NONCOMMUTATIVE CONVEXITY VS LMI'S

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Abstract: Most linear control problems convert directly to matrix inequalities, MIs. Many of these are badly behaved but a classical core of problems convert to linear matrix inequalities (LMIs). In many engineering systems problems convexity has all of the advantages of a LMI. Since LMIs have a structure which is seemingly much more ridged than convexity, there is the hope that a convexity based theory will be less restrictive than LMIs.
A dimensionless MI is a MI where the unknowns are matrices and appear in the formula in a manner which respects matrix multiplication. This holds for most of the classic MIs of control theory. The results presented here suggest the surprising conclusion that for dimensionless MIs convexity offers no greater generality than LMIs. In fact, we prove, for a class of model situations, that a convex dimensionless MI is equivalent to an LMI.

Keywords: Algebraic approaches, Convex optimization, Linear control systems, LMI, Linear matrix inequality

1. INTRODUCTION

Arguably, the biggest revolution in linear control theory of the 1990's is the realization that most linear control problems convert directly to matrix inequalities, abbreviated MIs. Many of these are badly behaved but a classical core of problems convert to linear matrix inequalities (LMIs) which are nicely behaved. Systems problems very directly and by routine methods produce lists of messy matrix inequalities, so it would be valuable to develop computer algebra methods to convert them to nice ones or prove this impossible. Typically nice means "convex". A basic question in light of preponderance in the systems literature of LMIs and the fact that convexity, a seemingly much weaker condition than being an LMI, guarantees numerical success is: *How much more restricted are LMIs than Convex MIs?*

There are two fundamentally different classes of linear systems problems. Ones whose statements do depend on the dimension of the system "explicitly" and ones whose statements do not. **Dimension dependent systems problems** lead to traditional semialgebraic geometry problems, while

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dimensionless systems problems lead directly to a new area which might be called noncommutative semialgebraic geometry, cf. (J. W. Helton and Vinnikov, preprint). The classic results of control such as laid out in the book of Zhou-Doyle-Glover, or for LMIs in the 1997 book of Skelton-Iwasaki-Grigoriadis, have dimensionless matrix unknowns, so lead to what we are calling noncommutative problems. In this article we shall emphasize dimensionless systems problems in which case formulas involve matrices of noncommutative polynomials or rational functions.

1.1 A Familiar Example

For sake of orientation with respect to noncommutativity let us begin with the most ancient and ubiquitous formula in classical linear control theory: the Riccati inequality,

$$AX + XA^T - XBB^TX + C^TC \geq 0, \qquad (1)$$

where " $\geq$ " means positive semidefinite as a matrix. Also there is the LMI (linear in X)

$$\begin{pmatrix} AX + XA^T + C^TC & XB \\ B^TX & I \end{pmatrix} \geq 0. \qquad (2)$$

We could think of the left side as being polynomials (resp. a matrix containing linear polynomials) in noncommuting variables A, B, C, X. The inequalities (1) and (2) are equivalent in that when matrices A, B, C are plugged in for the corresponding noncommuting variables they have the same set of solution matrices X. Distinguishing between A as a variable and A as a matrix might sound pedantic, but we must do this, since we are interested in developing computer algebra algorithms to handle A without breaking it into entries A_{ij}; of course we aim at formulas which are vastly more complicated than Riccatis. Soon our notation will enshrine this in that lower case a will denote the variable and capital A will denote the matrix.

What is done here bears on properties of algorithms programmed to run under our package NCAlgebra which is the main general noncommutative algebra package running under Mathematica. See §6.

2. DIMENSIONLESS CASE: CONVEXITY

This section focuses on symmetric polynomials in noncommuting variables and sets the stage for more general results.

2.1 NC Polynomials

We consider noncommutative (hereafter denoted NC) polynomials with real numbers as coefficients in variables $x = \{x_1, \cdots x_g\}$. While this class does not include Riccatis the theory, even for the case we treat, is new, rich, very challenging and indicative of Riccatis and much more general cases

Example 2.1. Some such polynomials are:

$$p(x) = x_1x_2x_1 + x_1x_2 + x_2x_1 \quad x_1^T = x_1 \;\; x_2^T = x_2.$$

Here we took the variables x_j to be formally symmetric. Next

$$p(x) = x_1^Tx_2x_1 + x_1^Tx_2 + x_2x_1 \quad x_2^T = x_2.$$

Here we took the variable x_2 to be formally symmetric, but x_1 is not.

A NC polynomial p is symmetric provided that it is formally symmetric with respect to the involution ^{T}. Often we shall substitute $n \times n$ matrices $X_1, \cdots, X_g$ into p for the variables $x_1, \cdots, x_g$. If the x_j are designated as symmetric variables the matrices $X_1, \cdots, X_g$ must be taken to be symmetric; then the resulting matrix $p(X_1, \cdots, X_g)$ is symmetric. The variables x_j which are not declared symmetric, if substituted by the matrix X_j, also result in the variables x_j^T being substituted by X_j^T. Henceforth, to save space and confusion, the formal definitions and theorems we state are all for functions in symmetric variables.

Denote the $n \times n$ matrices with real entries by $\mathbb{R}^{n\times n}$, and the subspace of symmetric $n \times n$ matrices by $\mathbb{SR}^{n\times n}$. Similarly, denote the set of all g-tuples $X = (X_1, \ldots, X_g)$ of (resp. symmetric) $n \times n$ matrices by $(\mathbb{R}^{n\times n})^g$ (resp. $(\mathbb{SR}^{n\times n})^g$).

A symmetric NC polynomial in symmetric variables is **matrix-convex** provided that for any tuples $X = (X_1, \ldots, X_g)$ and $Y = (Y_1, \ldots, Y_g)$ of square symmetric matrices of the same dimension n and any $0 \leq t \leq 1$, the matrix

$$tp(X) + (1-t)p(Y) - p(tX + (1-t)Y) \qquad (3)$$

is positive semidefinite. We emphasize that we test with matrices of all dimensions n. A less stringent condition is matrix convexity in a NC open set, where an example of this is **matrix- convex near 0**. This means there is a positive number ε such that if tuples X, Y each lie in the ball

$$\mathcal{B}_\varepsilon := \{Z \in (\mathbb{SR}^{n\times n})^g : Z_1^2 + \cdots + Z_g^2 < \varepsilon^2 I\},$$

then the inequality (3) holds. **Matrix convexity** is defined analogously for NC rational functions, a class of expressions discussed below, see §3.1.

2.2 NC Linear Pencils

At the core of an LMI is a linear pencil. Given a matrix W with entries W_{ij} and a variable x_ℓ, let $Wx_\ell = x_\ell W$ denote the matrix with entries given by

$$(Wx_\ell)_{ij} = W_{ij}x_\ell.$$

A $m \times d$ **NC linear pencil** (in g indeterminantes) is an expression of the form

$$M(x) := M_0 + M_1x_1 + \ldots + M_gx_g$$

and $M_0, M_1, \ldots, M_g$ are $m \times d$ matrices. A **monic** pencil is one with $M_0 = I$. As an example, for

$$M_0 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad M_1 := \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \quad M_2 := \begin{pmatrix} 5 & 4 \\ 4 & 2 \end{pmatrix},$$

the pencil is

$$M(x) = \begin{pmatrix} 1+3x_1+5x_2 & 2x_1+4x_2 \\ 2x_1+4x_2 & -1+x_1+2x_2 \end{pmatrix}.$$

We reiterate that our set up in this brief note does not allow terms linear in x_j like $ax_1 + x_1a^T + c^Tc$ because the coefficients are indeterminants a, c^Tc, however we think our theory is indicative of the behavior there.

An "ordinary", meaning commutative, pencil is one where the variables $x_j's$ all commute.

2.3 NC Convex Polynomials are Trivial

Theorem 2.2. (Helton and McCullough, 2004*a*) *Every symmetric NC polynomial which is matrix convex near 0 has degree two or less. Indeed,*

(1) r must be matrix convex everywhere, and
(2) r is a Schur Complement of an associated NC linear pencil.

Even in the one variable ($g = 1$) case this result, due to Ando twenty years ago, has content since the X and Y in the definition of convex need not commute. For an idea of how to prove Theorem 2.2 see §4.1.

3. CONVEXITY FOR NC RATIONALS

Since LMIs are linear it is easily proved that formulas based on them via Schur complements are convex. One would think that the reverse is not true, but in this section we give a result which presents strong evidence that the reverse is true; namely, *"dimensionless" systems problems which are convex correspond precisely to LMIs.* Specifically, we describe how the convex NC polynomial theorem extends to NC rational functions which are convex near the origin, see (J. W. Helton and Vinnikov, preprint).

3.1 NC Rational Functions

We shall discuss the notion of a NC rational function in terms of rational expressions. There is a technicality,"analytic at 0", which we include for sake of precision which casual readers can ignore.

A **NC rational expression analytic at** 0 is defined recursively. NC polynomials p are NC rational expressions as are all sums and products of NC rational expressions. If r is a NC rational expression and $r(0) \neq 0$, then the inverse of r is a rational expression.

The notion of the **formal domain of a rational expression** r, denoted $\mathcal{F}_{r,\text{formal}}$, and the evaluation $r(X)$ of the rational expression at a tuple $X \in (\mathbb{SR}^{n\times n})^g \cap \mathcal{F}_{r,\text{formal}}$ are also defined recursively [4]. Example 3.1 below is illustrative.

An example of a NC rational expression is, the Riccati expression for discrete-time systems:

$$r = a^Tpa - p + c^Tc$$
$$+(a^Tpb + c^Td)(I - d^Td - b^Tpb)^{-1}(b^Tpa + d^Tc)$$

Here some variables are symmetric some are not. A difficulty is two different expressions, such as

$$r_1 = x_1(1-x_2x_1)^{-1} \quad \text{and} \quad r_2 = (1-x_1x_2)^{-1}x_1$$

that can be converted into each other with algebraic manipulation. Thus they are the same function and one needs to specify an equivalence relation on rational expressions to arrive at what are typically called **NC rational functions**. (This is standard and simple for commutative (ordinary) rational functions.) There are many alternate ways to describe the NC rational functions and they go back 50 years or so in the algebra literature. For engineering purposes one need not be too concerned, since what happens is that two expressions r_1 and r_2 are equivalent whenever the usual manipulations you are accustomed to with matrix expressions convert r_1 to r_2.

For $\mathfrak{r}$ a rational function, that is, an "equivalence class of rational expressions r", define its **domain** by

$$\mathcal{F}_{\mathfrak{r}} := \cup_{\{r \text{ represents } \mathfrak{r}\}} \mathcal{F}_{r,\text{formal}}.$$

Let $\mathcal{F}^0_{\mathfrak{r}}$ denote the arcwise connected component of $\mathcal{F}_{\mathfrak{r}}$ containing 0. We call $\mathcal{F}^0_{\mathfrak{r}}$ the **principal component** of $\mathcal{F}_{\mathfrak{r}}$. Henceforth we do not distinguish between the rational functions $\mathfrak{r}$ and rational expressions r, since this causes no confusion.

Example 3.1.

$$r(x_1, x_2) = (1 + x_1 - (3 + x_2)^{-1})^{-1},$$

where we take $x_1 = x_1^T, x_2 = x_2^T$ is a symmetric NC rational expression. The domain $\mathcal{F}_{r,\text{formal}}$ is

$$\cup_{n>0}\{X_1, X_2 \text{ in } \mathbb{SR}^{n\times n} :$$
$$1 + X_1 - (3 + X_2)^{-1} \text{ and } 3 + X_2 \text{ are invertible}\}.$$

Its principal component $\mathcal{F}^0_r$ is

$$\{X_1, X_2 :\ 1+X_1-(3+X_2)^{-1} > 0 \text{ and } 3+X_2 > 0\}$$

3.2 Convexity vs LMIs

Theorem 3.2. (J. W. Helton and Vinnikov, preprint) *Suppose r is a NC symmetric rational function which is convex near the origin. Then*

(1) r has a representation

$$r(x) = r_0 + r_1(x) + \ell(x)\ell(x)^T \qquad (4)$$
$$+\Lambda(x)(I - L(x))^{-1}\Lambda(x)^T,$$

[4] The formal domain of a polynomial p is all of $(\mathbb{SR}^{n\times n})^g$ and $p(X)$ is defined just as before. The formal domain of sums and products of rational expressions is the intersection of their respective formal domains. If r is an invertible rational expression analytic at 0 and $r(X)$ is invertible, then X is in the the formal domain of r^{-1}.

where

$$L(x),\ \ell(x),\ \Lambda(x),\ r_0 + r_1(x)$$

are linear pencils in $x_1, \cdots, x_g$ satisfying

$$L(0) = 0,\ \ell(0) = 0,\ \Lambda(0) = 0,\ r_1(0) = 0.$$

In addition L and r_1 are symmetric, for example, $L(x)$ has the form $L(x) = A_1x_1 + \cdots + A_gx_g$ for symmetric matrices A_j.
Thus for γ any real number $r - \gamma$ is a Schur complement of the noncommutative linear pencil $\mathcal{M}_\gamma(x) :=$

$$\begin{pmatrix} -1 & 0 & \ell(x)^T \\ 0 & -(I - L(x)) & \Lambda(x)^T \\ \ell(x) & \Lambda(x) & r_0 - \gamma + r_1(x) \end{pmatrix}.$$

(2) The principal component of the domain of r is a convex set, indeed it is the positivity set of the pencil $I - L(x)$. Indeed this holds for any r of the form (4).

This gives a tight enough correspondence between properties of the pencil and properties of r, so that

Corollary 3.3. For any $\gamma \in \mathbb{R}$, principal component, $\mathcal{G}^0_\gamma$, of the set of solutions X to the matrix inequality

$$r(X) < \gamma I$$

equals the set of solutions to an LMI based on a certain linear pencil $\mathcal{M}_\gamma(x)$.

That is, **numerically solving matrix inequalities based on r is equivalent to numerically solving an LMI associated to r.**

3.3 Proof of Corollary 3.3

By item (2) of Theorem 4.2 the upper 2×2 block of $\mathcal{M}_\gamma(X)$ is negative definite if and only if $I - L(X) > 0$ if and only if X is in the component of 0 of the domain of r. Given that the upper 2×2 block of $\mathcal{M}_\gamma(X)$ is negative, definite, by the LDL^T (Cholesky) factorization , $0 > \mathcal{M}_\gamma(X)$ is negative definite if and only if $\gamma I > r(X)$. □

3.4 Proof of Theorem 3.2

We will say little about the proof, since at this early stage of the subject it requires about 50 pages. However, a point of general interest is that the technique for obtaining a representation for r as a Schur Complement of a linear pencil is classical. In fact the following representation of any symmetric NC rational expression r is the symmetric version of the one due originally to Schützenberger and Fliess (who were motivated by automata and formal languages, and bilinear systems) and extended recently by C. Beck and by Ball–Groenewald–Malakorn. See (J. W. Helton and Vinnikov, preprint) for references.

Theorem 3.4. r has a minimal **Descriptor Realization***, namely,*

$$r(x) = r_0 + C(J - \sum_j^g \mathcal{A}_j x_j)^{-1} C^T, \qquad (5)$$

with J is a "signature matrix", that is, J is symmetric and $J^2 = I$, and $\mathcal{A}_j \in \mathbb{R}^{n \times n}$ are symmetric.

For example, a polynomial of degree 10 can be represented in this form. However, convexity of r forces J to be within rank one of I and then algebraic manipulations give Theorem 3.2 item (1). The proof of item (2) is yet more gruelling.

4. NC SEMIALGEBRAIC GEOMETRY (SAG)

A NC symmetric polynomial p is **matrix positive polynomial** provided $p(X_1, \cdots, X_g)$ is a positive semidefinite for every $X \in (\mathbb{SR}^{n \times n})^g$ (and every n). An example of a matrix positive NC polynomial is a **Sum of Squares** of NC polynomials, meaning an expression of the form

$$p(x) = \sum_{j=1}^c h_j(x)^T h_j(x).$$

Clearly, when X in $(\mathbb{SR}^{n \times n})^g$ is substituted into $p(X)$ we get $\sum_{j=1}^c h_j(X)^T h_j(X)$ which is a PSD matrix. Remarkably these are the only positive NC polynomials:

Theorem 4.1. Every matrix positive NC polynomial is a sum of squares.

Those familiar with conventional "commutative" semialgebraic geometry will recognize that this NC behavior is much cleaner. See (Parrilo, 2000; Lasserre, 2001) for a beautiful treatment of applications of commutative SAG.

This theorem is just a sample of the structure of NC semialgebraic geometry. Indeed, there is a strong NC Positivstellensatz recently proved in conjunction with M. Putinar. References containing this and the results alluded to in this section can be found in the bibliography of (Helton and McCullough, 2004*b*).

4.1 Application of NC SAG to NC Convexity

To give an idea of how one proves Theorem 2.2, we prove that the symmetric NC polynomial $p(x) = x^4$ is not matrix convex. While far weaker than the full Theorem 2.2 it is a good illustration of how NC positivity (i.e., NC semialgebraic geometry) is related to NC convexity. It is easy to show that matrix convexity of an NC rational function on a "convex domain" is equivalent to its NC second directional derivative being matrix positive. This is the link between NC convexity and NC positivity. First we define NC directional derivatives, then we turn to convexity of $p(x) = x^4$.

Symbolic Differentiation of NC Functions The *first directional derivative* of a noncommutative rational function $r(x)$ with respect to x in the direction h is defined in the usual way

$$Dr(x)[h] := \frac{d}{dt} r(x+th)\Big|_{t=0}.$$

Likewise, the *second directional derivative* is

$$\mathcal{H}r(x)[h] = \frac{d^2}{dt^2} r(x+th)\Big|_{t=0}. \quad (6)$$

For example, take $x = \{x_1\}$ and $p(x) = x^4$. Get

$$Dp(x)[h] = hxxx + xhxx + xxhx + xxxh$$

and get $\mathcal{H}r(x)[h]$

$$= 2(hhxx + hxhx + hxxh + xhhx + xhxh + xxhh)$$

Nonconvexity of x^4 If $p(x) = x^4$ is matrix convex, then $\mathcal{H}r(x)[h]$ is matrix positive and by Theorem 4.1 is a sum of squares,

$$hhxx + hxhx + hxxh + xhhx + xhxh + xxhh$$
$$= f_1(x,h)^T f_1(x,h) + \cdots + f_k(x,h)^T f_k(x,h).$$

One can show that each $f_j(x,h)$ is linear in h. Thus one term, say $f_1^T f_1$, must contain $hhxx$. Thus $f_1 = hxx + h +$ *more*. However then

$$f_1^T f_1 = xxhhxx + \text{ much more}.$$

Moreover, the term $xxhhxx$ can not be cancelled out, contradicting that $\mathcal{H}(x)[h]$ is of degree 4. □

5. DIMENSION DEPENDENT CASE: CONVEXITY

In this section only we treat commutative x and corresponding linear pencils $M(x)$ in commutative variables $x \in \mathbb{R}^g$.

5.1 Two Closely Related Questions

Q1. We say a set $\mathcal{C} \subset \mathbb{R}^g$ has a **Linear Matrix Inequality (LMI) Representation** provided that there is a linear pencil $M(x)$ for which

$$\{x \in \mathbb{R}^g : M(x) \quad \text{is PosSemiDef}\} \quad (7)$$

equals the set $\mathcal{C}$. Without loss of generality we may take the pencil to be monic, that is $M(0) = I$. Parrilo and Sturmfels (see[PSpre] in (Helton and Vinnikov, preprint)) formally ask:
Which sets have an LMI representation?

Q2. Define a polynomial $\check{p}$ by

$$\check{p}(x) := \det[I + M_1 x_1 + \cdots + M_g x_g]. \quad (8)$$

where the M_j are symmetric $d \times d$ matrices. This is said to be a **monic determinantal representation** for $\check{p}$ and d is called the **size** of the representation.
Which polynomials have a monic determinantal representation?

As we shall see an answer to Q2 also answers Q1. This section describes work in (Helton and Vinnikov, preprint) which settles the questions for $g = 2$.

5.2 Obvious Necessary Conditions

5.2.1. p Satisfies the Real Zeroes Condition For a determinantal representation to exist there is an obvious condition. Observe from (8) that

$$\check{p}(\mu x) := \mu^d \det[\frac{I}{\mu} + M_1 x_1 + \cdots + M_g x_g].$$

Since (for real numbers x_j) all eigenvalues of the symmetric matrix $M_1 x_1 + \cdots + M_g x_g$ are real we see that, while $p(\mu x)$ is a complex valued function of the complex number μ, it vanishes only at μ which are real numbers. This condition is critical enough that we formalize it in a definition.

A **Real Zero polynomial** (**RZ** polynomial) is defined to be a polynomial in g variables satisfying, for each $x \in R^g$,

$$p(\mu x) = 0 \quad \textit{for } \mu \textit{ a complex number}$$

implies

(**RZ**) $\quad\mu$ *is actually a real number*

Example 5.1. $p(x) := 1 - (x_1^4 + x_2^4)$. Then

$$p(\mu x) = [1 - \mu^2 (x_1^4 + x_2^4)^{\frac{1}{2}}][1 + \mu^2 (x_1^4 + x_2^4)^{\frac{1}{2}}]$$

which for any x_1, x_2 has 2 complex (not real zeroes) which are $\mu_\pm := i(x_1^4 + x_2^4)^{-\frac{1}{2}}$. Thus p does not satisfy the Real Zero condition.

5.2.2. LMI Representations Suppose we are given a monic pencil $M(x)$ which represents a set $\mathcal{C}$ as in (7). Clearly

(1) *$\mathcal{C}$ is a convex set (with 0 in $\mathcal{C}$) .*
(2) *The boundary of $\mathcal{C}$ is contained in the zero set of the polynomial $\check{p}(x) := detM(x)$, so the boundary of $\mathcal{C}$ lies on an algebraic curve.*

In addition, $\mathcal{C}$ is what we call an **Algebraic Interior**, that is, it is a set $\mathcal{C}$ in R^g for which there is a polynomial p in g variables (normalized by $p(0) = 1$), such that $\mathcal{C}$ equals the closed arcwise connected set containing 0, with $p(x) > 0$ on its interior and $p(x) = 0$ on its boundary. Such a p is called a **defining polynomial** for $\mathcal{C}$.

5.3 Main Representation Results

It is shown in (Helton and Vinnikov, preprint) that:

The minimal degree defining polynomial p for an Algebraic Interior $\mathcal{C}$ is unique;

let d denote the degree of p and say that $\mathcal{C}$ is an **Algebraic Interior of Degree** d. Moreover, $\mathcal{C}$ will be called **rigidly convex** provided that its minimal degree defining polynomial satisfies the Real Zero condition.

Theorem 5.2. If $\mathcal{C}$ has an LMI representation, then it is rigidly convex.

The converse is true in two dimensions, that is, when $g = 2$. Furthermore, in this case there exists an LMI representation of the size equal to the degree of $\mathcal{C}$.

Conjecture 5.3. We conjecture that for any dimension g if $\mathcal{C}$ is rigidly convex, then it has an LMI representation.

Remark 5.4. When $g > 2$, the minimal size of an LMI representation in Conjecture 5.3 is in general larger than the degree of $\mathcal{C}$.
Conjecture 5.3 follows from

Conjecture 5.5. [(Helton and Vinnikov, preprint)] Every Real Zero polynomial has a monic determinantal representation.
For $g = 2$ this is proved true in (Helton and Vinnikov, preprint).

This might be called a modified Lax conjecture in that Lewis, Parrilo and Ramana (A. S. Lewis and Ramana, to appear) settled a 1958 conjecture of Peter Lax affirmatively for $g = 2$, using the $g = 2$ monic determinantal representation mentioned above. They proved the Lax conjecture false for $g > 2$ but our conjecture is a natural modification.

6. SOFTWARE

Here is a list of software running under NCAlgebra (which runs under Mathematica) that implements and experiments on symbolic algorithms pertaining to NC Convexity and LMIs.

http://www.math.ucsd.edu/ ∼ ncalg

LMI producing: A symbolic algorithm of N. Slinglend has been implemented by J. Shopple under NCAlgebra to construct the linear pencil $\mathcal{M}$ in Theorem 3.2 symbolically from r. While it provably always associates an LMI with a convex r it works now only on small problems. Also to flexibly handle control problems more generality is needed and work is in progress. However, these results establish that the correspondence between dimensionless systems problems which are convex and LMIs is very strong.

Convexity Checker: Camino, Helton, Skelton have an (algebraic) algorithm for determining the region on which a rational expression is convex.

Classical Production of LMIs: There are two Mma notebooks by de Oliveira and Helton. The first is based on algorithms for implementing the 1997 approach of Skelton, Iwasaki and Grigonidas associating LMIs to many control problems. The second (requires C++) produces LMIs by symbolically implementing the 1997 change of variables method of C. Scherer et. al.

7. CONCLUSIONS

7.1 Commutative:
Semialgebraic geometry is a subject which goes back 75 years.

For sets convexity is different from having an LMI representation, which we conjecture is the same as rigid convexity. Parrilo speculated that any convex algebraic interior has a lift to an LMI representable set.

Change of variables to make a problem convex is classically analyzed (via Morse theory) and understood, but what one finds seems not to be profoundly practical.

7.2 Noncommutative:
NC semialgebraic geometry is only a few years old and while challenging is developing well into a mathematically rich area.

We conjecture that NC convexity is the same as having an NC LMI representation. While far from proved, evidence for that conclusion is strong.
Needed is a theory with "letters as coefficients" and for matrices with NC rational entries. The first of these while messy has been done in many situations, e.g., in some of the software.

Changing variables to make a NC problem convex is an open area. It is possible that NC changes of variables will behave better than classically.

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