time step in the pattern. This will allow us to use the same procedure defined in the previous section to analyze this communication pattern. A similar strategy was used in Hristu and Morgansen (1999) to represent a time-varying communication pattern.

This approach leads to an augmentation of the static communication system due to the fact that behavior across time-steps is now included. This can be represented using a "meta-state" defined for each vehicle i as:

$$\tilde{x_i}(k) = \begin{bmatrix} x_i(k) \\ x_i(k-1) \\ \vdots \\ x_i(k-n) \end{bmatrix} \tag{8}$$

where n represents the length of the communication pattern and $x_i(k)$ represents the state of vehicle i at time step k. Similarly, one can define a new $\tilde{z}$ variable representing the error dynamics of the sytem. In the static case, z was defined as in Equation (3) to represent the information available to each node. Due to the properties of the meta-graph, a similar definition can be used but will encompass information from past time steps. In fact, it is important to note that the definition of $\tilde{z}$ in the dynamic case encompasses a buffering strategy. By defining what information is available in the current time step, it is implied that nodes have some sort of "buffer" to store previous information. For example, $\tilde{z}$ may be defined to encompass information from the current and previous time steps as shown below:

$$\tilde{z}_i(k) = \frac{1}{|J_i(k) + J_i(k-1)|} \left(\sum_{j \in J_i(k)} (x_i(k) - x_j(k)) + \sum_{j \in J_i(k-1)} (x_i(k) - x_j(k-1)) \right) \tag{9}$$

Equation (9) now represents the error dynamics at a step k. Note that the current error dynamics are considered as well as those from the previous time step. Analogously to the manner in which z can be written in terms of the Laplacian as shown in Equation (6), the dynamic $\tilde{z}$ can be written in terms of a new "meta-Laplacian". This meta-Laplacian is defined just as in Equation (4) but uses the new "meta-adjacency" matrix of the dynamic setting.

$$A_k = \begin{bmatrix} A(k) & A(k-1) & 0 & 0 \\ 0 & A(k-1) & A(k-2) & \vdots \\ 0 & 0 & \ddots & \ddots \\ 0 & 0 & 0 & A(k-n) \end{bmatrix} \tag{10}$$

Continuing with the example where only the current and previous time steps are considered, one can write the meta-adjacency graph as shown in Equation (10). Here, $A(j)$ represents the adjacency graph of the network topology at step j. This leads to the following definitions:

$$D_k = \begin{bmatrix} D(k) + D(k-1) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & D(k-n) + D(k-n-1) \end{bmatrix}$$
$$L_k = I - D_k^{-1} A_k$$
$$\tilde{z}(k) = L_k \tilde{x}(k) \tag{11}$$

Using this definition, it is clear that the new error states for a vehicle at step k depend on the state information it is receiving from other vehicles at step k as well as the state information it received at step $k-1$.

To further illustrate this idea, consider again the example shown in Fig. 3. Using the above definitions, the meta-Laplacian and $\tilde{z}$ are

$$L_2 = \begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\tilde{z}(2) = \begin{bmatrix} x_1(2) - \frac{1}{2}x_3(2) - \frac{1}{2}x_2(1) \\ x_2(2) - x_3(1) \\ x_3(2) - x_1(1) \\ x_1(1) - x_2(1) \\ x_2(1) - x_3(1) \\ x_3(1) - x_1(1) \end{bmatrix}$$

where $x_i(k)$ is the state of vehicle i at time step k. It is important to note that the definition of the meta-Laplacian will vary if different buffering schemes are used. In the next section, a periodic pattern will be examined in which all of the states for the current period are used. This results in a slightly different meta-Laplacian than the one used above. Whatever the definition of the meta-Laplacian, it must accurately portray the error dynamics and buffering scheme of the proposed dynamic communications sequence. If this is done, then Equation (7) can be extended to produce the following theorem.

Theorem 3.1. Consider a set of N vehicles with identical dynamics as defined in Equation (1) and a sequence of communication patterns between these vehicles represented by the set of graphs $\mathcal{G} = \{G_1, G_2, ..., G_k\}$. A candidate control stabilizes this system if and only if it stabilizes the set of systems:

$$\begin{aligned} \tilde{x} &= A_k\tilde{x} + B_k\tilde{u} \\ \tilde{y} &= C_k\tilde{x} \\ \tilde{z} &= \tilde{\lambda}_i C_{2_k}\tilde{x} \end{aligned} \tag{12}$$

where $\tilde{\lambda}_i$ are the eigenvalues of the matrix L_k as defined in Equation (11), $\tilde{x} = [\tilde{x}_1^T ... \tilde{x}_N^T]^T$ is defined as in Equation (8), and A_k, B_k and C_k are block-diagonal matrices with A, B and C repeated along the diagonal, respectively.

Proof: The proof of this theorem follows directly from the proof of Equation (7) given in Fax and Murray (2002*a*) using the "meta-states" and "meta-Laplacian" defined in this paper. This proof uses the fact that combining the vehicle dynamics for all N vehicles across all k time-steps with the controller dynamics leads to a block-diagonal system where the only non-diagonal element is the Laplacian matrix which defines the z_i values. This is true regardless of the specific meta-Laplacian definition used, given the constraint that it meets the definition of the Laplacian given in Equations (3) and (6). We therefore have

$$\frac{d}{dt}\begin{bmatrix}\tilde{x}\\ \tilde{v}\end{bmatrix} = \begin{bmatrix}\hat{A}_k + \widehat{B_k J_1 C_k} + \widehat{B_k J_2 C_{2_k}} L_k & \widehat{B_k H} \\ \widehat{G_1 C_k} + \widehat{G_2 C_{2_k}} L_k & \hat{F}\end{bmatrix}\begin{bmatrix}\tilde{x}\\ \tilde{v}\end{bmatrix} \tag{13}$$

The "hat" notation (e.g., $\hat{A}$) represents the given matrix repeated N times along the diagonal. In essence, the dynamics are augmented twice—once across k time steps and again over N vehicles. This means that if the original vehicle state vector contains n states, $\tilde{x}$ will contain $n \times k \times N$ states. Transforming the system such that the Laplacian is diagonalized leads to a system which is purely diagonal with the eigenvalues of the Laplacian being added to the dynamics. Therefore, stabilizing the formation across time steps is equivalent to stabilizing the systems defined in Theorem 3.1. ■

For the case where the individual vehicles are SISO, the results of Fax and Murray can be extended to show a stability requirement for the formation across time steps using the Nyquist plot. Using Theorem 2 from Fax and Murray (2002*a*), a new theorem for the dynamic communication case can be stated.

Theorem 3.2. Given individual vehicle dynamics $G(s)$ which are SISO, a controller $K(s)$ stabilizes the relative formation dynamics across time steps for a given communication sequence iff the net encirclement of $-\lambda_i^{-1}$ by the Nyquist plot of $K(s)G(s)$ is zero for all nonzero λ_i, where λ_i are the eigenvalues of the meta-Laplacian defined for the communication sequence.

Proof: This theorem follows directly from Theorem 3.1 and the fact that the key point of interest is in the encirclements of (-1,0) in the complex plane by $\lambda_i K(s)G(s)$. ■

By examining the eigenvalue locations for the graph of a communications sequence, Theorem 3.2 can predict stability for the system. By generalizing eigenvalue locations for certain types of graphs, one can derive an a priori understanding of whether the system will be stable for a given communications sequence. In Fax and Murray (2002*a*), Nyquist locations of the eigenvalues for certain types of graphs are given. Due to the fact that our meta-graph is simply another type of graph, the same types of meta-graphs will have eigenvalues in the same locations:

$$\lambda_i(L) = 1 - e^{\frac{2\pi j}{N} i} \tag{14}$$

Of these types, the most troubling from a stability standpoint is the periodic graph, which has eigenvalues located as shown in Equation (14). Here, N is the number of vehicles in the formation. These eigenvalues, once plotted as $-1/\lambda_i$, tend to live near the imaginary axis and can potentially reduce or eliminate stability margins. A periodic graph is defined as one in which all cycle lengths within the graph have a common divisor greater than 1. Therefore, as a meta-graph approaches a periodic graph, we can expect stability margins to decrease. This becomes an important consideration when dealing with periodic communication patterns as described in the next section.

4. PERIODIC COMMUNICATION PATTERNS

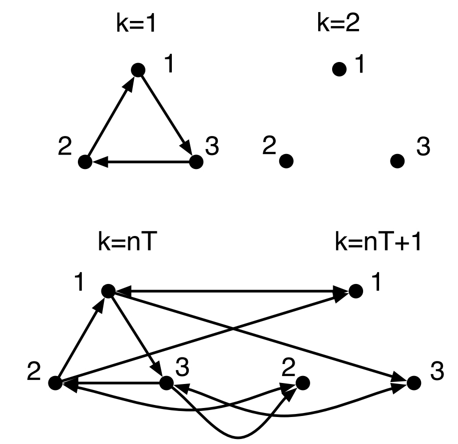

Fig. 4. Periodic communications pattern and its associated meta-graph.

As defined in the previous section, the meta-Laplacian grows without bound as the communication sequence continues. However, the meta-Laplacian can be described in a finite closed form for a periodic communication sequence. Because the sequence is periodic, the communication pattern is simply switching between a finite set of graphs representing each step in the sequence.

In Fig. 4, the communication pattern switches between two patterns, and the meta-graph can be fully represented with only six nodes. In this case, information flows back and forth between these two patterns as denoted by the bi-directional edges contained in the graph. This leads to a modified definition for the meta-Laplacian as shown below, where T represents the period of the sequence.

$$
\begin{aligned}
L_k &= I - D_k^{-1} A_k \\
D_k &= \begin{bmatrix} D(T) + ... + D(1) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & D(T) + ... + D(1) \end{bmatrix} \\
A_k &= \begin{bmatrix} A(T) & A(T-1) & \cdots & A(1) \\ \vdots & \vdots & \vdots & \vdots \\ A(T) & A(T-1) & \cdots & A(1) \end{bmatrix}
\end{aligned}
\tag{15}
$$

As mentioned in Sec. 3, periodic graphs can lead to decreased stability margins. For meta-graphs describing a periodic communications sequence as shown in Fig. 4, we will naturally have cycles which have a length equal to the period of the sequence. In order to maintain stability margins, it will become important to ensure that individual communication patterns do not contain cycles with a length equal to an integer multiple of the period of the overall sequence. As all cycles in the meta-graph tend toward integer multiples of each other, the entire meta-graph becomes more periodic and the eigenvalues of the meta-Laplacian will migrate toward the imaginary axis. This may lead to an unstable system once the eigenvalues cross over the Nyquist plot for the dynamics of an individual vehicle as can be seen in the unstable example given in Sec. 5.

5. MOTIVATING EXAMPLE

In order to verify the above results, consider building a periodic sequence from stable and unstable static communication patterns. Depending on the relative instability, one should be able to alternate between stable and unstable patterns to form an overall sequence which maintains the stability of the formation. The eigenvalues of the meta-Laplacian for the sequence will predict the stability of the system.

Consider the stable and unstable communication networks of six vehicles as given in Fax and Murray (2002*a*) with individual dynamics $P(s) = e^{-s}\frac{1}{s^2}$. A controller which stabilizes the individual vehicles is $K(s) = 0.6s + 0.15$. Two communications networks, one stable and one unstable, are shown in Fig. 5. Fig. 6 shows the Nyquist plot for the system $K(s)P(s)$ along with the negative inverse eigenvalues, $-\lambda_i^{-1}$, of the Laplacian for each communication pattern as required by Theorem 3.2. The 'o' eigenvalues denote the stable communications pattern while the 'x' values are from the unstable communications pattern. Consider two communications sequences built from the given

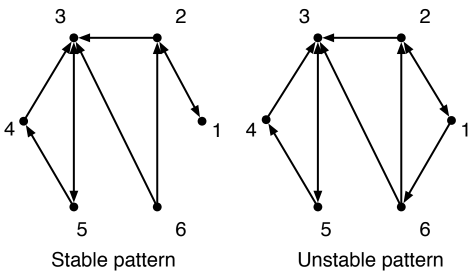

Fig. 5. Stable and unstable communication pattern from Fax and Murray.

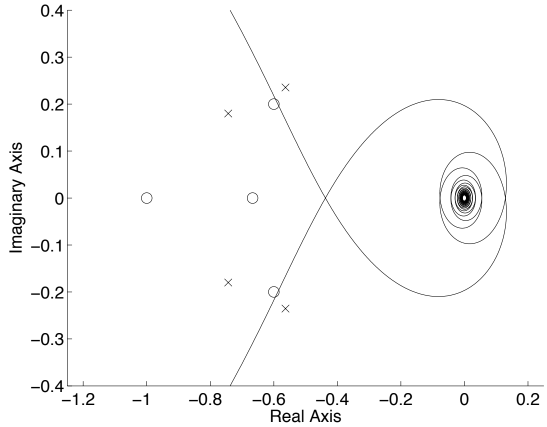

Fig. 6. Nyquist plot with static Laplacian eigenvalues.

patterns. The first is composed of seven instances of the stable network followed by an instance of the unstable network. The meta-Laplacian eigenvalues for

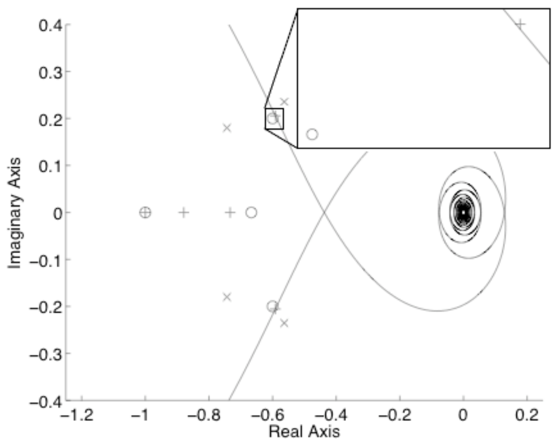

Fig. 7. Meta-Laplacian eigenvalues for the sequence of 7 stable networks followed by an unstable one.

the sequence are shown in Fig. 7, represented by a '+'. From the detail box in the upper right corner, it is clear that the system should be marginally unstable. A simulation was built which used this communication sequence and buffering strategy defined by the periodic meta-Laplacian. The response of the system is shown in Fig. 8. Here, the positions of all six vehicles are superimposed. One can see that the vehicles oscillate around each other with increasing amplitude.

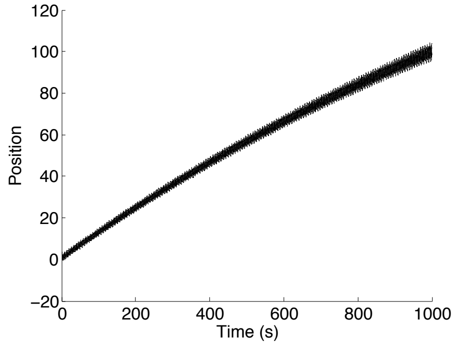

Fig. 8. System response for the sequence of 7 stable networks followed by an unstable one.

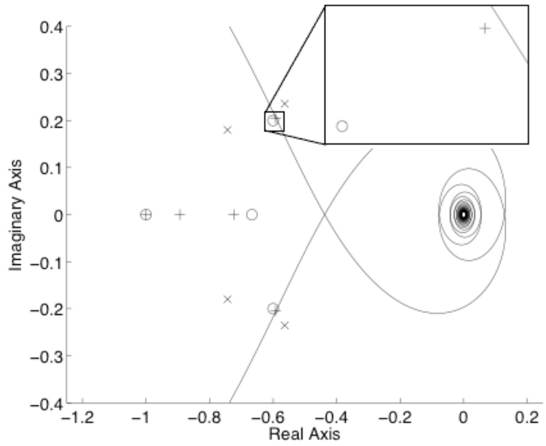

Fig. 9. Meta-Laplacian eigenvalues for the sequence of 8 stable networks followed by an unstable one.

It is clear that the z_i variables are unbounded and will tend to infinity as time increases. Therefore, the formation is unstable and the simulation agrees with the prediction based on the eigenvalues.

Consider now the sequence of eight stable graphs followed by an unstable one. The eigenvalues for this meta-Laplacian are shown in Fig. 9. From the detail box in the upper right corner, it is clear that the eigenvalues have crossed to the other side of the Nyquist plot and the system should now be stable. The system response for the simulation of this communication pattern is shown in Fig. 10. In this case, the positions of the six vehicles differ by a decreasing amount. Therefore, the z_i variables will eventually go to zero (albeit slowly) and the formation is stable as predicted. In fact, the slow rate of convergence is also expected due to the marginal stability of the eigenvalues as shown in Fig. 9. In this manner, one can see how the meta-Laplacian can be used as a tool to construct dynamic communication sequences which remain stable although individual communication patterns may be unstable.

6. CONCLUSIONS AND FUTURE WORK

One can envision examining both optimality and robustness using these methods. Using Theorem 3.1 as a common characteristic that all stable control and communication pattern pairs must have, it may be possible to work backwards and determine a minimally communicative case. This could lead to potential improvements in system robustness and/or efficiency. Finally, it may be possible to compensate for communication or vehicle failure by modifying communication patterns such that the eigenvalues of L_k remain stable and Theorem 3.1 still holds. Finding an algorithm which would allow a formation to auto-stabilize in these situations would be ideal and could prove very useful. In addition, these results could potentially be extended to heterogeneous vehicle formations where each vehicle has its own unique dynamics.

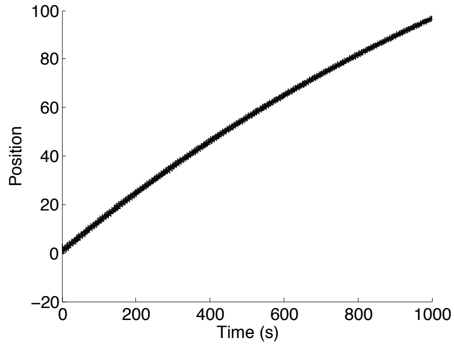

Fig. 10. System response for the sequence of 8 stable networks followed by an unstable one.

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