VSS ERROR FEEDBACK REGULATOR FOR LINEAR SYSTEMS

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Abstract: This paper focuses on the design of an error feedback sliding mode regulator able to achieve the asymptotic tracking of a reference trajectory for linear systems. It is assumed that the reference trajectory is generated by means of neutrally stable unforced systems. The solution conditions are derived for linear systems presented in general, Regular and Block controllable forms.

Keywords: Sliding-mode control, Regulator theory, Observers.

1. INTRODUCTION[1]

The regulator problem, in the classical setup (Francis, 1977), consists of designing a continuous state or error feedback controller such that the output of a system tracks a reference signal possibly in the presence of a disturbance signal. An alternative approach to deal with this problem is the use of the sliding mode technique to decompose and simplify the regulator design procedure and impose robustness properties (Utkin and Young, 1978).

The underlying idea is to design a sliding surface on which the dynamics of the system are constrained to evolve by means of a discontinuous control law, instead of designing a continuous stabilizing feedback, as in the case of the classical regulator problem. The sliding manifold contains the steady-state surface, and the dynamics of the systems tend asymptotically, along the sliding manifold to the steady-state behavior.

In the case of static state feedback sliding mode, a regulator design has been investigated in (Loukianov, *et al.*, 1999a) and (Utkin V.A., 2001). To overcome the limiting requirement of full information knowledge, a dynamic discontinuous error feedback strategy has been designed for linear systems in general case in (Edwards and Spurgeon, 1996) and for Regular form in (Loukianov, *et al.*, 1999b). However, the design of the error feedback sliding mode Regulator for linear systems in the Block Controllable form (Drakunov, *et al.*, 1990) is an open problem.

In this paper, the stability condition of the sliding mode error feedback regulator is reformulated. Instead of a closed loop asymptotic stability condition of the equilibrium point as presented in (Loukianov, *et al.*, 1999b), two stability conditions are introduced: a sliding mode stability condition that requires finite time convergence to the proposed

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sliding manifold and an asymptotic stability condition of the sliding mode dynamics of the equilibrium point. With these issues in mind, the error feedback sliding mode regulator for linear systems in the general case and in the so-called Regular form are reformulated and for the Block Controllable form for linear systems is designed.
To formalize the ideas, the basic facts on regulation theory are briefly recalled. Consider a linear system

$$\dot{x} = Ax + Bu + Dw \quad (1)$$
$$y = Cx \quad (2)$$

where $x \in R^n$, $u \in R^m$ and $y \in R^p$. The output tracking error is defined as

$$e = Cx - Qw \quad (3)$$

where the reference signal, $y_{ref} = Qw$, is generated by a given external system described by

$$\dot{w} = Sw \quad (4)$$

with state $w \in R^S$. This system is characterized by the following assumption:

H1). The matrix S has all eigenvalues on the imaginary axis.

It is assumed also that only the components of the error *e* are available for measurement. It has been shown that the control action to (1) can be provided by an error feedback dynamic system (Francis, 1977):

$$\dot{\xi} = F\xi + Ge \quad (5)$$
$$u = [K \quad (\Gamma - K\Pi)]\xi \quad (6)$$

and the solvability of the Error Feedback Regulator Problem (EFRP) under assumption **H1**, can be stated in terms of the existence of a pair of matrices Π and Γ that solve the Sylvester matrix equation

$$A\Pi + B\Gamma + D = \Pi S \quad (7)$$
$$C\Pi - Q = 0 \quad (8)$$

In fact, the conditions (7) and (8) are added by the following trivially necessary conditions:

H2). The pair $\{A,B\}$ is stabilizable and

H3). The pair $\left\{\begin{bmatrix} C & Q \end{bmatrix}, \begin{bmatrix} A & D \\ 0 & S \end{bmatrix}\right\}$ is detectable.

In the following the regulator problem from a sliding mode viewpoint is presented. We define the problem and give the conditions for the existence of a solution.

2. ERROR FEEDBACK SLIDING MODE CONTROL PROBLEM

Analogously to EFRP, the ***Error Feedback Sliding Mode Regulation Problem*** (EFSMRP) is defined as the problem of finding a sliding manifold

$$\sigma(\xi) = 0, \qquad \sigma \in R^m \quad (9)$$

and a discontinuous controller

$$\dot{\xi} = \eta(\xi, u, e) \quad (10)$$

$$u_i = \begin{cases} u_i^+(\xi) & \sigma_i(\xi) > 0 \\ u_i^-(\xi) & \sigma_i(\xi) < 0 \end{cases} \qquad i = 1,\dots,m \quad (11)$$

where $u_i^+(\xi)$, $u_i^-(\xi)$ and $\sigma_i(\xi)$, $i = 1,\dots,m$ are chosen so that the following conditions hold:

- **(SMS)**. (*Sliding Mode Stability*). The state of the closed-loop system formed from closing the loop in the system (1)-(2), with the controller (9)-(11) converges to the manifold (9) in a finite time.
- **(SS)**. The equilibrium point $x = 0$ of the sliding mode dynamics

  $$\dot{x} = Ax + Bu_{eq} + Dw, \sigma(\xi) = 0 \quad (12)$$
  $$\dot{\xi} = \eta(\xi, u_{eq}, e) \quad (13)$$

  is asymptotically stable, where u_{eq} is the equivalent control defined as a solution of $\dot{\sigma} = 0$ (Utkin and Young, 1978).
- **(SR)**. The output tracking error goes asymptotically to zero, i.e.

  $$\lim_{t\to\infty} e(t) = 0 . \quad (14)$$

Note, that the conditions for sliding motion to occur on $\sigma_i(\xi) = 0$ may be stated in numerous ways. We need $\lim_{\sigma_i \to 0^+} \dot{\sigma}_i < 0$ and $\lim_{\sigma_i \to 0^-} \dot{\sigma}_i > 0$ in the neighborhood $\sigma_i(\xi) = 0$, $i = 1,\dots,m$, (Utkin and Young, 1978). In the following, a solution for this problem will be presented.

3. SOLVABILITY CONDITIONS

Analogously to EFRP, in this section, the EFSMRP solvability conditions will be derived for linear systems in the general form. Considering the linear system (1) - (3), the steady-state error is defined as $z = x - \Pi w$ where Π is a matrix to be defined later and thus rewrite the original equations as

$$\dot{\zeta} = \bar{A}\zeta + \bar{B}u \quad (15)$$
$$e = \bar{C}\zeta \quad (16)$$

with $\zeta = \begin{pmatrix} z \\ w \end{pmatrix}, \bar{A} = \begin{pmatrix} A & A\Pi - \Pi S + D \\ 0 & S \end{pmatrix}, \bar{B} = \begin{pmatrix} B \\ 0 \end{pmatrix}$

$\bar{C} = \begin{pmatrix} C & (C\Pi - Q) \end{pmatrix}$, $rankB = m$. Then the system (10) can be designed in this case as an observer for ζ. For asymptotic stabilization of the closed-loop system via error feedback the following assumption is introduced:

H4. The pair $\{C, A\}$ is detectable.

Under this assumption, the system (10) with state $\xi = (\hat{z}, \hat{w})^T$ is designed as the observer:

$$\dot{\xi} = \bar{A}\xi + \bar{B}u + L(e - \hat{e}),\ e = \bar{C}\xi \tag{17}$$

where ξ is the estimate of $\zeta = (z, w)^T$, and the matrix $L = (L_1, L_2)^T$ is chosen to stabilize the error dynamics:

$$\dot{\varepsilon} = \left(\bar{A} - L\bar{C}\right)\varepsilon \tag{18}$$

with $\varepsilon = \xi - \zeta = (\varepsilon_1, \varepsilon_2)^T$. Once the observer is designed, a sliding manifold $\hat{\sigma}(\xi) = 0$ has to be chosen to satisfy the stability conditions. To this end, the sliding manifold is chosen as

$$\hat{\sigma}(\xi) = \begin{pmatrix}\Sigma & 0\end{pmatrix}\xi = \Sigma\hat{z} = 0 \tag{19}$$

where an appropriately chosen design matrix Σ will determine the dynamic response of the system on (19). To investigate the stability on this sliding manifold, the following lemma is first proved:

Lemma 1. Let the operator P be defined as $P = [I_n - B(\Sigma B)^{-1}\Sigma]$. Then the relation

$$P(A\Pi - \Pi S + D) = 0 \tag{20}$$

is true if and only if there are matrices Π and Λ, such that

$$A\Pi - \Pi S + D = B\Lambda \tag{21}$$

Proof of Lemma1. The operator P is a projection operator along the space of the rank of B over the Σ null space, i.e., $PB = [I_n - B(\Sigma B)^{-1}\Sigma]B = 0$, $Pz = z \quad \forall z \in \mathrm{N},\ \mathrm{N} = \{z \in R^n \mid \Sigma z = 0\}$.

Thus, if condition (21) holds, then it follows that $P(A\Pi - \Pi S + D) = PB\Sigma = 0$. Conversely, if condition (20) is satisfied, then (AΠ-ΠS+D) must be in the image of B, i.e., $A\Pi - \Pi S + D = B\Lambda$ for some matrix Λ. From this result, a condition for a solution of the discontinuous regulator problem can be deduced.

Proposition 1. Suppose that assumptions **H1**, **H2** and **H**4 hold, and there exists a matrix Π which solves the linear equations

$$A\Pi - \Pi S + D = B\Lambda \tag{22}$$

$$C\Pi - Q = 0 \tag{23}$$

for some matrix Λ. Then the EFSMRP for linear system is solvable.

Proof of proposition 1 . Choose the control u as

$$\begin{aligned} u &= k(\Sigma B)^{-1} sign(\hat{\sigma}) \\ \hat{\sigma} &= \Sigma\hat{z},\ k > 0 \end{aligned} \tag{24}$$

Using the Lyapunov function $V = \frac{1}{2}\hat{\sigma}^T\hat{\sigma}$, from the derivative of V taken along the trajectories of (17) and (24) the condition $k > \left\|(\Sigma B)u_{eq}\right\|$ is obtained, that guarantees the (**SMS**) condition. The equivalent control u_{eq} is calculated from $\dot{\hat{\sigma}} = 0$ as

$$\begin{aligned} u_{eq} &= -(\Sigma B)^{-1}\Sigma\begin{bmatrix} A\hat{z} + (A\Pi - \Pi S + D)\hat{w} \\ +L_1 C\varepsilon_1 \end{bmatrix} \\ &= -(\Sigma B)^{-1}\Sigma\begin{bmatrix} Az + (A\Pi - \Pi S + D)w \\ -(A - L_1 C)\varepsilon_1 \\ -((A\Pi - \Pi S + D) \\ -L_1(C\Pi - Q))\varepsilon_2 \end{bmatrix}. \end{aligned} \tag{25}$$

The reduced order sliding mode dynamics on $\hat{\sigma} = \sigma - \Sigma\varepsilon_1 = 0$ are obtained by substituting (25) in (15), to yield:

$$\dot{\zeta} = \tilde{A}\zeta + E\varepsilon,\ \ \Sigma(z - \varepsilon_1) = 0 \tag{26}$$

$$\dot{\varepsilon} = \left(\bar{A} - L\bar{C}\right)\varepsilon \tag{27}$$

$$e = \begin{pmatrix} C & (C\Pi - Q) \end{pmatrix}\zeta \tag{28}$$

where $\tilde{A} = \begin{pmatrix} PA & R \\ 0 & S \end{pmatrix}, E = \begin{pmatrix} (I_n - P)E_{11} & (I_n - P)E_{12} \\ 0 & 0 \end{pmatrix}$,
$E_{11} = (A - L_1 C), E_{12} = \begin{pmatrix} (A\Pi - \Pi S + D) \\ -L_1(C\Pi - Q) \end{pmatrix}$, with P already defined in Lemma 1, and $R = P(A\Pi - \Pi S + D)$. Using condition (22) and Lemma 1 it yields that R=0. Then, assuming that the observer estimation error decays rapidly by appropriate choice of L_1 and L_2 (under assumption **H4**), equation (26) reduces to

$$\dot{z} = PAz|_{\Sigma z = 0} \tag{29}$$

Since the matrix Σ in (29) by assumption **H2** can be chosen such that (ΣB) is invertible, and the (n-m) eigenvalues of PA are arbitrarily placed in C^- (Chesawi, *et al*., 1983), then $z(t) \to 0$ as $t \to \infty$, satisfying condition (**SS**). Now, if the tracking error equation (28) satisfies condition (23), then, $e(t) \to 0$ as $t \to \infty$, satisfying condition (**SR**). Comparing the conditions (7) and (22), the steady state matrices Π and Γ for the state x and control u, respectively, in equation (7) have to be calculated. On the contrary, in the second case (22) only the matrix Π needs to be calculated such that the perturbation satisfies the matching condition (Dragenovic, 1969). The structure of equation (22) can be put in evidence using the decomposition of a linear system to Regular form.

4. **REGULAR FORM**

In order to show the explicit form of condition (22) and sliding dynamics (29), the linear system (1) is first converted into Regular form (Utkin and Young, 1978):

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}=\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}+\begin{pmatrix}0\\ B_2\end{pmatrix}u+\begin{pmatrix}D_1\\ D_2\end{pmatrix}w \quad (30)$$
$$e = C_1x_1 + C_2x_2 - Qw \quad (31)$$

where $x_1 \in R^{n-m}$, $x_2 \in R^m$, and $rankB_2 = m$. See (Utkin and Young, 1978) for details of matrices appearing in (30) and (31). Defining $z_1 = x_1 - \Pi_1 w$ and $z_2 = x_2 - \Pi_2 w$ with Π_1 and Π_2 constant matrices of proper dimension, the system (30) in the new variables z_1 and z_2 obeys the following dynamics:

$$\begin{pmatrix}\dot{z}_1\\ \dot{z}_2\end{pmatrix}=\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}\begin{pmatrix}z_1\\ z_2\end{pmatrix}+\begin{pmatrix}0\\ B_2\end{pmatrix}u+\begin{pmatrix}R_1\\ R_2\end{pmatrix}w \quad (32)$$
$$e = C_1z_1 + C_2z_2 + (C_1\Pi_1 + C_2\Pi_2 - Q)w \quad (33)$$

with $R_1 = A_{11}\Pi_1 + A_{12}\Pi_2 - \Pi_1 S + D_1$ and $R_2 = A_{21}\Pi_1 + A_{22}\Pi_2 - \Pi_2 S + D_2$. Now, the state in system (10) is chosen as $\xi = (\hat{z}_1, \hat{z}_1\hat{w})^T$, therefore, system (10) takes the following form:

$$\dot{\xi} = A'\xi + B'u + L'(e - \hat{e}), \; \hat{e} = C'\xi \quad (34)$$

where

$$A' = \begin{pmatrix}A_{11} & A_{12} & R_1\\ A_{21} & A_{22} & R_2\\ 0 & 0 & S\end{pmatrix}, B' = \begin{pmatrix}0\\ B_2\\ 0\end{pmatrix}, L' = \begin{pmatrix}L_1\\ L_2\\ L_3\end{pmatrix},$$

ξ is the estimate of $\zeta = (z_1, z_2, w)^T$, and $C = \begin{pmatrix}C_1 & C_2 & (C_1\Pi_1 + C_2\Pi_2 - Q)\end{pmatrix}$. The observer gain matrix L' is chosen to stabilize the observer error state $\varepsilon = \xi - \zeta = (\varepsilon_1, \varepsilon_2, \varepsilon_3)^T$, a dynamics of which are governed by

$$\dot{\varepsilon} = (A' - L'C')\varepsilon\,. \quad (35)$$

The following assumption is thus necessary to guarantee the stability of the system (35):

H5). The pair $\{C', A'\}$ is detectable.

Proposition 2. Suppose that assumptions **H1**, **H2** and **H5** hold, and there exist matrices Π_1 and Π_2 which solve the linear equations

$$A_{11}\Pi_1 + A_{12}\Pi_2 - \Pi_1 S + D_1 = 0 \quad (36)$$
$$C_1\Pi_1 + C_2\Pi_2 - Q = 0\,. \quad (37)$$

Then the EFSMRP for linear system in the Regular form is solvable.

Proof of proposition 2. We first specify the sliding manifold (19) in terms of the estimated states as $\hat{\sigma} = \hat{z}_2 - \Sigma_1\hat{z}_1 = z_2 - \Sigma_1 z_1 - (\varepsilon_2 - \Sigma_1\varepsilon_1) = 0$ where $\Sigma_1 \in R^{m\times(n-m)}$. The proposed sliding control law is given as $u = k(B_2)^{-1}sign(\hat{\sigma}),\; k > 0$. Then the requirement (**SMS**) is fulfilled if $k > \left\|B_2 u_{eq}\right\|$, where u_{eq} is calculated from $\dot{\hat{\sigma}} = 0$ and has the following form:

$$u_{eq} = -(B_2)^{-1}\begin{bmatrix} -\Sigma_1\begin{pmatrix}A_{11}\hat{z}_1 + A_{12}\hat{z}_2 + R_1\hat{w}\\ +L_1C'\varepsilon\end{pmatrix}\\ +\begin{pmatrix}A_{21}\hat{z}_1 + A_{22}\hat{z}_2 + R_2\hat{w}\\ +L_2C'\varepsilon\end{pmatrix}\end{bmatrix}$$
$$= -(B_2)^{-1}\begin{bmatrix} -\Sigma_1\begin{pmatrix}A_{11}z_1 + A_{12}z_2 + R_1 w\\ +(L_1C' - G_1)\varepsilon\end{pmatrix}\\ +\begin{pmatrix}A_{21}z_1 + A_{22}z_2 + R_2 w\\ +(L_2C' - G_2)\varepsilon\end{pmatrix}\end{bmatrix}$$

with $G_1 = \begin{pmatrix}A_{11} & A_{12} & R_1\end{pmatrix}$ and $G_2 = \begin{pmatrix}A_{21} & A_{22} & R_2\end{pmatrix}$. By condition (36) it follows that $R_1 = 0$ in (32), therefore, the reduced order sliding mode equation can be obtained as

$$\dot{z}_1 = A_{11}z_1 + A_{12}z_2,\; z_2 = \Sigma_1 z_1 + (\varepsilon_2 - \Sigma_1\varepsilon_1) \quad (38)$$
$$\dot{w} = Sw\,,\; \dot{\varepsilon} = (A' - L'C')\varepsilon$$
$$e = (C_1 - C_2\Sigma_1)z_1 + (C_1\Pi_1 + C_2\Pi_2 - Q)w \quad (39)$$

It is known (Utkin and Young, 1978) that if the pair $\{A,\, B\}$ is stabilizable then the pair $\{A_{11}, A_{12}\}$ is stabilizable as well. Therefore there exists a matrix Σ_1 such that the matrix $(A_{11} + A_{12}\Sigma_1)$ in (38) is stable and hence $z_1(t)$ asymptotically tends to zero, satisfying condition (**SS**). In consequence, thanks to condition (37) the output tracking error $e(t)$ in (39) tends to zero too and condition (**SR**) is satisfied. Note that the conditions (22) and (23) are modified as (36) and (37), respectively. On the other hand, the equation (36) as well as the system (38) can be further decomposed if the system (1) or (30) is represented in BC-form.

5. BLOCK CONTROLLABLE FORM

In this section a discontinuous regulator is proposed using the Block Control technique (Drakunov, *et al*., 1990). The underlying idea is first to reduce system (1) to a Block Controllable form (BC-form) in the presence of perturbations by means of a non singular transformation, then, using the BC-technique, to design a sliding surface on which the unperturbed part of the dynamics of the system is stable. Finally, the condition for the solution of the corresponding EFSMRP is derived. The essential feature of the proposed method is the transformation of (1) into BC-form consisting of *r* blocks of the form:

$$\dot{x}_1 = A_{11}x_1 + B_1x + D_1w$$
$$\dot{x}_i = \sum_{j=1}^{i} A_{ij}x_j + B_ix_{i+1} + D_iw,\; i = 2,\ldots,r-1 \quad (40)$$
$$\dot{x}_r = \sum_{j=1}^{r} A_{rj}x_j + B_ru + D_rw$$

$$e = \sum_{k=1}^{r} M_k x_k - Qw \quad (41)$$

where the transformed vector $\bar{x}$ is decomposed as $\bar{x} = (x_1, ..., x_r)^T$, and $x_i \in R^{n_i}$, $i = 1,...,r$. The integers $(n_1, ..., n_r)$ characterize the structure of the system (40) by the condition $(n_1 \le n_2 \le \cdots \le n_r \le m)$ with $\sum_{i=1}^{r} n_i = n$. See (Drakunov, *et al.*, 1990) for details of matrices appearing in (40) and (41). It was shown that a necessary condition to transform the system (1) into BC-form (40), is that the pair $\{A, B\}$ must be controllable (Drakunov, *et al.*, 1990).

Introducing the steady-state $\Gamma_i w$ for the state vectors x_i the steady-state error z_i is defined as

$$z_i = x_i - \Gamma_i w, \quad i = 1,...,r \quad (42)$$

Then, the states in (42) are obtained from the evolution of (40) of the following form:

$$\dot{z}_1 = A_{11} z_1 + B_1 z_2 + R_1 w$$

$$\dot{z}_i = \sum_{j=1}^{i} A_{ij} z_j + B_i z_{i+1} + R_i w, \ \ i = 2,...,r-1 \quad (43)$$

$$\dot{z}_r = \sum_{j=1}^{r} A_{rj} z_j + B_r u + R_r w$$

with $e = \sum_{k=1}^{r} M_k z_k + \left(\sum_{k=1}^{r} M_k \Gamma_k - Q \right) w$, where

$$R_1 = A_{11}\Gamma_1 + A_{12}\Gamma_2 + D_1 - \Gamma_1 S,$$

$$R_i = \sum_{j=1}^{i} A_{ij}\Gamma_j + B_i \Gamma_{i+1} + D_i - \Gamma_i S, \ \ i = 2,...,r \quad (44)$$

The system (10) with state $\xi = (\hat{z}_1, ..., \hat{z}_r, \hat{w})^T$ is designed as follows:

$$\dot{\hat{z}}_1 = A_{11}\hat{z}_1 + B_1 \hat{z}_2 + L_1(e - \hat{e})$$

$$\dot{\hat{z}}_i = \sum_{j=1}^{i} A_{ij}\hat{z}_j + B_i \hat{z}_{i+1} + L_i(e - \hat{e}), \ \ i = 2,...,r-1$$

$$\dot{\hat{z}}_r = \sum_{j=1}^{r} A_{rj}\hat{z}_j + B_r u + R_r \hat{w} + L_r(e - \hat{e})$$

$$\dot{\hat{w}} = S\hat{w} + L_{r+1}(e - \hat{e}) \quad (45)$$

with $\hat{e} = \sum_{k=1}^{r} M_k \hat{z}_k + \left(\sum_{k=1}^{r} M_k \Gamma_k - Q \right) \hat{w}$, where $\xi = (\hat{z}_1, ..., \hat{z}_r, \hat{w})^T$ is the estimate of $\zeta = (z_1, ..., z_r, w)^T$, and $\tilde{L} = (L_1, ..., L_r, L_{r+1})^T$ is the observer gains matrix. Assuming that $R_i = 0, \ i = 1,...,r-1$, (44), then the observer error state $\varepsilon = (\varepsilon_1, ..., \varepsilon_r, \varepsilon_{r+1})^T$ obeys the following dynamics:

$$\dot{\varepsilon} = (\tilde{A} - \tilde{L}\tilde{C})\varepsilon \quad (46)$$

with $\tilde{A} = \begin{pmatrix} A_{11} & B_1 & 0 & \cdots & 0 & 0 \\ A_{21} & A_{22} & B_2 & \cdots & 0 & 0 \\ \vdots & & & \vdots & & \\ A_{r1} & A_{r2} & A_{r3} & \cdots & A_{rr} & R_r \\ 0 & 0 & 0 & \cdots & 0 & S \end{pmatrix}$,

$\tilde{C} = \begin{pmatrix} M_1 & \cdots & M_r & \left(\sum_{k=1}^{r} M_k \Gamma_k - Q \right) \end{pmatrix}$.

Similar to the previous case it is assumed that:

H6). The pair $\{\tilde{C}, \tilde{A}\}$ is detectable.

Proposition 3. Suppose that assumptions **H**2 and **H**6 hold, and there exist matrices Γ_i, $i = 1,...,r$ that solve the linear equations

$$A_{11}\Gamma_1 + B_1\Gamma_2 + D_1 = \Gamma_1 S, \quad (47)$$

$$\sum_{j=1}^{i} A_{ij}\Gamma_j + B_i\Gamma_{i+1} + D_i = \Gamma_i S, \ \ i = 2,...,r-1 \quad (48)$$

$$\sum_{k=1}^{r} M_k \Gamma_k - Q = 0 \quad (49)$$

Then the EFSMRP for a linear system in the BC-form is solvable.

Proof of proposition 3. Note first that if conditions (47)-(48) met, then, $R_i = 0, \ i = 1,...,r-1$ in (43), yielding to the same observer error dynamics founded in (46), which under assumption **H6**, can be stabilized by a proper choice of $\tilde{L}$.

A sliding manifold will be designed based on the system (45) considering the state $\hat{z}_{i+1}$, $i = 1,...,r-1$ as a fictitious control vector in the i^{th} block of (45), and the term $L_i(e - \hat{e})$ as the perturbation. This procedure is outlined as follows.

We start by defining a new variable $\chi_1 = \hat{z}_1$. Taking the derivative of χ_1 along (45) yields

$$\dot{\chi}_1 = A_{11}\hat{z}_1 + B_1\hat{z}_2 + L_1(e - \hat{e}) \quad (50)$$

As mentioned above, $\hat{z}_2$ is considered as a quasi-control in (50), and must force the desired dynamics, $K_1\chi_1$ with design stable matrix K_1 for this block by the anticipation of its dynamics of the following form:

$$\dot{\chi}_1 = A_{11}\hat{z}_1 + B_1\hat{z}_2 + L_1(e - \hat{e}) = K_1\chi_1 \quad (51)$$

Now, $\hat{z}_2$ is calculated from (51) as a desired state named $\hat{z}_2^d$. This desired state has the following form: $\hat{z}_2^d = -B_1^+(A_{11}\hat{z}_1 + L_1(e - \hat{e}) - K_1\chi_1)$, where $B_1^+ = B_1^T(B_1B_1^T)^{-1}$ denotes the right pseudo-inverse matrix of B_1. Proceeding in the same way, a second new variable χ_2 is defined as $\chi_2 = \hat{z}_2 - \hat{z}_2^d$. Taking the derivative of χ_2 and anticipating its dynamics, the next block is obtained

$$\dot{\chi}_2 = A_{21}\hat{z}_1 + A_{22}\hat{z}_2 + B_2\hat{z}_3 + L_2(e-\hat{e}) - \dot{\hat{z}}_2^d = K_2\chi_2 \quad (52)$$

The desired state of $\hat{z}_3$ is calculated from (52) as $\hat{z}_3^d = -B_2^+ \begin{pmatrix} A_{21}\hat{z}_1 + A_{22}\hat{z}_2 + \hat{z}_3 \\ +L_2(e-\hat{e}) - \dot{\hat{z}}_2^d - K_2\chi_2 \end{pmatrix}$, with $B_2^+ = B_2^T(B_2B_2^T)^{-1}$, and K_2 is a Hurwitz matrix. This procedure may be performed iteratively defining the i^{th} new state as $\chi_i = \hat{z}_i - \hat{z}_i^d$, and the i^{th} block as follows: $\dot{\chi}_i = \sum_{j=1}^{i} A_{ij}\hat{z}_j + B_i\hat{z}_{i+1} + L_i(e-\hat{e}) - \dot{\hat{z}}_i^d = K_i\chi_i$, $i = 3,...,r-1$, and the desired state as $\hat{z}_{i+1}^d = -B_i^+\left(\sum_{j=1}^{i} A_{ij}\hat{z}_j + L_i(e-\hat{e}) - \dot{\hat{z}}_i^d - K_i\chi_i\right)$ where, again, $B_i^+ = B_i^T(B_iB_i^T)^{-1}$, and K_i is a Hurwitz matrix. In the final step, $\hat{z}_r^d$ is known, and defining the last new variable $\chi_r = \hat{z}_r - \hat{z}_r^d$, the r^{th} block is transformed as follows: $\dot{\chi}_r = \sum_{j=1}^{r} A_{rj}\hat{z}_j + B_r u + L_r(e-\hat{e}) - \dot{\hat{z}}_r^d$. It should be noted that the new state $\chi = (\chi_1,...,\chi_r)^T$ is derived by the nonsingular transformation:

$$\chi_1 = \hat{z}_1,\ \chi_i = \hat{z}_i - \hat{z}_i^d,\ i = 2,...,r \quad (53)$$

This transformation simplifies system (45) to the following form:

$$\begin{aligned} \dot{\chi}_1 &= K_1\chi_1 + B_1\chi_2 \\ \dot{\chi}_i &= K_i\chi_i + B_i\chi_{i+1}, \quad i = 2,...,r-1 \\ \dot{\chi}_r &= \sum_{j=1}^{r} A_{rj}\hat{z}_j + B_r u + R_r\hat{w} + L_i(e-\hat{e}) - \dot{\hat{z}}_r^d. \end{aligned} \quad (54)$$

A natural choice of the switching function for system (54) is $\sigma = \chi_r$. In order to generate a sliding mode in (54), the control is chosen as $u = kB_r^+ sign(\sigma),\ k > 0,\ B_r^+ = B_r^T(B_rB_r^T)^{-1}$. If $k > \|B_r u_{eq}\|$, the condition (**SMS**) is guaranteed, and u_{eq} is calculated from $\dot{\sigma} = 0$ as $u_{eq} = -B_r^+\left(\sum_{j=1}^{r} A_{rj}\hat{z}_j + R_r\hat{w} + L_r(e-\hat{e}) - \dot{\hat{z}}_r^d\right)$. The sliding mode motion on $\sigma = \chi_r = 0$ is described by the reduced order system

$$\begin{aligned} \dot{\chi}_1 &= K_1\chi_1 + B_1\chi_2 \\ \dot{\chi}_i &= K_i\chi_i + B_i\chi_{i+1},\ i = 2,...,r-2 \\ \dot{\chi}_{r-1} &= K_{r-1}\chi_{r-1} \end{aligned} \quad (55)$$

$$\dot{w} = Sw \quad (56)$$

$$\dot{\varepsilon} = (\tilde{A} - \tilde{L}\tilde{C})\varepsilon \quad (57)$$

$$e = \sum_{k=1}^{r} M_k z_k + \left(\sum_{k=1}^{r} M_k\Gamma_k - Q\right)w. \quad (58)$$

Since the diagonal matrices K_i, $i = 1,...,r-1$ in (55) are Hurwitz, then the states of (55) tend asymptotically to zero, i.e. $\lim_{t\to\infty}\chi_i(t) = 0$, $i = 1,...,r-1$. Hence, by transformation (53) $\lim_{t\to\infty}\hat{z}_i(t) = 0$, $i = 1,...,r$. Now, by assumption **H6** there is a matrix $\tilde{L}$ in (57) such that $\lim_{t\to\infty}\varepsilon(t) = 0$, therefore $\lim_{t\to\infty} z_i(t) = 0$, $i = 1,...,r$, satisfying condition (**SS**). In consequence, thanks to condition (49) the output tracking error *e*(*t*) (58) tends asymptotically to zero, satisfying condition (**SR**). Note that the Regular form conditions (36) and (37) are represented for the BC-form as (47)-(48) and (49), respectively.

6. CONCLUSIONS

The EFSMRP has been reformulated. Solution conditions are derived for linear systems presented in the Regular and BC forms. In particular, the combination of Sliding Mode and BC-techniques allows straightforward solutions to be obtained, especially when compared to the classical solutions of the error feedback regulator problem. Additionally the sliding mode based controller achieves robustness with respect to the uncertainty.

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