CARTESIAN SLIDING PD FORCE-POSITION CONTROL FOR CONSTRAINED ROBOTS UNDER JACOBIAN UNCERTAINTY

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Abstract: *Joint* control requires to map, using ill-posed inverse kinematics, desired cartesian tasks into desired joint tasks, then it codes them into desired joint trajectories. To avoid this, *cartesian* control directly codes the cartesian task in cartesian coordinates, avoiding in this way any computation of inverse kinematics, which is relevant in particular for force control since the force task is always given in operational (cartesian) space. In this paper, a local *cartesian* exponential tracking control for constrained motion without using inverse kinematics is proposed. The novelty lies, besides its nontrivial extension from ODE (position) robots to DAE (force) robots, in the fact that fast *cartesian* tracking is obtained without using the model of the robot nor exact knowledge of inverse jacobian. The scheme shows a smooth control input. Simulations results shows the expected tracking performance.

Keywords: Force Control, Cartesian Control, Second Order Sliding Mode, Robot Manipulators

1. INTRODUCTION

Mode-based inverse dynamics (with and without coordinate partitioning, (McClamroch and Wang, 1998) , (Parra-Vega and Arimoto, 1996), respectively), and adaptive *joint* control for constrained system yield the simultaneous asymptotic convergence of position and force tracking errors, while the first order sliding mode control produce exponential tracking at the expense of chattering, whose discontinuity renders a high frequency controller that is impossible to implement in practice (Parra-Vega and Hirzinger, 2001).

To implement a joint robot control, the desired joint reference is computed from desired cartesian coordinates using inverse mappings and its derivatives up to second order. The main difficulty of computing inverse kinematics is represented by the fact of the ill-posed nature of the inverse kinematic mappings. In contrast, cartesian control does not require inverse kinematics mappings since it accepts directly desired cartesian coordinates. This saving is significant in real time applications because inverse kinematics are hard to compute on line. So, cartesian control arises as an option to circumvent the computation of inverse kinematics, and this is the subjacent interest of this scheme. Solving this problem would allow to design efficient and intuitive to tune controllers with very low computational cost.

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Remember that for constrained motion, the inverse kinematics are not only involved in mapping cartesian task into joint tasks, but also in checking the consistency of the constrained holonomic equation. Thus, the importance of cartesian control is more important for constrained motion (force control) than for free motion. In this work we *design a smooth cartesian control system that ensures fast tracking of constrained robot manipulators subject to unknown robot dynamics and uncertain inverse jacobian, assuming that the holonomic constraint is known.*

2. BACKGROUND

Constrained motion is a relevant problem in the robotics community because it stands as a fundamental scheme for a variety of practical constrained robotic tasks, such as griding, scribing, polishing, rubbering, walking, dynamic simulation, hand manipulation, teleoperation, to mention a few. Usually, the kinematic and dynamic models of all these tasks are very difficult to obtain in real applications. Moreover, cartesian control has attracted recently renewed attention, and some novel schemes for regulation and tracking of free motion robots have been proposed, however, these new schemes has not been extended to constrained motion, wherein a more interesting class of robotic tasks can be carried out.

On the other hand, joint model-based control of constrained robots allows simultaneous control of contact force and joint positions while moving along the surface of the object in operational (cartesian) coordinates. Moreover, the desired contact force profile is given in operational (cartesian) coordinates, therefore, it is interesting to design cartesian controllers for constrained robots, which guarantee analytically tracking without using the model, subject to jacobian uncertainty. This problem remains open in the literature. In this regard, in order to expand the applications of robot manipulators in many tasks, it is necessary to control both position of the end-effector and the constrained force between the end effector and the environment Additionally, there are several practical issues of concern like smooth control, fast trajectory tracking, and robustness, as well as simple control structure. In this paper, an alternative approach that satisfy the problem is a very simple cartesian PID-like force controller, which yields fast tracking through two orthogonalized sliding modes without computing inverse kinematics, nor computing cartesian robot dynamics. The chattering-free smooth sliding mode compensate largely for the robot dynamics, while the smooth control input produces locally exponential convergence of position and force tracking errors. The main characteristic of our scheme are: *i.* robot dynamics are not required;*ii.* very fast tracking of force and position trajectories is guaranteed;*iii.* smooth control activity arises;

3. ROBOT DYNAMICS

Consider a rigid manipulator with all revolute type joints constrained by a rigid environment. The dynamic model of a robot manipulator, when its end effector is in touch with a rigid surface, is given by

$$H(q)\ddot{q} + (B_0 + C(q,\dot{q}))\dot{q} + g(q) = \tau + J_{\varphi}^T(q)\lambda \quad (1)$$

$$\varphi(x) = 0 \quad (2)$$

where $q = (q_1, \ldots, q_n)^T$ is the generalized coordinates vector, $H(q) \in \Re^{nxn}$ denotes a symmetric positive definite inertial matrix, $B_0 \in \Re^{nxn}$ stands for a diagonal positive definite matrix composed of damping friction for each joint, the second term in the left side represents the Coriolis and centripetal forces $C(q,\dot{q}) \in \Re^n$, $g(q) \in \Re^n$ models the gravitational torques, $\tau \in \Re^{nxn}$ stands for the torque input, $\lambda \in \Re^r$ is constrained Lagrangian representing the magnitude of the contact force, and $J_\varphi(q) = \frac{(\partial\varphi(x)/\partial x)^T}{\|\partial\varphi(x)\partial x\|} J(q) : \Re^n \rightarrow \Re^{1xn}$ is the constrained jacobian. The infinitely rigid surface is described by a geometric function $\varphi(\nu,\omega) = 0 : \Re^n \rightarrow \Re^1$ where $\nu = (x,y,z)$ denotes the cartesian coordinates (task coordinated) fixed at the inertial reference frame, and $\omega = (\omega_1, \omega_2, \omega_3)^T$ its associated Euler angles. Since (1) can be parameterized linearly in terms of a nominal reference $(\dot{q}_r, \ddot{q}_r)^T \in \Re^{2n}$ consider

$$H(q)\ddot{q}_r + (B_0 + C(q,\dot{q}))\dot{q}_r + G(q) = Y_r\Theta \quad (3)$$

where the regressor $Y_r\Theta = Y_r(q,\dot{q},\dot{q}_r,\ddot{q}_r)$ is composed of known nonlinear functions, and $\Theta \in \Re^p$ is assumed to represent unknown but constant parameters with $(\dot{q}_r, \ddot{q}_r)$ *to be defined yet.* Then, if we add and subtract (3) into (1), the open loop error equation arises

$$H(q)\dot{S} + (B_0 + C(q,\dot{q}))S = \tau - Y_r\Theta + J_{\varphi}^T(q)\lambda \quad (4)$$

where the extended error

$$S = \dot{q} - \dot{q}_r \quad (5)$$

carries out changes of coordinates through $(\dot{q}_r, \ddot{q}_r)$.

4. ORTHOGONALIZATION OF OPEN LOOP ERROR EQUATION

When the end effector is in touch with the constrained surface, it holds that $\varphi(f(q)) = 0 \ \forall t$, then, from $x = f(q)$ for $f(.) \in \Re^n \rightarrow \Re^n$, it's differentiation is given as

$$\frac{d}{dt}\varphi(f(q)) = \frac{\partial\varphi(f(q))}{\partial q}\frac{dq}{dt} \equiv J_\varphi(q)\dot{q} = 0$$

and its orthogonal projection matrix of J_φ is

$$Q_{=}I - \frac{J_\varphi^T}{\|J_\varphi(x)\|^2}J_\varphi \quad (6)$$

This means that J_φ and Q are orthogonal, and this is known as the principle of orthogonalization. Notice that Q spans the tangent plane at the contact point, and this tangent plane is exactly wherein the vector $\overrightarrow{q}$ lies. On the other hand, notice that $\varphi(x) = 0$ as long as the manipulator end effector is in touch with the surface, then it holds that $\frac{d}{dt}\varphi(x) = 0 \Rightarrow J_\varphi(q)\dot{q} = 0$. This also means that $J_\varphi(q)$ and vector $\dot{q}$ are orthogonal, then necessarily $\dot{q}$ lies in the orthogonal complement of $J_\varphi(q)$, that is $\dot{q}$ lies in Q. Thus,

$$Q\dot{q} = \dot{q} \quad \text{and} \quad QJ_\varphi^T \equiv 0$$

Therefore, J_φ y Q are two orthogonal subspaces such that $\Re^n$ can be written as the direct sum, given as $rank(im(Q(q))) = m(= n - r)$ y $rank(im(J_\varphi)) = r$, such that $m + r = n$. This derivation constitutes the key to design passivity based force controllers since position and force subspaces decomposes the space, and this gives us hints how to fulfill the passivity inequality to design the controller. The first steep toward this, is the design of an error manifold.

4.1 Cartesian Error Manifold

The forward kinematic is generally a non lineal transformation that describe the relation between joint space and task space (cartesian coordinates). Notice that the differential kinematics establishes a mapping of velocities $\dot{X} = J(q)\dot{q}$, where $J(q) \in \Re^{nxm}$ is a manipulator jacobian matrix. the inverse kinematics can be expressed as follows

$$\dot{q} = J^{-1}\dot{X} \quad (7)$$

If we multiply (7) by Q, and as show before $\dot{q} = Q\dot{q}$, we have

$$Q\dot{q} = QJ^{-1}\dot{X} \Rightarrow \dot{q} = QJ^{-1}\dot{X} \quad (8)$$

Notice that (8) established a mapping of cartesian and joint velocities via inverse analytical Jacobian, and the orthogonal projection Q. Since the control objective is simultaneous cartesian position-force control, therefore, we need to design a $\dot{q}_r$ similarly to (8) but depending also on a nominal *force* reference. Similar to joint force control (Parra-Vega and Arimoto, 1996), a reasonable choice is

$$\dot{q}_r = QJ^{-1}\dot{X}_r + \beta J_\varphi^T(q)\{\underbrace{\Delta F - S_{df}}_{S_{qf}} + \gamma_f \int sgn(S_{qf})\} \quad (9)$$

where $S_{qf} = S_f - S_{df}, S_f = \Delta F, S_{df} = S_f(t_0)e^{-k_f t}, \Delta F = \int_{t_0}^t (\lambda - \lambda_d)(\zeta))d\zeta$, $\gamma_f = \gamma_f^T \in R_+^{n\times n}, \beta > 0$, function $sgn(y)$ is the function sign of vector y, ς_d denote the desired reference of ς [3] The new nominal reference *cartesian* position $\dot{X}_r$ is given as

$$\dot{X}_r = \dot{x}_d - \alpha\Delta x + S_{dp} - \gamma_p \int sgn(S_{qp}) \quad (10)$$

where $S_{qp} = S_p - S_{dp}, S_p = \Delta\dot{X} + \alpha\Delta X, S_{dp} = S_{qp}(t_0)e^{-k_p t}, \gamma_p > 0, \alpha = \alpha_f^T \in R_+^{n\times n}$. Now, equation (9) becomes, using (10),

$$\dot{q}_r = QJ^{-1}\{\dot{x}_d - \alpha\Delta x + S_{dp} - \gamma_p \int sgn(S_{qp})\} + \beta J_\varphi^T(q)\{S_{qf} + \gamma_f \int sgn(S_{qf}(\zeta))d\zeta\} \quad (11)$$

Finally, substituting (11) in (5) we obtain

$$S = QJ^{-1}S_{vp} - \beta J_\varphi^T(q)S_{vf} \quad (12)$$

where the extend orthogonalized manifolds of force S_{vf} and cartesian position S_{vp}, are defined as

$$S_{vp} = S_{qp} + \gamma_p \int sgn(S_{qp}) \quad (13)$$

$$S_{vf} = S_{qf} + \gamma_f \int sgn(S_{qf}) \quad (14)$$

5. UNCERTAIN JACOBIAN-BASED CONTROLLER

Note that when the jacobian is not exactly know, then the nominal reference (11) cannot be used since $\dot{q}_r = J^{-1}\dot{X}_r$ is not available. Let the new uncalibrated nominal reference when the jacobian is uncertain, now given as

$$\hat{\dot{q}}_r = Q\hat{J}^{-1}\dot{X}_r + \beta J_\varphi^T S_{vf} \quad (15)$$

with $\hat{J}^{-1}(q)$ stands for an estimated of $J^{-1}(q)$, such as rank $rank(J^{-1}(q)(q)) = n, \forall q \in \Omega$, where the robot workspace free of singularities is defined

[3] In the rest of the paper we denote as $\int_{t_0}^t sgn(z(\zeta))d\zeta \equiv \int sgn(z)$.

by $\Omega = \{q|rank(J(q)) = n\}$. Thus, similarly to (5), the *uncalibrated joint error surface* $\hat{S}_q$ arises, and then, after using (15) we obtain

$$\hat{S} = \dot{q} - \hat{\dot{q}}_r = QJ^{-1}\dot{X} - Q\hat{J}^{-1}\dot{X}_r - \beta J_\varphi^T S_{vf} \quad (16)$$

where $\hat{S}$ is available because $\dot{q}$ and $\hat{\dot{q}}_r$ are available. Using (15), the uncertain parametrization $Y_r\hat{\theta}$ arises

$$H(q)\hat{\ddot{q}}_r + C(q,\dot{q})\hat{\dot{q}}_r + g(q) = Y_r\hat{\theta} \quad (17)$$

Adding and subtracting (17) to (1), we obtain finally the *uncertain open loop error equation* expressed in terms

$$H(q)\dot{\hat{S}}_r = -C(q,\dot{q})\hat{S}_r + \tau + J_\varphi^T(q)\lambda - Y_r\hat{\theta} \quad (18)$$

Now we are ready to present the main result. **Theorem 1.** Assume that initial conditions and desired trajectories belong to Ω. Consider the closed-loop error dynamics (1), subject to parametric uncertainties on jacobian $J^{-1}(q)$, in closed loop with the controller

$$\tau = -K_d\hat{S}_r + J_{\varphi+}^T(q)\left[-\lambda_d + \eta\Delta F\right] + \gamma_F J_{\varphi+}^T(q)\left[\tanh(\mu S_{qF}) + \eta\int_{t_0}^{t} sgn(s_{qF})\right] \quad (19)$$

where $K_d = K_d^T \in R_+^{n\times n}, \gamma_F = \gamma_F \in R_+^{n\times n}, \eta > 0$, and λ_d the desired contact force. If K_d is large enough and an error of initial conditions are small enough, with $\lambda_{min}(\gamma_p) \geq \left\| \frac{d}{dt}\left\{J(q)\left[\hat{S}_r + (\hat{J}^{-1} - J^{-1})\dot{x}_r\right]\right\}\right\|$ *and* $\lambda_{min}(\gamma_F) \geq \left\|\frac{d}{dt}\left[\left(J_\varphi J_\varphi^T(q)\right)^{-1} J_\varphi\hat{S}_q\right]\right\|$, then exponential convergence of position and force tracking errors is guaranteed, where $\lambda_{min}(A)$ stands for the minimum eigenvalue of matrix A.

The closed loop dynamics between (18) and (19) yields

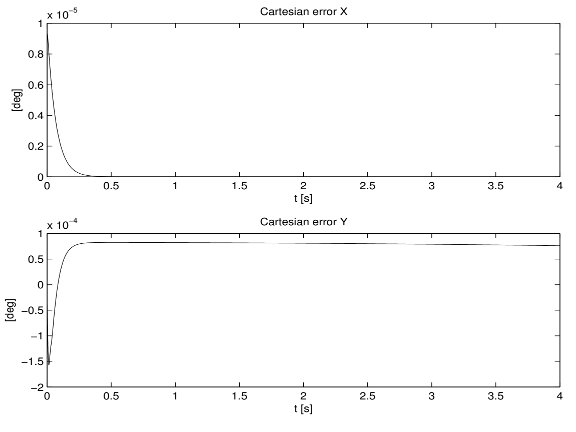

Fig. 1. Cartesian tracking errors jacobian uncertainty.

$$H(q)\dot{\hat{S}}_r = -\{K_d + B_0 + C(q,\dot{q})\}\hat{S}_r - Y_r\hat{\theta}_b + J_{\varphi+}^T(q)\left[\Delta\lambda + \gamma_F \tanh(\mu S_{qF})\right] + \eta J_{\varphi+}^T(q)\left[\Delta F + \gamma_F\int sgn(s_{qF})\right] \quad (20)$$

Now, we proceed to proof of Theorem 1.

Proof: See Appendix 1.

6. DISCUSSIONS

In contrast to first order sliding mode force control, our approach induces a sliding mode without high frequency commutation of the controller, and without knowledge of the regressor. In contrast to adaptive force control, our approach is faster and more robust, without any overparametrization, and without requiring the regressor. Concerning to PID-like force control, our approach guarantees tracking. The last important remark is that inverse kinematic computation is not required in our approach, which is a standard requirement on the usual approach on (joint) force control.

In the case when the jacobian is known, the stability proof is simplified and in the same way the control is smooth and the cartesian tracking error is obtained.

7. SIMULATIONS

In order to demonstrate usefulness of our controller, we present a digital simulations where the DAE solver is the stiff $4s$, ode23tb stiff/TR-BDF2 of Matlab 5.3, at $1ms$ sampling period. The set-up is a simple, but representative constrained task is simulated considering a 2 degrees of freedom robot. The end-effector is moving up and down along a rigid wall, while exerting $\lambda_d = 20 + 7.5sin(4.83t))$, with $10mm$ of initial error, and zero initial velocity, and there is 25% of parametric uncertainty on the jacobian. Robot parameters are $m_1 = 8.3kg, m_2 = 5kg, L_1 = 45, L_2 = 30, L_{c1} = 27cm, L_{c21} = 28cm$ with inertias $I_1 = 0.025, I_2 = 0.008$ Feedback gains are $\alpha = diag(20), K_d = diag(150), \eta = 10, \gamma_p = 5, \gamma_F = 4, \beta = diag(2)$. As expected, the end-effector draws the desired position and force trajectory. In Fig. 1 shows the good convergence of cartesian tracking error, while in Fig. 2 and Fig. 3 show smooth of control input, chattering free and the fast convergence of force, respectively. After a short transient, due to numerical problems of the DAE solver, exponential tracking is established. We have not obtained a systematic procedure to tune the control gains basically because of the nonlinear nature of the closed-loop system. Thus, it is usually done in this cases, feedback gains are tuned in trial-and-error-basis, according to the interplay of each gain in the closed-loop system.

8. CONCLUSIONS

A new cartesian, model-free, state feedback controller for constrained motion robots is proposed. It is based on inverse jacobian, and it is studied parametric uncertainty on the jacobian. Local exponential convergence of position and force tracking errors arise through a second order error sliding mode. Passivity considerations established stability in the sense of Lyapunov.

APPENDIX 1: PROOF OF THEOREM 1

The prove is divided in three parts: firstly, we prove that above equation shows boundedness of all system trajectories; secondly, we show the conditions to induce sliding modes, and thirdly, conditions of exponential convergence of tracking errors are shown.

Part I. Boundedness of Closed Loop Trajectories. Consider the time derivative of the following Lyapunov candidate function

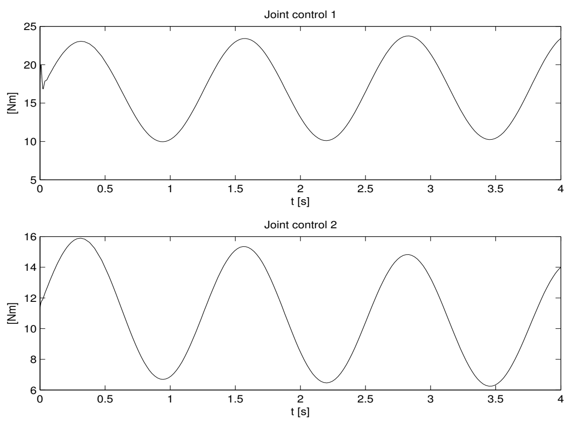

Fig. 2. Joint control under jacobian uncertainty.

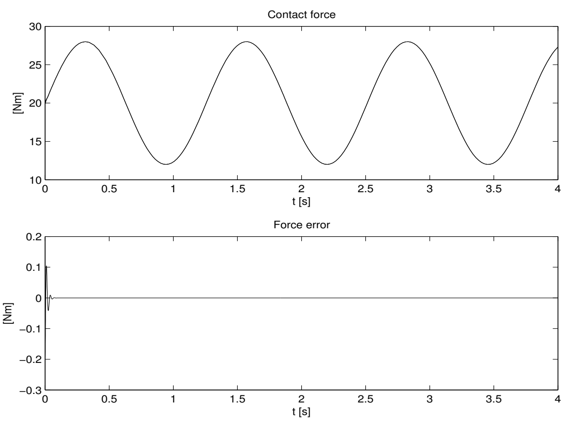

Fig. 3. Real and desired contact force (above) and errors, under jacobian uncertainty.

$$V = \frac{1}{2}\left[\hat{S}^T H\left(q\right)\hat{S} + \beta S_{vf}^T S_{vf}\right]$$

along the solutions of (20), then it yields

$$\dot{V} \leq -K_d \left\|\hat{S}\right\|_2^2 - \eta\beta \left\|S_{vf}\right\|_2^2 + \|\hat{S}\|\psi$$

where ψ is a functional depending on the state and error manifolds (Parra-Vega and Arimoto, 1996),(V. Parra-Vega and Akella, 2003). Now, if K_d and β are large enough and the initial errors are small enough, we conclude the seminegative definiteness of $\dot{V}$ outside of hyperball $\varepsilon_0 = \left\{\widehat{S}|\dot{V} \leq 0\right\}$ centered at the origin, such that the following properties of the state of closed loop system arise

$$\hat{S}, S_{vf} \in \mathcal{L}_\infty \tag{21}$$

Since desired trajectories are $\mathcal{C}^2$ and feedback gains are bounded, the right hand side of (20) shows that there exists $\varepsilon_1 > 0$ such that

$$\left\|\dot{\hat{S}}\right\| \leq \varepsilon_1$$

This result shows only local stability of $\hat{S}$ and $\dot{\hat{S}}$. To prove convergence of tracking errors, the sliding modes condition must be verified. To this end, adding and subtracting $Q\hat{J}^{-1}\dot{x}_r$ to (16), we obtain

$$\hat{S} = Q\left\{J^{-1}S_{vp} - \Delta\hat{J}^{-1}\dot{X}_r\right\} - \beta J_\varphi^T S_{vf} \tag{22}$$

where $\Delta\hat{J}^{-1} = \hat{J}^{-1} - J^{-1}$. Since $\hat{S} \in \mathcal{L}_\infty$, and $\hat{J}^{-1}$ and Q are bounded, then $Q\hat{J}^{-1}S_{vp}$ is bounded, and due to $\varphi(q)$ is smooth and lies in the reachable robot space Ω, and $S_{vf} \to 0$, then $\beta J_\varphi^T S_{vf} \to 0$. Now, taking into account that $\dot{\hat{S}}$ is bounded, then $\frac{d}{dt}\hat{J}^{-1}QS_{vp}$ and $\frac{d}{dt}\beta J_\varphi^T S_{vf}$ are bounded (this is possible because $\dot{J}_\varphi^T$ is bounded and so $\dot{Q}$ is). All this chain of conclusions proves that there exists bounded constants $\varepsilon_2 > 0$ and $\varepsilon_3 > 0$ such that

$$\left|\dot{S}_{vp}\right| < \varepsilon_2, \left|\dot{S}_{vf}\right| < \varepsilon_3$$

Now, we have to prove that for a proper selection of feedback gains γ_p and γ_F, sliding modes are established at the subspaces of position Q and force $J_\varphi^T\left(q\right)$.

Part II.a: Sliding modes for the velocity subspace Q. Considering that operator QJ_{Rinv} spans the vector $\hat{S}$ in its image $im\left\{QJ^{-1}\left(S_{vp}\right)\right\} \equiv S_{vp}^{im}$ and the operator βJ_φ^T spans the same vector in its image $im\left\{\beta J_\varphi^T\left(S_{vf}\right)\right\} \equiv S_{vf}^{im}$, this implies that

$$\hat{S} = Q\left\{J^{-1}S_{vp} - \Delta J^{-1}\dot{X}_r\right\} - \beta J_\varphi^T S_{vf}$$

$$= (S_{vp}^{im} - im\left\{\Delta J^{-1}\dot{X}_r\right\}) - S_{vf}^{im} \quad (23)$$

where $S_{vp}^{im} - im\left\{\Delta J^{-1}\dot{X}_r\right\}$ and S_{vf}^{im} belongs each other to a orthogonal complements, that means $\left\langle S_{vp}^{im} - im\left\{\Delta J^{-1}\dot{X}_r\right\}, S_{vf}^{im}\right\rangle = 0$. That is, we are able to analyze the residual dynamics $S_{vp}^{im} - im\left\{\Delta J^{-1}\dot{X}_r\right\}$, independently of S_{vf}^{im}, since S_{vf}^{im} belongs to the kernel of Q, therefore, if we multiply (23) by Q^T

$$Q^T\hat{S} = Q^T Q\left\{J^{-1}S_{vp} - \Delta J^{-1}\dot{X}_r\right\} - \underbrace{\beta Q^T J_\varphi^T S_{vf}}_{=0}$$

$$= S_{vp}^{im} - im\left\{\Delta J^{-1}\dot{X}_r\right\} \quad (24)$$

since Q is idempotent $\left(Q^T Q = Q\right)$ and $QJ_\varphi = 0$. It is important to notice that if $Ax = Ay$ for any square nonsingular matrix A and any couple of vectors x, y, then $x \equiv y$. Thus, (24) means that in the subspace Q, the equality $\hat{S} = Q\left\{J^{-1}S_{vp} - \Delta J^{-1}\dot{X}_r\right\}$ is valid within span of Q. Notice that Q is not full rank, then this equality is valid locally, not globally. In this local neighborhood, if we multiply the equality $\hat{S} = Q\left\{J^{-1}S_{vp} - \Delta J^{-1}\dot{X}_r\right\}$ by J, we have

$$J(q)\hat{S} = S_{qp} + \gamma_s \int sign(S_{qp}) - J(q)\left\{\Delta J^{-1}\dot{X}_r\right\}$$

Multiply the time derivative of the above equation by S_{qp}^T, we obtain

$$\begin{aligned} S_{qp}^T\dot{S}_{qp} &= -\gamma_s S_{qp}^T sign(S_{qp}) + \\ &\quad S_{qp}^T\frac{d}{dt}\left[J(q)(\hat{S}_r + \Delta J^{-1}\dot{X}_r)\right] \\ &\le -\mu_s\left|S_{qp}\right| \end{aligned} \quad (25)$$

where $\varepsilon_4 = \frac{d}{dt}\left[J(q)(\hat{S}_r + \Delta J^{-1}\dot{X}_r)\right]$, and $\mu_s = \gamma_s - \varepsilon_4$, Thus, a sliding mode at $S_{qp} = 0$ arises at $t_s = \frac{|S_{qp}(t_0)|}{\mu_s} \equiv 0$ since $S_{qp}(t_0) = 0$, then $S_{qp}(t) = 0$ is guaranteed for all time.

Part II.b: Sliding modes for the force subspace. If we multiply $\hat{S}$ for J_φ, we obtain

$$J_\varphi\hat{S} = \underbrace{J_\varphi Q\left\{J^{-1}S_{vp} - \Delta J^{-1}\dot{X}_r\right\}}_{=0} - \beta J_\varphi J_\varphi^T S_{vf}$$

$$\beta^{-1}J_\varphi^{\#}\hat{S} = -\beta J_\varphi J_\varphi^T S_{vf}$$

$$J_\varphi^{\#}\hat{S} = S_{qf} + \gamma_F\int sign(S_{qf}) \quad (26)$$

where $J_\varphi^{\#}(q) = \left(J_\varphi J_\varphi^T(q)\right)^{-1}J_\varphi$. The time derivative (26), multiplied for S_{qF}^T, becomes

$$\begin{aligned} S_{qf}^T\dot{S}_{qf} &= -\gamma_F\left|S_{qf}\right| + S_{qf}^T\frac{d}{dt}\left(J_\varphi^{\#}(q)\hat{S}\right) \\ &\le -\gamma_F\left|S_{qf}\right| + \left|S_{qf}\right|\frac{d}{dt}\left(J_\varphi^{\#}(q)\hat{S}\right) \\ &\le -\mu_F\left|S_{qf}\right| \end{aligned} \quad (27)$$

where $\mu_F = \gamma_F - \varepsilon_5$, and $\varepsilon_5 = \frac{d}{dt}\left[\left(J_\varphi J_\varphi^T(q)\right)^{-1}J_\varphi\hat{S}\right]$. If $\gamma_F > \varepsilon_5$, then a sliding mode at $S_{qf}(t) = 0$ is induced for all time, because $t_f \le \frac{|S_{qf}(t_0)|}{\mu_F} \equiv 0$ since $S_{qf}(t_0) = 0$.

Part III.a: Position tracking errors. Since a sliding mode exists for all time at $S_{qp}(t) = 0$, then, we have

$$S_p = S_{dp}\forall t \to \Delta\dot{X} = -\alpha\Delta X + S_{dp}(t_0)e^{-\kappa_p t}$$

implying that position tracking errors locally exponentially tends to zero, this is $X \to X_d, \dot{X} \to \dot{X}_d$.

Part III.b: Force tracking errors. Since a sliding mode at $S_{qf}(t) = 0$ is induced for all time, this means $\Delta F = \Delta F(t_0)e^{-\kappa_f t}$. Moreover, (Y. H. Liu and Kitagaki, 1997) shows that if $\Delta F \to 0$, then convergence of force tracking errors arises, thus $\lambda \to \lambda_d$ exponentially fast. **QED.**

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