NONLINEAR ROBUST CONTROL VIA APPROXIMATE FEEDBACK LINEARIZATION

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Abstract: This paper develops a technique of robust approximate tracking and regulation for nonlinear systems that do not satisfy the restrictive regularity assumptions required by exact feedback linearization approach. This technique achieves closed-loop stability and reasonable performance in the presence of time-varying parametric uncertainties or unknown nonlinearities. Regarding the uncertainties, though the knowledge of bounds and satisfaction of matching condition are assumed, no linear dependence on the system dynamics or conic continuity on the growth of system nonlinearities are required. The design is developed for the input tracking and state regulation problems separately.

Keywords: nonlinear systems, feedback linearization, approximate analysis, robust control, stability analysis, uncertainty.

1. INTRODUCTION

Different efforts have been devoted to substituting much milder assumptions such as existence of a robust relative degree, slightly minimum phase property, high-order approximate involutivity, and local Lipschitz condition, in order to loose stringent regularity conditions that are required by nonlinear geometric approaches (Hauser, *et al.*, 1992; Banaszuk and Hauser, 1993; Ghanadan, 1994). Several attempts have been also made to robustify the feedback linearization approach against modeling errors and system uncertainties (Sastry and Isidori, 1989; Pomet and Praly, 1992; Kanellakopoulos, *et al.*, 1991).

The scheme introduced by Ghanadan and Blankenship (1996) is an attempt to resolve both of the drawbacks of exact feedback linearization method. It approximates a nonlinear system to the highest degree possible, and then applies an indirect adaptive scheme to eliminate parametric uncertainties. Despite the advantages of this technique, it does not consider unknown nonlinear functions and deals with the ideal case of parametric uncertainties only. Nonlinearities of the system are assumed known and unknown parameters are assumed to appear linearly with respect to these known functions. Egardt (1979) stated that such adaptive schemes may result in growing the parametric error and ultimately destabilizing the system when bounded disturbances are present. Rohrs, *et al.* (1985) also explained that other perturbations such as time-varying parameters and un-modeled dynamics may result in instability. Zhang and Bitmead (1990) clarified another drawback of these methods that poor initial parameter estimates may result in poor transient behavior.

In this paper, a robust approximate controller is designed on a parallel with the adaptive one introduced by Ghanadan and Blankenship (1996), based on a continuous approximation of the min-max control law. It is assumed that only the nominal dynamic equations are approximately feedback linearizable. Comparing to the earlier adaptive one, the technique of this paper removes the linear dependence of unknown parameters. It can attenuate the effect of modeling errors coming from both parametric uncertainties and unknown nonlinearities. It can also guarantee a reasonable transient performance. As well, the approximate linearizability is only required for the nominal system, not for the true system affected by unknown uncertainties in a family of operating envelopes.

2. SYSTEM SPECIFICATION

Consider a SISO nonlinear system as

$$\dot{\boldsymbol{x}} = \boldsymbol{f}\big(\boldsymbol{x},\boldsymbol{\theta}(t)\big) + \boldsymbol{g}\big(\boldsymbol{x},\boldsymbol{\theta}(t)\big)u \quad , \quad y = h(\boldsymbol{x}) \tag{1}$$

where $\boldsymbol{x} \in \Re^{\mathrm{n}}$ is the state, $u \in \Re$ is the input, $y \in \Re$ is the output, $\boldsymbol{\theta}(t) \in \Re^{\mathrm{p}}$ is the vector of unknown time-varying uncertainties that takes values in an admissible set $\Omega \subset \Re^{\mathrm{p}}$, and $\boldsymbol{f}$, $\boldsymbol{g}$ and h are smooth functions in a region $\mathrm{M} \subset \Re^{\mathrm{n}}$, $\forall \boldsymbol{\theta}(t) \in \Omega$. Without loss of generality, it is assumed that $\boldsymbol{f}(\mathbf{0},\boldsymbol{\theta}) = \mathbf{0}$, $\boldsymbol{g}(\mathbf{0},\boldsymbol{\theta}) \neq \mathbf{0}$ and $h(\mathbf{0}) = 0$, $\forall \boldsymbol{\theta}(t) \in \Omega$. Suppose that only a nominal constant value $\boldsymbol{\theta}_{\mathrm{N}}$ for the uncertainty vector $\boldsymbol{\theta}(t)$ is known and perturbations about $\boldsymbol{\theta}_{\mathrm{N}}$ are represented by $\boldsymbol{\theta}(t) = \boldsymbol{\theta}_{\mathrm{N}} + \delta\boldsymbol{\theta}(t)$. The model used for design and stability analysis is an evaluation of system (1) at $\boldsymbol{\theta}(t) = \boldsymbol{\theta}_{\mathrm{N}}$, so-called the nominal system, as

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}_{\mathrm{N}}) + \boldsymbol{g}(\boldsymbol{x},\boldsymbol{\theta}_{\mathrm{N}})u \quad , \quad y = h(\boldsymbol{x}) \tag{2}$$

It is assumed that the nonlinear functions $\boldsymbol{f}$ and $\boldsymbol{g}$ are analytic in $\boldsymbol{\theta}(t)$ about $\boldsymbol{\theta}_{\mathrm{N}}$. Therefore, $\dot{\boldsymbol{x}}$ can be expanded using a Taylor series in $\boldsymbol{\theta}(t)$ about $\boldsymbol{\theta}_{\mathrm{N}}$ as

$$\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}_{\mathrm{N}}) + \boldsymbol{g}(\boldsymbol{x},\boldsymbol{\theta}_{\mathrm{N}})u \\ &\quad + \left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\theta}} + \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{\theta}}u\right)_{\boldsymbol{\theta}_{\mathrm{N}}} \delta\boldsymbol{\theta}(t) + \mathrm{O}(\delta\boldsymbol{\theta})^2\end{aligned} \tag{3}$$

where $\mathrm{O}(\delta\boldsymbol{\theta})^2$ denotes terms of $\delta\boldsymbol{\theta}$ whose orders are higher than or equal to 2. With the exception of $\delta\boldsymbol{\theta}(t)$, all the terms in (3) are known, because they are evaluated about the nominal vector $\boldsymbol{\theta}_{\mathrm{N}}$. Note that no assumption has been made regarding the linearity or nonlinearity of uncertainties. For a special case, where the functions $\boldsymbol{f}$ and $\boldsymbol{g}$ can be parameterized linearly in $\boldsymbol{\theta}(t)$, expansion (3) would be exact to the first order in $\delta\boldsymbol{\theta}(t)$.

Assumption 1 (Robust Relative Degree): The nominal system (2) has a robust relative degree of γ on $\mathrm{U}_{\varepsilon}(\boldsymbol{x}_{\mathrm{e}})$, a family of operating envelopes about the equilibrium $\boldsymbol{x}_{\mathrm{e}}$, i.e., $\forall \boldsymbol{x} \in \mathrm{U}_{\varepsilon}(\boldsymbol{x}_{\mathrm{e}}) \subset \mathrm{M}$

$$\mathrm{L}_{\boldsymbol{g}(\boldsymbol{x},\boldsymbol{\theta}_{\mathrm{N}})}\mathrm{L}^{i}_{\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}_{\mathrm{N}})}h(\boldsymbol{x}) = 0 \quad , \quad i = 0,\ldots,\mathrm{r}-2$$

$$\mathrm{L}_{\boldsymbol{g}(\boldsymbol{x},\boldsymbol{\theta}_{\mathrm{N}})}\mathrm{L}^{j}_{\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}_{\mathrm{N}})}h(\boldsymbol{x})\cdot u \quad , \quad j = \mathrm{r}-1,\ldots,\gamma-2$$

are uniformly higher order on $\mathrm{U}_{\varepsilon} \times \mathrm{B}_{\sigma}$

$$\mathrm{L}_{\boldsymbol{g}(\boldsymbol{x},\boldsymbol{\theta}_{\mathrm{N}})}\mathrm{L}^{\gamma-1}_{\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}_{\mathrm{N}})}h(\boldsymbol{x}) \neq 0$$

where B_{σ} is a ball of radius σ centered at the origin and $0 < \mathrm{r} < \gamma$ is the relative degree of (2) outside $\mathrm{U}_{\varepsilon}(\boldsymbol{x}_{\mathrm{e}})$ but not necessarily well defined at every point inside $\mathrm{U}_{\varepsilon}(\boldsymbol{x}_{\mathrm{e}})$ (see Hauser, *et al.* (1992) for details and definitions).

The robust relative degree of (2) is equal to the relative degree of its Jacobian linearization as a linear system (Hauser, *et al.*, 1992). Therefore, if the nominal system (2) does not have a well defined relative degree but it is linearly controllable, it can still be approximated with an input-output linearized one. Moreover, the robust relative degree of (2) is invariant under a state dependent change of control coordinates as $u = \alpha(\boldsymbol{x},\boldsymbol{\theta}_{\mathrm{N}}) + \beta(\boldsymbol{x},\boldsymbol{\theta}_{\mathrm{N}})\,v$.

Functions $\xi_i(\boldsymbol{x}) = \mathrm{L}^{i-1}_{\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}_{\mathrm{N}})}h(\boldsymbol{x})$ $i = 1,2,\ldots,\gamma$ that approximate the output and its derivatives are independent in a neighborhood of the equilibrium $\boldsymbol{x}_{\mathrm{e}}$ (Hauser, *et al.*, 1992). With the γ independent functions ξ_i in hand, the nonlinear change of coordinates can, by the Frobenius theorem, be completed with functions $\eta_i(\boldsymbol{x})$ $i = 1,\ldots,\mathrm{n}-\gamma$ such that $\mathrm{L}_{\boldsymbol{g}(\boldsymbol{x},\boldsymbol{\theta}_{\mathrm{N}})}\eta_i(\boldsymbol{x}) = 0$. Therefore, the state coordinate transformation $\boldsymbol{\Phi}(\boldsymbol{x},\boldsymbol{\theta}_{\mathrm{N}}) : \mathrm{M} \to \Re^{\mathrm{n}}$ is

$$\boldsymbol{\Phi} = \begin{pmatrix}\boldsymbol{\Phi}_{\xi}(\boldsymbol{x},\boldsymbol{\theta}_{\mathrm{N}}) \\ \boldsymbol{\Phi}_{\eta}(\boldsymbol{x},\boldsymbol{\theta}_{\mathrm{N}})\end{pmatrix} = \begin{pmatrix}\boldsymbol{\xi} \\ \boldsymbol{\eta}\end{pmatrix} \;,\; \begin{aligned}\boldsymbol{\xi} &= [\,\xi_1,\ldots,\xi_{\gamma}\,]^{\mathrm{T}} \\ \boldsymbol{\eta} &= [\,\eta_1,\ldots,\eta_{\mathrm{n}-\gamma}\,]^{\mathrm{T}}\end{aligned} \tag{4}$$

In the new coordinates $(\boldsymbol{\xi},\boldsymbol{\eta})$ the nominal system (2) can approximately be linearized from the new input v to the output y up to the terms of $\mathrm{O}(\boldsymbol{x},u)^2$ by

$$u(\boldsymbol{x}) = \frac{v - \mathrm{L}^{\gamma}_{\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}_{\mathrm{N}})}h(\boldsymbol{x})}{\mathrm{L}_{\boldsymbol{g}(\boldsymbol{x},\boldsymbol{\theta}_{\mathrm{N}})}\mathrm{L}^{\gamma-1}_{\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}_{\mathrm{N}})}h(\boldsymbol{x})} \tag{5}$$

Therefore $\alpha(\boldsymbol{x},\boldsymbol{\theta}_{\mathrm{N}}) = -\mathrm{L}^{\gamma}_{\boldsymbol{f}}h(\boldsymbol{x}) \big/ \mathrm{L}_{\boldsymbol{g}}\mathrm{L}^{\gamma-1}_{\boldsymbol{f}}h(\boldsymbol{x})$ and $\beta(\boldsymbol{x},\boldsymbol{\theta}_{\mathrm{N}}) = 1 \big/ \mathrm{L}_{\boldsymbol{g}}\mathrm{L}^{\gamma-1}_{\boldsymbol{f}}h(\boldsymbol{x})$, where $\boldsymbol{f}$ and $\boldsymbol{g}$ are evaluated about the nominal vector $\boldsymbol{\theta}_{\mathrm{N}}$. Here, the term $\mathrm{O}(\boldsymbol{x},u)^{\mathrm{d}}$ denotes a uniformly higher order function of the form $\mathrm{O}(\boldsymbol{x})^{\mathrm{d}} + \mathrm{O}(\boldsymbol{x})^{\mathrm{d}-1}\cdot u$. The new approximately linearized system can be represented by a compact form as

$$\dot{\boldsymbol{\xi}} = \mathbf{A}\,\boldsymbol{\xi} + \mathbf{b}\,v + \boldsymbol{\psi}_{\mathrm{N}}(\boldsymbol{x},u) \quad , \quad \dot{\boldsymbol{\eta}} = \boldsymbol{q}(\boldsymbol{\xi},\boldsymbol{\eta}) \tag{6}$$

where $(\mathbf{A},\mathbf{b})$ are in Brunovsky canonical form, $q_i(\boldsymbol{\xi},\boldsymbol{\eta})$ is $\mathrm{L}_{\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}_{\mathrm{N}})}\eta_i(\boldsymbol{x})$ expressed in $(\boldsymbol{\xi},\boldsymbol{\eta})$ coordinates, and

$$\boldsymbol{\psi}_{\mathrm{N}}(\boldsymbol{x},u) = [\,0,\ldots,\psi_{\mathrm{N}\,\mathrm{r}},\ldots,\psi_{\mathrm{N}(\gamma-1)},0\,]^{\mathrm{T}}$$

$$\psi_{\mathrm{N}\,i}(\boldsymbol{x},u) = \mathrm{L}_{\boldsymbol{g}(\boldsymbol{x},\boldsymbol{\theta}_{\mathrm{N}})}\mathrm{L}^{i-1}_{\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}_{\mathrm{N}})}h(\boldsymbol{x})\cdot u$$

Note that the form (6) is a perturbation of the normal form of exact feedback linearization when $\boldsymbol{\psi}_{\mathrm{N}}(\boldsymbol{x},u) = 0$.

Assumption 2 (Slightly Non-minimum Phase): The zero dynamics of approximate nominal system are locally exponentially stable and $\boldsymbol{q}$ is Lipschitz continuous function of $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ on $\boldsymbol{\Phi}(\mathrm{U}_\varepsilon)$.

The above mentioned dynamics, i.e. $\boldsymbol{q}(\mathbf{0},\boldsymbol{\eta})$, are in fact the dynamics of true nominal system (2), or equivalently transformed system (6), when the output and its derivatives are approximately constrained to zero by the input. The stability of these dynamics may be satisfied by slightly non-minimum phase nominal systems. The nominal system (2) is said slightly non-minimum phase if its zero dynamics are not stable but the zero dynamics of its approximate input-output model, obtained by neglecting $\boldsymbol{\psi}_\mathrm{N}$ in (6), are stable (Pomet and Praly, 1992).

Assumption 3 (Matching Condition): For $\forall \boldsymbol{x} \in \mathrm{M}$ and $\forall \boldsymbol{\theta} \in \Omega$, it is required that

$$\left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\theta}}\right)_{\boldsymbol{\theta}_\mathrm{N}} \delta\boldsymbol{\theta}(t) \ \textit{and} \ \left(\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{\theta}}\right)_{\boldsymbol{\theta}_\mathrm{N}} \delta\boldsymbol{\theta}(t) \in \textit{span}\,\{\, \boldsymbol{g}(\boldsymbol{x},\boldsymbol{\theta}_\mathrm{N})\,\}$$

Loosely speaking, the matching condition implies that the input and uncertainties have a same reachable part of the stable space.

3. ROBUST INPUT TRACKING

Consider the nominal system (2) as the known compartment of the uncertain system (1). It is supposed to be approximately feedback linearizable, in the form of (6), by a nonlinear change of coordinates (4) and a choice of linearizing control (5). Applying the transformation (4) to the expanded model (3), subject to the state feedback (5), yields

$$\begin{aligned}\dot{\boldsymbol{\xi}} &= \mathbf{A}\,\boldsymbol{\xi} + \mathbf{b}\,v + \boldsymbol{\psi}_\mathrm{N}(\boldsymbol{x},u)\\ &\quad + \frac{\partial \boldsymbol{\Phi}_\xi}{\partial \boldsymbol{x}}\left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\theta}} + \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{\theta}}u\right)_{\boldsymbol{\theta}_\mathrm{N}} \delta\boldsymbol{\theta}(t) + \mathrm{O}(\delta\boldsymbol{\theta})^2 \end{aligned} \tag{7}$$

$$\begin{aligned}\dot{\boldsymbol{\eta}} &= \boldsymbol{q}(\xi,\boldsymbol{\eta})\\ &\quad + \frac{\partial \boldsymbol{\Phi}_\eta}{\partial \boldsymbol{x}}\left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\theta}} + \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{\theta}}u\right)_{\boldsymbol{\theta}_\mathrm{N}} \delta\boldsymbol{\theta}(t) + \mathrm{O}(\delta\boldsymbol{\theta})^2 \end{aligned} \tag{8}$$

The uncertain terms of (7) may be rewritten as

$$\begin{aligned}&\frac{\partial \boldsymbol{\Phi}_\xi}{\partial \boldsymbol{x}}\left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\theta}} + \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{\theta}}u\right)_{\boldsymbol{\theta}_\mathrm{N}} \delta\boldsymbol{\theta} + \mathrm{O}(\delta\boldsymbol{\theta})^2\\ &\qquad = \boldsymbol{\psi}_\delta(\boldsymbol{x},u,\delta\boldsymbol{\theta}) + \mathbf{b}\,\Delta(\boldsymbol{x},u,\delta\boldsymbol{\theta})\end{aligned}$$

where

$$\boldsymbol{\psi}_\delta(\boldsymbol{x},u,\delta\boldsymbol{\theta}) = [\,0,\ldots,\psi_{\delta\,\mathrm{r}},\ldots,\psi_{\delta\,(\gamma-1)},0\,]^\mathrm{T}$$

$$\psi_{\delta\, i} = \frac{\partial \Phi_{\xi\, i}}{\partial \boldsymbol{x}}\left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\theta}} + \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{\theta}}u\right)_{\boldsymbol{\theta}_\mathrm{N}} \delta\boldsymbol{\theta} + \mathrm{O}(\delta\boldsymbol{\theta})^2$$

$$\Delta(\boldsymbol{x},u,\delta\boldsymbol{\theta}) = \frac{\partial \boldsymbol{\Phi}_{\xi\,\gamma}}{\partial \boldsymbol{x}}\left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\theta}} + \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{\theta}}u\right)_{\boldsymbol{\theta}_\mathrm{N}} \delta\boldsymbol{\theta} + \mathrm{O}(\delta\boldsymbol{\theta})^2 \tag{9}$$

Under the matching condition as in Assumption 3, it is easy to show the validity of

$$\left(\frac{\partial^i \boldsymbol{f}}{\partial \boldsymbol{\theta}^i}\right)_{\boldsymbol{\theta}_\mathrm{N}} (\delta\boldsymbol{\theta})^i \ \textit{and} \left(\frac{\partial^i \boldsymbol{g}}{\partial \boldsymbol{\theta}^i}\right)_{\boldsymbol{\theta}_\mathrm{N}} (\delta\boldsymbol{\theta})^i \in \textit{span}\left\{\boldsymbol{g}(\boldsymbol{x},\boldsymbol{\theta}_\mathrm{N})\right\}$$

for $i = 1, 2, \ldots$. According to $\mathrm{L}_{\boldsymbol{g}(\boldsymbol{x},\boldsymbol{\theta}_\mathrm{N})}\eta_i(\boldsymbol{x}) = 0$, the above expression implies that all terms on the right-hand side of (8) become zero except for the first. It means that the internal dynamics of input-output model under the matching condition stay independence from the uncertainties. Therefore, the expanded and transformed form of the true system (1), specified by (7) and (8), may be represented in a compact form as

$$\begin{aligned}\dot{\boldsymbol{\xi}} &= \mathbf{A}\,\boldsymbol{\xi} + \mathbf{b}\,v + \mathbf{b}\,\Delta(\boldsymbol{x},u,\delta\boldsymbol{\theta}) + \boldsymbol{\psi}(\boldsymbol{x},u,\delta\boldsymbol{\theta})\\ \dot{\boldsymbol{\eta}} &= \boldsymbol{q}(\xi,\boldsymbol{\eta})\end{aligned} \tag{10}$$

where $\boldsymbol{\psi}(\boldsymbol{x},u,\delta\boldsymbol{\theta}) = \boldsymbol{\psi}_\mathrm{N}(\boldsymbol{x},u) + \boldsymbol{\psi}_\delta(\boldsymbol{x},u,\delta\boldsymbol{\theta})$. Under the matching condition as in Assumption 3 and considering the robust relative degree defined by Assumption 1, it is clear that all nonlinear functions $\psi_{\mathrm{N}\,i}(\boldsymbol{x},u)$ and $\psi_{\delta\, i}(\boldsymbol{x},u,\delta\boldsymbol{\theta})$ are uniformly higher order with respect to $\boldsymbol{x}$ and u on $\mathrm{U}_\varepsilon \times \mathrm{B}_\sigma$, and thus $\boldsymbol{\psi} = \boldsymbol{\psi}_\mathrm{N} + \boldsymbol{\psi}_\delta$ is also uniformly higher order on $\mathrm{U}_\varepsilon \times \mathrm{B}_\sigma$.

Consider $v = \hat{v} + v_\mathrm{R}$, where $\hat{v}$ assigns the poles of the approximate input-output model of nominal system to the proper places, and v_R is an additional control term to eliminate the effect of uncertainties. The approximate tracking of desired trajectory y_d is achieved by

$$\hat{v} = y_\mathrm{d}^\gamma + \mathrm{a}_{\gamma-1}(y_\mathrm{d}^{\gamma-1} - \xi_\gamma) + \cdots + \mathrm{a}_0(y_\mathrm{d} - \xi_1) \tag{11}$$

where a_i are chosen such a way that $\mathrm{s}^\gamma + \mathrm{a}_{\gamma-1}\,\mathrm{s}^{\gamma-1} + \cdots + \mathrm{a}_0$ is a Hurwitz polynomial. Choose

$$v_\mathrm{R} = -\frac{\rho^2(\mathbf{c}\boldsymbol{e})}{\rho\left|\mathbf{c}\,\boldsymbol{e}\right| + \omega} \quad , \quad \dot{\omega} = -\kappa\,\omega \tag{12}$$

where $\rho(\boldsymbol{x},t)$ is a positive function and $\boldsymbol{e} = \boldsymbol{\xi} - \boldsymbol{y}_\mathrm{d}$ is an error vector with $\boldsymbol{y}_\mathrm{d} = [\,y_\mathrm{d}, \dot{y}_\mathrm{d}, \ \ldots, y_\mathrm{d}^{\gamma-1}\,]^\mathrm{T}$. $\mathbf{c} \in \Re^\gamma$ is a constant row vector chosen such that $\mathbf{c}\,(\mathrm{s}\,\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{b}$ is a strictly positive real function. Suppose that $\rho(\boldsymbol{x},t) + \bar{\rho} \ge |\Delta|$, where $\bar{\rho}$ is a constant sufficiently small. Replacing the control input (5) with $v = \hat{v} + v_\mathrm{R}$ in (9) and according to $|v_\mathrm{R}| \le \rho$, the triangle inequality gives

$$\begin{aligned}\rho(\boldsymbol{x},t) &= \left(1 - \left|\frac{\partial \boldsymbol{\Phi}_{\xi\,\gamma}}{\partial \boldsymbol{x}} \cdot \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{\theta}}\,\beta\right|_{\boldsymbol{\theta}_\mathrm{N}} (\,\underset{\boldsymbol{\theta}}{\textit{sup}}\,|\delta\boldsymbol{\theta}|\,)\right)^{-1}\\ &\times\left(\left|\frac{\partial \boldsymbol{\Phi}_{\xi\,\gamma}}{\partial \boldsymbol{x}}\left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\theta}} + \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{\theta}}\,\alpha + \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{\theta}}\,\beta\,\hat{v}\right)\right|_{\boldsymbol{\theta}_\mathrm{N}} (\,\underset{\boldsymbol{\theta}}{\textit{sup}}\,|\delta\boldsymbol{\theta}|\,)\right)\end{aligned}$$

where $\bar{\rho} \ge \underset{\boldsymbol{\theta}}{\textit{sup}}\left|\mathrm{O}(\delta\boldsymbol{\theta})^2\right|$.

It is important to note in the above expression that everything on the right-hand side is known and easily determinable, once the deviation range of $\delta\boldsymbol{\theta}(t)$ is specified.

Assumption 4 (Reference Signal): The desired trajectory $y_d(t)$ and its first γ derivatives are bounded, i.e., $|\boldsymbol{y}_d| \le b_d$ for some $b_d > 0$.

Theorem 1 (Robust Approximate Tracking): Consider the uncertain nonlinear system (1) with the nonlinear functions $\boldsymbol{f}$ and $\boldsymbol{g}$ satisfying the matching condition as in Assumption 3. Suppose that the nominal system (3) has a robust relative degree as in Assumption 1 and is slightly non-minimum phase as in Assumption 2. Then for ε sufficiently small and for the desired trajectories satisfying Assumption 4, the control law (5), subject to $v = \hat{v} + v_R$ with (11) and (12), yields a closed-loop system that its states are bounded and its input tracking error converges to a ball of order ε.

Proof: Using $v = \hat{v} + v_R$ in (10), the error equation may be represented by

$$\begin{aligned} \dot{\boldsymbol{e}} &= \mathbf{A}_c\,\boldsymbol{e} + \mathbf{b}\,v_R(\boldsymbol{e}) + \mathbf{b}\,\Delta(\boldsymbol{x},u,\delta\boldsymbol{\theta}) + \boldsymbol{\psi}(\boldsymbol{x},u,\delta\boldsymbol{\theta}) \\ \dot{\boldsymbol{\eta}} &= \boldsymbol{q}(\boldsymbol{\xi},\boldsymbol{\eta}) \end{aligned} \tag{13}$$

with $\mathbf{A}_c = \mathbf{A} - \mathbf{b}\cdot[a_0,\ldots,a_{\gamma-1}]^T$. Assumption 4 gives

$$|\boldsymbol{\xi}| \le |\boldsymbol{e}| + b_d \tag{14}$$

From Assumption 2, the converse Lyapunov theorem assures the existence of a Lyapunov function $V_2(\boldsymbol{\eta})$ for the zero dynamics of approximate nominal system, such that

$$\begin{gathered} k_1|\boldsymbol{\eta}|^2 \le V_2(\boldsymbol{\eta}) \le k_2|\boldsymbol{\eta}|^2 \\ \frac{\partial V_2}{\partial \boldsymbol{\eta}}\boldsymbol{q}(\mathbf{0},\boldsymbol{\eta}) \le k_3|\boldsymbol{\eta}|^2 \\ \left|\frac{\partial V_2}{\partial \boldsymbol{\eta}}\right| \le k_4|\boldsymbol{\eta}| \end{gathered}$$

for some positive constants k_1, k_2, k_3 and k_4. Since $\boldsymbol{\Phi}$ is a local diffeomorphic transformation between $\boldsymbol{x}$ and $(\boldsymbol{\xi},\boldsymbol{\eta})$, $\boldsymbol{x}$ is locally Lipschitz in $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ with the Lipschitz constant ℓ_x, which from (14) gives

$$|\boldsymbol{x}| \le \ell_x(|\boldsymbol{\xi}| + |\boldsymbol{\eta}|) \le \ell_x(|\boldsymbol{e}| + b_d + |\boldsymbol{\eta}|) \tag{15}$$

From Assumption 2, $\boldsymbol{q}$ is locally Lipschitz in $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ with the Lipschitz constant ℓ_q, and so

$$|\boldsymbol{q}(\xi_1,\eta_1) - \boldsymbol{q}(\xi_2,\eta_2)| \le \ell_q\left(|\xi_1 - \xi_2| + |\eta_1 - \eta_2|\right)$$

$$\begin{aligned} \frac{\partial V_2}{\partial \boldsymbol{\eta}}\boldsymbol{q}(\boldsymbol{\xi},\boldsymbol{\eta}) &= \frac{\partial V_2}{\partial \boldsymbol{\eta}}\boldsymbol{q}(\mathbf{0},\boldsymbol{\eta}) + \frac{\partial V_2}{\partial \boldsymbol{\eta}}\left(\boldsymbol{q}(\boldsymbol{\xi},\boldsymbol{\eta}) - \boldsymbol{q}(\mathbf{0},\boldsymbol{\eta})\right) \\ &\le -k_3|\boldsymbol{\eta}|^2 + k_4\ell_q|\boldsymbol{\eta}|\left(|\boldsymbol{e}| + b_d\right) \end{aligned} \tag{16}$$

In order to show that $\boldsymbol{e}$ and $\boldsymbol{\eta}$ are bounded, consider a Lyapunov function for system (13) as

$$V(\boldsymbol{e},\boldsymbol{\eta}) = \boldsymbol{e}^T\,\mathbf{P}\,\boldsymbol{e} + 2\kappa^{-1}\omega + \mu V_2(\boldsymbol{\eta})$$

where $\mathbf{P} = \mathbf{P}^T$ is a positive definite solution to the Lyapunov equation $\mathbf{P}\mathbf{A}_c + \mathbf{A}_c^T\mathbf{P} + \mathbf{Q} = \mathbf{0}$, $\mathbf{P}\mathbf{b} = \mathbf{c}^T$, and constant $\mu > 0$ is determined later. The derivative of $V(\boldsymbol{e},\boldsymbol{\eta})$ along the solution trajectories of (13) is

$$\begin{aligned} \dot{V} &= -\boldsymbol{e}^T\,\mathbf{Q}\,\boldsymbol{e} + 2\boldsymbol{e}^T\,\mathbf{P}\mathbf{b}(v_R + \Delta) + 2\kappa^{-1}\dot{\omega} \\ &\quad + 2\boldsymbol{e}^T\,\mathbf{P}\,\boldsymbol{\psi}(\boldsymbol{x},u,\delta\boldsymbol{\theta}) + \mu\frac{\partial V_2}{\partial \boldsymbol{\eta}}\boldsymbol{q}(\boldsymbol{\xi},\boldsymbol{\eta}) \\ &\le -\lambda_m|\boldsymbol{e}|^2 + 2\boldsymbol{e}^T\,\mathbf{P}\mathbf{b}\,v_R + 2\left|\boldsymbol{e}^T\,\mathbf{P}\mathbf{b}\right|.|\Delta| - 2\omega \\ &\quad + \left|2\boldsymbol{e}^T\,\mathbf{P}\,\boldsymbol{\psi}(\boldsymbol{x},u,\delta\boldsymbol{\theta})\right| + \mu\frac{\partial V_2}{\partial \boldsymbol{\eta}}\boldsymbol{q}(\boldsymbol{\xi},\boldsymbol{\eta}) \end{aligned}$$

where λ_m is the minimum eigenvalue of $\mathbf{Q}$. Since $\boldsymbol{\psi}(\boldsymbol{x},u,\delta\boldsymbol{\theta})$ is uniformly higher order, and therefore $O(\boldsymbol{x},u)^2$, on $U_\varepsilon \times B_\sigma$ and $\delta\boldsymbol{\theta}$ is bounded, there exists for some $\sigma > 0$, a monotone increasing function of ε, k_ε such that

$$\begin{aligned} |2\mathbf{P}\boldsymbol{\psi}(\boldsymbol{x},u,\delta\boldsymbol{\theta})| &\le k_\varepsilon(|\boldsymbol{x}|^2 + |\boldsymbol{x}|\cdot|u|) \quad \forall \boldsymbol{x},\boldsymbol{y}_d : |\boldsymbol{x}| \le \varepsilon, |u| \le \sigma \\ &\le \varepsilon k_\varepsilon(|\boldsymbol{x}| + |u|) \le \varepsilon k_\varepsilon \ell_u(|\boldsymbol{x}| + b_d) \end{aligned} \tag{17}$$

where $\ell_u > 0$ is a constant. Note that σ depends on ε and b_d, since u is a smooth function of $\boldsymbol{x}$ and $\boldsymbol{y}_d$. Hence the poles assigned by $\hat{v}$ should be chosen such that the state magnitude $|\boldsymbol{x}|$ stays inside the approximating region. Considering (12) with $\rho + \bar{\rho} \ge |\Delta|$ and inequalities (15), (16) and (17) gives

$$\begin{aligned} \dot{V} &\le -\lambda_m|\boldsymbol{e}|^2 - 2(\mathbf{c}\boldsymbol{e})\frac{\rho^2(\mathbf{c}\boldsymbol{e})}{\rho|\mathbf{c}\boldsymbol{e}| + \omega} + 2|\mathbf{c}\boldsymbol{e}|\rho - 2\omega \\ &\quad + \varepsilon|\boldsymbol{e}|k_\varepsilon\ell_u\left(\ell_x(|\boldsymbol{e}| + b_d + |\boldsymbol{\eta}|) + b_d\right) + 2\bar{\rho}|\mathbf{c}|.|\boldsymbol{e}| \\ &\quad + \mu\left(-k_3|\boldsymbol{\eta}|^2 + k_4\ell_q|\boldsymbol{\eta}|(|\boldsymbol{e}| + b_d)\right) \\ &\le -2\omega\left(1 - \frac{\rho|\mathbf{c}\boldsymbol{e}|}{\rho|\mathbf{c}\boldsymbol{e}| + \omega}\right) \\ &\quad - \lambda_m\left(\frac{|\boldsymbol{e}|}{2} - \varepsilon k_\varepsilon\ell_u b_d\lambda_m^{-1}(\ell_x + 1) - \rho_1\lambda_m^{-1}\right)^2 \\ &\quad - \lambda_m\left(\frac{|\boldsymbol{e}|}{2} - \lambda_m^{-1}(\varepsilon k_\varepsilon\ell_u\ell_x + \mu k_4\ell_q)|\boldsymbol{\eta}|\right)^2 \\ &\quad - \left(\mu k_4\ell_q b_d|\boldsymbol{\eta}| - \frac{1}{2}\right)^2 - \left(\frac{\lambda_m}{2} - \varepsilon k_\varepsilon\ell_u\ell_x\right)|\boldsymbol{e}|^2 \\ &\quad - \Big(-\lambda_m^{-1}(\varepsilon k_\varepsilon\ell_u\ell_x + \mu k_4\ell_q)^2 - (\mu k_4\ell_q b_d)^2 \\ &\quad + \mu k_3\Big)|\boldsymbol{\eta}|^2 + \lambda_m^{-1}\left(\varepsilon k_\varepsilon\ell_u b_d(\ell_x + 1) + \rho_1\right)^2 + \frac{1}{4} \end{aligned}$$

where $\rho_1 = 2\bar{\rho}|\mathbf{c}|$. Neglecting the strictly negative terms yields

$$\dot V \le -\left(\frac{\lambda_m}{2} - \varepsilon\, k_\varepsilon \ell_u \ell_x\right)|\boldsymbol{e}|^2$$
$$-\Big(-\lambda_m^{-1}(\varepsilon\, k_\varepsilon \ell_u \ell_x + \mu\, k_4 \ell_q)^2 - (\mu\, k_4 \ell_q b_d)^2$$
$$+\mu k_3 \Big)|\boldsymbol{\eta}|^2 + \lambda_m^{-1}\left(\varepsilon\, k_\varepsilon \ell_u b_d(\ell_x + 1) + \rho_1\right)^2 + \frac{1}{4}$$

For $\mu \le \dfrac{k_3/2}{\lambda_m^{-1}(k_\varepsilon \ell_u \ell_x + k_4 \ell_q)^2 + (k_4 \ell_q b_d)^2}$ and $\varepsilon \le$ $min\left(\lambda_m / 4\, k_\varepsilon \ell_u \ell_x\ ,\ \mu\right)$, it is concluded that

$$\dot V \le -\frac{\lambda_m}{4}|\boldsymbol{e}|^2 - \frac{\mu\, k_3}{2}|\boldsymbol{\eta}|^2$$

Thus $\dot V(\boldsymbol{e},\boldsymbol{\eta}) < 0$ whenever $|\boldsymbol{e}|$ or $|\boldsymbol{\eta}|$ is enough large. It implies that $\boldsymbol{e}$ and $\boldsymbol{\eta}$ are bounded and hence $\boldsymbol{x}$ and u are bounded too. Using the continuity of v_R and Δ, it is seen that (13) is an exponentially stable linear system with stable internal dynamics, under an $\varepsilon-$order perturbation $\boldsymbol{\psi}$. Therefore e_1 converges to a ball of order ε, i.e. $|y - y_d| < k\varepsilon$ for some constant k and the proof is complete. ■

Remark 1: Reducing $\overline{\rho}$ will reduce the convergence ball of tracking error. For this goal, a way is to consider higher order terms of $\delta\boldsymbol{\theta}$ with $\rho(\boldsymbol{x},t)$, e.g. considering second order terms of $\delta\boldsymbol{\theta}$ yields

$$\rho(\boldsymbol{x},t) = \Bigg(1 - \left|\frac{\partial \boldsymbol{\Phi}_{\xi\gamma}}{\partial \boldsymbol{x}}\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{\theta}}\beta\right|_{\boldsymbol{\theta}_N}(\sup_{\boldsymbol{\theta}}|\delta\boldsymbol{\theta}|)$$
$$- \frac{1}{2}\left|\frac{\partial \boldsymbol{\Phi}_{\xi\gamma}}{\partial \boldsymbol{x}}\frac{\partial^2 \boldsymbol{g}}{\partial \boldsymbol{\theta}^2}\beta\right|_{\boldsymbol{\theta}_N}(\sup_{\boldsymbol{\theta}}|\delta\boldsymbol{\theta}|)^2\Bigg)^{-1}$$
$$\times\Bigg(\left|\frac{\partial \boldsymbol{\Phi}_{\xi\gamma}}{\partial \boldsymbol{x}}\left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\theta}} + \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{\theta}}\alpha + \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{\theta}}\beta\,\hat v\right)\right|_{\boldsymbol{\theta}_N}(\sup_{\boldsymbol{\theta}}|\delta\boldsymbol{\theta}|)$$
$$+ \frac{1}{2}\left|\frac{\partial \boldsymbol{\Phi}_{\xi\gamma}}{\partial \boldsymbol{x}}\left(\frac{\partial^2 \boldsymbol{f}}{\partial \boldsymbol{\theta}^2} + \frac{\partial^2 \boldsymbol{g}}{\partial \boldsymbol{\theta}^2}\alpha + \frac{\partial^2 \boldsymbol{g}}{\partial \boldsymbol{\theta}^2}\beta\,\hat v\right)\right|_{\boldsymbol{\theta}_N}(\sup_{\boldsymbol{\theta}}|\delta\boldsymbol{\theta}|)^2\Bigg)$$

This, however, increases the on-line computation of $\rho(\boldsymbol{x},t)$. Another way is to add a constant value to $\rho(\boldsymbol{x},t)$. But this value makes the control effort be extra large. Therefore, choosing the figure of $\rho(\boldsymbol{x},t)$ will be a compromise among several goals, i.e. minimizing the tracking error, shortening the on-line computation and reducing the control effort.

3. ROBUST STATE REGULATION

In the state regulation problem, the attempting is approximately linearize the nonlinear system to the highest order possible and then design a controller such that the closed-loop system is asymptotically stable. In this case, the robust relative degree is equal to n, i.e. the approximate system has no zero dynamics but the true system may be non-minimum phase.

Consider the nonlinear system (1) with no output specified as

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}(t)) + \boldsymbol{g}(\boldsymbol{x},\boldsymbol{\theta}(t))u \qquad (18)$$

which has a nominal system as

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}_N) + \boldsymbol{g}(\boldsymbol{x},\boldsymbol{\theta}_N)\,u \qquad (19)$$

and an expanded model as (3).

Suppose that the nominal system (19) is pproximately linearizable to order d on $B_\varepsilon(\mathbf{0})$, i.e.

- Distribution $\{\boldsymbol{g}, ad_{\boldsymbol{f}}\boldsymbol{g},\ldots, ad_{\boldsymbol{f}}^{n-2}\boldsymbol{g}, ad_{\boldsymbol{f}}^{n-1}\boldsymbol{g}\}_{\boldsymbol{\theta}_N}$ has an order d local basis at $\mathbf{0}$;
- Distribution $\{\boldsymbol{g}, ad_{\boldsymbol{f}}\boldsymbol{g},\ldots, ad_{\boldsymbol{f}}^{n-2}\boldsymbol{g}\}_{\boldsymbol{\theta}_N}$ is order d involutive at $\mathbf{0}$.

Then, Krener (1984) stated that there exists a local diffeomorphic state transformation $\boldsymbol{z} = \boldsymbol{T}(\boldsymbol{x},\boldsymbol{\theta}_N)$, $\boldsymbol{T}(\mathbf{0},\boldsymbol{\theta}_N) = \mathbf{0}$ and a nonlinear feedback control law $u(\boldsymbol{x},\boldsymbol{\theta}_N) = \alpha(\boldsymbol{x},\boldsymbol{\theta}_N) + \beta(\boldsymbol{x},\boldsymbol{\theta}_N)v$ that can transform the nominal system (19) into the approximate linear system as

$$\dot z_i = z_{i+1} + O(\boldsymbol{x},u)^{d+1} \quad i = 1,2,\ldots,n-1$$
$$\dot z_n = v + O(\boldsymbol{x},u)^{d+1}$$

The state transformation and the linearizing feedback are given by

$$\boldsymbol{z} = \left[\lambda(\boldsymbol{x}), L_{\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}_N)}\lambda(\boldsymbol{x}),\ldots, L_{\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}_N)}^{n-1}\lambda(\boldsymbol{x})\right]$$
$$u(\boldsymbol{x}) = \frac{v - L_{\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}_N)}^{n}h(\boldsymbol{x})}{L_{\boldsymbol{g}(\boldsymbol{x},\boldsymbol{\theta}_N)}L_{\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}_N)}^{n-1}h(\boldsymbol{x})}$$

where the scalar function $\lambda(\boldsymbol{x})$ with $\lambda(\mathbf{0}) = 0$ is a solution of the following partial differential equations,

$$L_{ad_{\boldsymbol{f}}^{i}\boldsymbol{g}}\lambda(\boldsymbol{x}) = O(\boldsymbol{x})^{d} \quad i = 0,1,\ldots,n-2$$
$$L_{ad_{\boldsymbol{f}}^{n-1}\boldsymbol{g}}\lambda(\boldsymbol{x}) \ne 0$$

where $\boldsymbol{f}$ and $\boldsymbol{g}$ are evaluated about the nominal vector $\boldsymbol{\theta}_N$. It is easy to show that the above partial differential equations are equivalent to the d-th order approximate linearization conditions, given by

$$L_{\boldsymbol{g}(\boldsymbol{x},\boldsymbol{\theta}_N)}L_{\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}_N)}^{i}\lambda(\boldsymbol{x}) = O(\boldsymbol{x})^{d} \quad i = 0,1,\ldots,n-2$$
$$L_{\boldsymbol{g}(\boldsymbol{x},\boldsymbol{\theta}_N)}L_{\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}_N)}^{n-1}\lambda(\boldsymbol{x}) \ne 0$$

When the matching condition is satisfied as in Assumption 3, applying the same transformation $\boldsymbol{T}(\boldsymbol{x},\boldsymbol{\theta}_N)$ to the expanded model (3), subject to the same control law $u(\boldsymbol{x},\boldsymbol{\theta}_N)$ with $v = \mathbf{k}\,\boldsymbol{z} + v_R$, yields

$$\dot{\boldsymbol{z}} = \mathbf{A}_c\,\boldsymbol{z} + \mathbf{b}\,v_R + \mathbf{b}\,\Delta(\boldsymbol{x},u,\delta\boldsymbol{\theta}) + O(\boldsymbol{x},u)^{d+1} \qquad (20)$$

where $\mathbf{A}_c = \mathbf{A} + \mathbf{bk}$ is a stable matrix, $\mathbf{k} \in \Re^n$ is a row vector that assigns the poles of nominal linear system $(\mathbf{A}, \mathbf{b})$ to the proper places, and

$$\Delta(\boldsymbol{x}, u, \delta\boldsymbol{\theta}) = \frac{\partial \boldsymbol{T}(\boldsymbol{x}, \boldsymbol{\theta}_N)}{\partial \boldsymbol{x}} \left(\frac{\partial \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} + \frac{\partial \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \cdot u \right)_{\boldsymbol{\theta}_N} \times \delta\boldsymbol{\theta}(t) + \mathrm{O}(\delta\boldsymbol{\theta})^2$$

Choose

$$v_R = -\frac{\rho^2(\mathbf{c}\,\boldsymbol{z})}{\rho|\mathbf{c}\,\boldsymbol{z}| + \omega} \quad , \quad \dot{\omega} = -\kappa\omega \qquad (21)$$

where $\rho(\boldsymbol{x}, t)$ is a positive function and $\mathbf{c} \in \Re^n$ is a constant row vector chosen such that $\mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$ is a strictly positive real function. Assume that $\rho + \bar{\rho} \geq |\lambda|$, where $\bar{\rho}$ is a constant sufficiently small and ρ is specified by

$$\rho(\boldsymbol{x}, t) = \left(\frac{\partial \boldsymbol{T}(\boldsymbol{x}, \boldsymbol{\theta}_N)}{\partial \boldsymbol{x}} \left(\frac{\partial \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} + \frac{\partial \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \alpha(\boldsymbol{x}, \boldsymbol{\theta}_N) + \frac{\partial \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \beta(\boldsymbol{x}, \boldsymbol{\theta}_N)\mathbf{k}\,\boldsymbol{z} \right)_{\boldsymbol{\theta}_N} (\sup_{\boldsymbol{\theta}} |\delta\boldsymbol{\theta}|) \right) \times \left(1 - \left| \frac{\partial \boldsymbol{T}(\boldsymbol{x}, \boldsymbol{\theta}_N)}{\partial \boldsymbol{x}} \frac{\partial \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \beta(\boldsymbol{x}, \boldsymbol{\theta}_N) \right|_{\boldsymbol{\theta}_N} (\sup_{\boldsymbol{\theta}} |\delta\boldsymbol{\theta}|) \right)^{-1}$$

Theorem 2 (Robust Regulation): Consider the uncertain nonlinear system (18) satisfying the matching condition as in Assumption 3. Suppose that the nominal system (19) is approximately linearizable to order d on $\mathrm{B}_\varepsilon(\mathbf{0})$. Then for ε sufficiently small, the equilibrium $\boldsymbol{x} = \mathbf{0}$ of the true system (18) is exponentially stable.

Proof: Consider a Lyapunov function as $V(\boldsymbol{z}) = \boldsymbol{z}^T \mathbf{P}\boldsymbol{z} + 2\kappa^{-1}\omega$ with $\mathbf{P} = \mathbf{P}^T > \mathbf{0}$, such that $\mathbf{P}\mathbf{A}_c + \mathbf{A}_c^T\mathbf{P} + \mathbf{Q} = \mathbf{0}$, $\mathbf{P}\mathbf{b} = \mathbf{c}^T$. The derivative of $V(\boldsymbol{z})$ along the solution trajectories of (20) is

$$\dot{V} = -\boldsymbol{z}^T \mathbf{Q}\boldsymbol{z} + \boldsymbol{z}^T \mathbf{P}\mathbf{b}\big(v_R + \Delta(\boldsymbol{x}, u, \delta\boldsymbol{\theta})\big) + 2\kappa^{-1}\dot{\omega} + 2\boldsymbol{z}^T \mathbf{P} \cdot \mathrm{O}(\boldsymbol{x}, u)^{d+1}$$

Replacing (21) and adopting inequality manipulations similar to those in the proof of Theorem 1, give

$$\dot{V} \leq -\lambda_m |\boldsymbol{z}|^2 + 2\boldsymbol{z}^T \mathbf{P} \cdot \mathrm{O}(\boldsymbol{x}, u)^{d+1}$$

The bounds can be obtained as

$$|\boldsymbol{x}| \leq \ell_x |\boldsymbol{z}|$$

$$\mathrm{O}(\boldsymbol{x}, u)^{d+1} \leq \ell_u |\boldsymbol{x}|^{d+1}$$

$$\dot{V} \leq -\lambda_m |\boldsymbol{z}|^2 + \ell_s |\boldsymbol{z}|^{d+1}$$

Thus $\dot{V} \leq (-\lambda_m + \ell_s \varepsilon^d) \cdot |\boldsymbol{z}|^2$, $\forall \boldsymbol{z} \in \mathrm{B}_\varepsilon(\mathbf{0})$. Hence $\dot{V}$ is a negative definite for ε sufficiently small and consequently, (20) is exponentially stable. Since $\boldsymbol{z} = \boldsymbol{T}(\boldsymbol{x})$ is a local diffeomorphic transformation on $\mathrm{B}_\varepsilon(\mathbf{0})$ with $\boldsymbol{T}(\mathbf{0}) = \mathbf{0}$, the true system (18) is also exponentially stable and the proof is complete. ■

4. CONCLUSION

An approach was presented for control of a class of nonlinear systems via approximate feedback linearization. Its principal advantages were that the time-varying uncertainties in the system were possible and the linear parameterization was not necessary. Moreover, the (approximate) feedback linearizability was required only for the nominal system not for the whole family of uncertain systems. The nominal system could be with weakly relative degree, slightly non-minimum phase or non-involutive. The output approximate tracking and the state regulation problems were solved if certain assumptions were satisfied. The scheme can be extended to MIMO nonlinear systems too.

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