ROBUST ITERATIVE LEARNING CONTROL ON FINITE TIME INTERVALS

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Abstract: This paper considers the problem of designing optimal robust Iterative Learning Control (ILC) algorithms for LTI processes. Given a multiplicative uncertainty model of the plant, learning operators are designed to minimize the ultimate tracking error and convergence rate, while guaranteeing robust convergence. The optimal learning operators are shown to be noncausal. After the controller is optimized in the frequency domain, a finite-time implementation of the algorithm is shown to achieve the same performance and robustness.

Keywords: Iterative learning control; Robust control; Optimal control; Noncausal; Truncation effects, Linear systems.

1. INTRODUCTION

Iterative Learning Control (ILC) improves trajectory following by modifying control inputs based on past errors. A wide survey of the ILC field is presented in (Moore, 1999). Unlike feedback control, which is necessarily causal, ILC can process past error signals using noncausal operators. Continuous-time analyses of noncausal ILC algorithms have appeared in (Chen and Moore, 2000; Pandit and Buchheit, 1999).

Although a central purpose of ILC is to overcome plant uncertainty through learning, few ILC analyses model plant uncertainty explicitly. General formulations of the robust ILC design problem for LTI plants have appeared in (Tayebi and Zaremba, 2003; Goldsmith, 2003), where plant uncertainty is modelled as a multiplicative perturbation in the frequency domain, as is commonly done in classical feedback control design (Doyle *et al.*, 1992). A major limitation of the design proposed in (Tayebi and Zaremba, 2003) is that it uses causal operators in the ILC algorithm. It is well known that ILC restricted to proper causal operators cannot converge to zero error for plants having relative degree greater than one (Ghosh and Paden, 2001). Moreover, for every causal ILC, there exists a feedback control (constructed from the ILC operators alone) that achieves the ultimate tracking error of the ILC, without any iterations (Goldsmith, 2002).

A robust design problem formulated in the frequency domain was recently solved using noncausal ILC (Goldsmith, 2003). However, the results of such analyses on the infinite time interval do not extend trivially to trials of finite duration, as noncausal operators require future information that is unavailable beyond the end of the trial (Padieu and Su, 1990).

This paper extends the idealized design methodology of (Goldsmith, 2003) to the real situation where trials have finite duration. A design modification ensures that robustness is preserved when the ILC is implemented on a finite time interval. In Section 2, we formulate the general ILC design problem for LTI plants. Properties of noncausal operators are described in Section 3. The optimal design is presented in Section 4. In Section 5, we propose a finite-time implementation of the optimal noncausal ILC and prove its robustness. Section 6 concludes the paper.

2. ITERATIVE LEARNING CONTROL

For clarity, we shall restrict ourselves to SISO systems. Signals are in L_{2e}, the extension of $L_2(-\infty,\infty)$ to include unstable signals. The process to be controlled is modelled as

$$y_i = Pu_i, \tag{1}$$

where $i \in \{0,1,2,...\}$ is the trial number, y_i is the output, u_i is the control input, and P is a proper causal LTI operator. We shall use the same symbol to denote a signal and its Laplace transform (and similarly for signal operators and their transfer functions), as the meaning should be clear from the context.

It is desired that y_i follow a reference signal $y_d \in L_2$, and that the tracking accuracy improve as the number of trials increases. The error in trial i is

$$e_i = y_d - y_i. \tag{2}$$

Substitution of (1) into (2) gives

$$e_i = y_d - Pu_i. \tag{3}$$

The ILC algorithm is given by

$$u_i = Ce_i + Fu_{i-1} + Le_{i-1}, \tag{4}$$

where C, F, and L are proper LTI operators. Note that the feedback operator C is necessarily causal since it is implemented in real time, but F and L can be noncausal. Feedback is essential if P is unstable.

A fixed point (e_∞, u_∞) of the iterative system (3) and (4) has the property that $e_i = e_{i-1} = e_\infty$ and $u_i = u_{i-1} = u_\infty$. Hence

$$e_\infty = y_d - Pu_\infty, \tag{5}$$

and

$$u_\infty = Ce_\infty + Fu_\infty + Le_\infty, \tag{6}$$

which may be solved to give

$$e_\infty = Ry_d, \tag{7}$$

where

$$R = \frac{1-F}{1-F+(C+L)P}. \tag{8}$$

The goal of ILC is to make the system converge to a small e_∞. Although $F=1$ gives $e_\infty = 0$, we must also ensure that e_i converges to e_∞. If we define $\bar{e}_i = e_i - e_\infty$ and $\bar{u}_i = u_i - u_\infty$, then (3) and (5) give

$$\bar{e}_i = -P\bar{u}_i. \tag{9}$$

Similarly, (4) and (6) give

$$\bar{u}_i = C\bar{e}_i + F\bar{u}_{i-1} + L\bar{e}_{i-1}, \tag{10}$$

Substituting (9) at i and at $i-1$ into (10) gives

$$\bar{u}_i = H\bar{u}_{i-1}, \tag{11}$$

where

$$H = S(F - LP), \tag{12}$$

and

$$S = (1+CP)^{-1}. \tag{13}$$

Applying P to both sides of (11) and substituting in (9) at i and at $i-1$ gives

$$\bar{e}_i = H\bar{e}_{i-1}, \tag{14}$$

We shall use $\|\cdot\|$ to mean the L_2 norm of a time signal and the induced L_2 norm of an LTI operator, which equals the ∞-norm of its transfer function. If H is stable, then a sufficient condition for $\bar{u}_i$ and $\bar{e}_i$ to converge to zero is that

$$|H(j\omega)| < 1 \tag{15}$$

for all ω. A slightly stronger condition, which guarantees uniform convergence, is that there exists $\mu < 1$ such that

$$|H(j\omega)| \le \mu \tag{16}$$

for all ω. This is equivalent to

$$\|H\| < 1. \tag{17}$$

Uncertainty in P is modelled as

$$P = P_0(1 + \Delta W_2), \tag{18}$$

where P_0 is a nominal model of the plant, W_2 is a known stable causal transfer function, and Δ is an unknown stable causal transfer function satisfying $\|\Delta\| < 1$.

3. NONCAUSAL OPERATORS

An ILC with causal F and L is limited to the performance and robustness of feedback control (without iterations). If the causal ILC converges to zero ($F = 1$), then the high gain feedback $u = Ke$ with $K = C + k(C+L)$ gives $\lim_{k\to\infty} e =$

0 (Goldsmith, 2002). This is possible only for minimum phase plants of relative degree 0 or 1. For more general plants, if the causal ILC converges to e_∞, then the feedback control $K = (1 - F)^{-1}(C + D)$ gives $e = e_\infty$ (without iterations). Since this 'equivalent feedback' does not depend on the plant, it is as robust as the ILC: if the plant is perturbed and the ILC still converges, then so does the equivalent feedback.

An operator is *causal* if its output at any time r depends only on its input at times $t \leq r$. An operator C is causal if, for each $r \in (-\infty, \infty)$,

$$T_r C T_r = T_r C, \qquad (19)$$

where T_r is the right truncation operator, defined by

$$(T_r u)(t) = u(t) \quad t \leq r \qquad (20)$$
$$= 0 \quad \text{otherwise}, \qquad (21)$$

for $t \in (-\infty, \infty)$. A causal operator C also has the property that for each $l \in (-\infty, \infty)$,

$$T_l C T_l = C T_l, \qquad (22)$$

where T_l is the left truncation operator defined by

$$(T_l u)(t) = u(t) \quad t \geq l \qquad (23)$$
$$= 0 \quad \text{otherwise}, \qquad (24)$$

for $t \in (-\infty, \infty)$.

An operator is *anticausal* if its output at any time l depends only on its input at times $t \geq l$. An operator A is anticausal if, for each $l \in (-\infty, \infty)$,

$$T_l A T_l = T_l A. \qquad (25)$$

An anticausal operator A also has the property that for each $r \in (-\infty, \infty)$,

$$T_r A T_r = A T_r. \qquad (26)$$

These properties will be useful in Section 5 when we consider the effect of truncation on finite-time ILC.

The term *noncausal* is used in the same manner as *nonlinear*. It can mean either *not causal* or *not necessarily causal*, depending on the context. In the latter sense, causal is a special case of noncausal.

A concept closely related to causal operators is causal signals. A *causal signal* is a signal $u(t)$ that is zero for $t < 0$, while an *anticausal signal* is zero for $t > 0$. A proper LTI operator is causal if its impulse response is causal and anticausal if its impulse response is anticausal. The impulse response of a proper noncausal operator is the sum of a causal signal and an anticausal signal.

Of particular interest are symmetric operators whose anticausal and causal parts are reflections about $t = 0$. An example is a symmetric noncausal low-pass filter N whose impulse response is

$$n(t) = 1/2\omega_0 e^{-|\omega_0 t|}, \qquad (27)$$

where $\omega_0 > 0$. This signal is stable, decaying as t approaches $-\infty$ and $+\infty$. The (two-sided) Laplace transform of $n(t)$ is

$$N(s) = \frac{1}{\left(1 + \frac{s}{\omega_0}\right)\left(1 - \frac{s}{\omega_0}\right)} \qquad (28)$$

$$-\omega_0 < Re(s) < \omega_0, \qquad (29)$$

Note that this transfer function is stable, even though it has a RHP pole at $s = \omega_0$. This is because its domain (29) lies to the left of this pole. Taking the inverse Laplace transform of $N(s)$ along a vertical contour in (29) transforms the RHP pole of $N(s)$ back into a *stable* anticausal signal (given by (27) for $t \leq 0$) instead of an *unstable* causal signal.

Setting $s = j\omega$ in (28) gives

$$N(j\omega) = \frac{1}{1 + \left(\frac{\omega}{\omega_0}\right)^2}, \qquad (30)$$

showing that $N(j\omega)$ is a real-valued function, and thus has zero phase at all frequencies.

We will see that the optimal F and L in the ILC (4) are noncausal. They should also be proper, since improper operators such as differentiation (Arimoto *et al.*, 1984)) amplify high-frequency noise.

4. OPTIMAL ILC DESIGN

The general case of designing C, F and L is considered in (Goldsmith, 2003). Here, we shall focus on the case when $C = 0$ and P is stable. For unstable P, this design can be applied to the stabilized closed-loop plant SP resulting from suitable design of C, as long as ΔW_2 is then redefined as the uncertainty in SP.

Our design goal is to minimize the nominal value of e_∞ while guaranteeing ILC convergence in the face of plant uncertainty. With $S = 1$ (since $C = 0$ in (13)), (12) becomes

$$H = F - LP. \qquad (31)$$

Substitution of (18) into (31) gives

$$H = F - LP_0(1 + \Delta W_2). \qquad (32)$$

First, we shall consider the case when $\|W_2\| \leq 1$. The optimal ILC in this case is

$$F = 1 \tag{33}$$

$$L = NP_0^{-1}, \tag{34}$$

where N is a stable symmetric noncausal low-pass filter having relative degree at least equal to that of P_0 (to make L proper) and satisfying

$$0 < N(j\omega) \leq 1. \tag{35}$$

Substitution of (33) and (34) into (32) gives, at every frequency

$$|H| = |1 - N(1 + \Delta W_2)| \tag{36}$$

$$= |1 - N - N\Delta W_2| \tag{37}$$

$$< |1 - N| + |N| \tag{38}$$

$$= 1, \tag{39}$$

where (38) follows from $\|\Delta W_2\| < 1$, and (39) follows from (35). Hence, the convergence condition (15) is satisfied. Substituting (33) into (7) gives $e_\infty = 0$.

Although this design is optimal ($e_\infty = 0$ is the smallest error possible), the optimal filter N is not unique. The bandwidth of N may be chosen to achieve a desired tradeoff between nominal convergence rate and noise attenuation. Let H_0 denote the value of H when the uncertainty $W_2 = 0$. Then (36) gives

$$|H_0| = |1 - N|. \tag{40}$$

Fast nominal convergence of (11) is achieved by choosing N near 1 at lower frequencies, while good noise attenuation is achieved by choosing N small at high frequencies. A tradeoff occurs at intermediate frequencies.

Now let us consider the case $\|\Delta W_2\| > 1$, since in most real plants $|W_2| > 1$ at high frequencies. Then $\|e_\infty\| = 0$ is not possible, which can be seen as follows. By (7), $\|e_\infty\| = 0$ implies $F = 1$. If $N = 0$, then $H = 1$ in (36), and the ILC does not converge. If $N \neq 0$, then at a frequency where $|\Delta W_2| > 1$, the phase of $1 + \Delta W_2$ can be made equal to any value by varying the phase of Δ. Hence $N(1 + \Delta W_2)$ in (36) can be made negative real, implying $|H| > 1$.

To design F and L for the case $\|W_2\| > 1$, we select the following design specifications for robustness, convergence rate, and nominal performance:
i) robust convergence
ii) instant nominal convergence
iii) minimized nominal $\|e_\infty\|$

We again parameterize L as $L = NP_0^{-1}$ with N a new design parameter, which must have relative degree at least equal to that of P so that L is proper. Equation (32) becomes

$$H = F - N(1 + \Delta W_2), \tag{41}$$

and the nominal value of H is

$$H_0 = F - N. \tag{42}$$

Instant nominal convergence of (11) implies that $F = N$, and hence

$$H = -N\Delta W_2. \tag{43}$$

Since $\|\Delta\| < 1$, there exists $\mu < 1$ such that

$$|\Delta(j\omega)| \leq \mu \quad \forall\omega \in R \tag{44}$$

Taking μ to be the largest such value and substituting (44) into (43) gives

$$|H(j\omega)| \leq \mu |N(j\omega)W_2(j\omega)|, \quad \forall\omega \in R \tag{45}$$

This satisfies the convergence condition (16) iff

$$|N(j\omega)W_2(j\omega)| \leq 1, \quad \forall\omega \in R \tag{46}$$

Meanwhile, the nominal value of e_∞ is given by (7) as

$$e_\infty = (1 - N)y_d \tag{47}$$

At low frequencies, where $|W_2| \leq 1$, the choice $N = 1$ gives $e_\infty = 0$ in (47) while satisfying (46). At frequencies where $|W_2| > 1$, the N satisfying (46) and minimizing $\|e_\infty\|$ in (47) is positive real (i.e. a symmetric noncausal filter), and is given by

$$N(j\omega) = |W_2(j\omega)|^{-1}, \quad \forall\omega \in R. \tag{48}$$

The optimal ILC is thus

$$u_i = Nu_{i-1} + NP_0^{-1}e_{i-1}. \tag{49}$$

5. TRUNCATION EFFECTS

A major limitation of the foregoing analysis is that it assumes trials of infinite duration. Here we shall consider the real situation in which ILC trials are performed on a finite time interval $t \in [l, r]$.

Let

$$T = T_l T_r, \tag{50}$$

where T_r and T_l are defined by (21) and (24). The truncated signal $Tu \in L_{2e}$ uniquely represents a signal in $L_2[l, r]$.

Given any $L : L_{2e} \to L_{2e}$, TLT is called the compression of L to $L_2[l, r]$. It applies L to a truncated input and then truncates the output. If L is linear

$$\|TLT\| \leq \|L\|, \tag{51}$$

since $\|T\| = 1$. If C and P are both causal, then (19), (22), and (50) imply that

$$TCTPT = TCPT. \tag{52}$$

Truncation is not problematic for causal operators such as feedback. If a feedback C is applied to a plant P, the truncated error response to a truncated reference Ty_d is $e = TSTy_d$, which is smaller in norm than the infinite-time error response $e = STy_d$.

We assume P does not include a pure time delay, since any delay is handled trivially by defining time zero ($l = 0$) of the error signal as the time the plant starts responding to the reference. This amounts to inverting a time delay with a time advance. With no (remaining) time-delay in P, a causal (but generally improper) P^{-1} exists.

The initial conditions of the plant are assumed to be zero. This assumption is not restrictive, since non-zero initial conditions x_0 become zero under the state tranformation $x \to x - x_0$. Such zeroing should be performed at the start of each ILC trial. However, if the same sensors are used to reset the system on subsequent trials, this new zero must *not* be used in the resetting procedure, otherwise any systematic resetting error present would accumulate. Actual reset errors must be reduced by improving the resetting procedure, not the ILC.

In a real ILC implementation, since the plant operates only when $t \in [0, r]$, the real plant is the compression TPT of the L_{2e} operator P, where $l = 0$ in (50). Hence, (9) becomes

$$\bar{e}_i = -TPT\bar{u}_i. \tag{53}$$

Similarly, the finite-time implementation of (4) is

$$u_i = TFTu_{i-1} + TLTe_{i-1}, \tag{54}$$

or

$$\bar{u}_i = TFT\bar{u}_{i-1} + TLT\bar{e}_{i-1} \tag{55}$$

with respect to the fixed point. Substituting (53) at i and at $i-1$ into (55) gives

$$\bar{u}_i = H_T\bar{u}_{i-1}, \tag{56}$$

where

$$H_T = TFT - TLTPT. \tag{57}$$

By (56) and (53), $\bar{u}_i$ and $\bar{e}_i$ converge to zero if $\|H_T\| < 1$.

Based on our optimal solution for the (usual) case when $\|W_2\| > 1$, we set $F = N_T$ and $L = N_TTP_0^{-1}$, where N_T is a parameter to be designed. Substituting these into (57) gives

$$\begin{aligned} H_T &= TN_TT - TN_TTP_0^{-1}TPT && (58)\\ &= TN_TT - TN_TTP_0^{-1}PT && (59)\\ &= TN_TT - TN_TT(1-\Delta W_2)T && (60)\\ &= TN_TT\Delta W_2T, && (61)\end{aligned}$$

where (59) follows from (52) and the fact that P and P_0^{-1} are causal. If $W_2 = 0$, (61) gives $H_T = 0$, showing that this ILC design yields instant nominal convergence (specification (ii) in Section 4) when applied to a finite time interval. However, guaranteeing robust convergence requires careful construction of N_T.

The symmetric noncausal filter N with magnitude response given by (48) may be factored as $N = N_aN_c$, where N_c is a causal minimum-phase lowpass filter having the same magnitude response as N, while N_a is an anticausal all-pass filter of unit norm and opposite in phase to N_c. We choose N_T as

$$N_T = N_aTN_c. \tag{62}$$

Substituting (62) into (61) gives

$$\begin{aligned} \|H_T\| &= \|TN_aTN_cT\Delta W_2T\| && (63)\\ &= \|TN_aTN_c\Delta W_2T\| && (64)\\ &\le \|TN_aT\| \cdot \|\Delta\| \cdot \|T\| && (65)\\ &\le \mu < 1, && (66)\end{aligned}$$

Here, (64) follows from (52) and the fact that N_c and ΔW_2 are causal, (65) follows from $|N_c(j\omega)| = |N(j\omega)|$ and (48), and (66) follows from $\|T\| = \|N_a\| = 1$ and $\|\Delta\| = \mu < 1$. This ILC design thus guarantees robust convergence when applied on a finite time interval. The complete update law for the case $C = 0$ is

$$u_i = TN_aTN_cTu_{i-1} + TN_aTN_cTP_0^{-1}Te_{i-1}. \tag{67}$$

Let us now consider nominal performance. Substituting the fixed point of (67) into $e_\infty = Ty_d - TPTu_\infty$ with $P = P_0$ yields

$$\begin{aligned} e_\infty &= (T - TN_aTN_cT)y_d && (68)\\ &= (T_r - T_lN_aT_rN_c)y_d, && (69)\end{aligned}$$

where (69) follows from $T = T_lT_r$, from properties (19), (22), (25), and (26), and from $T_ly_d = y_d$ (since $l = 0$ and $y_d(t) = 0$ for $t < 0$).

We wish to compare this e_∞ with that produced on $[0, r]$ by the infinite-time ILC (4) (with $C = 0$). The latter gives e_∞ as

$$\begin{aligned} e_\infty^* &= T(1-N)y_d && (70)\\ &= (T_r - TN_aN_c)y_d, && (71)\end{aligned}$$

where we have again used the fact that $T_ly_d = y_d$. If we let $x = e_\infty - e_\infty^*$, then (69) and (71) give

$$x = (TN_aN_c - T_lN_aT_rN_c)y_d \quad (72)$$

$$= TN_a(1 - T_r)N_cy_d, \quad (73)$$

where in (73) we have used the fact that $N_aT_r = T_rN_aT_r$. The signal x represents the *truncation error*, which is the difference between the $e_\infty(t)$ observed on $t \in [0, r]$ when the ILC is implemented on $(-\infty, \infty)$ and that observed when the ILC is implemented on $[0, r]$.

The time-dependence of the truncation error x may be obtained as follows. The anticausal allpass filter N_a has the form $N_a = \pm 1 + A$, where A is a stable strictly- proper anticausal filter, and the sign of the identity depends on the order of the filter. The identity in N_a is annihilated in (73) since $T(1-T_r) = TlT_r(1-T_r) = Tl(T_r - T_r) = 0$. Therefore (73) becomes

$$x = TA(1 - T_r)y_{df}, \quad (74)$$

where $y_{df} = N_cy_d$ denotes the (causally) filtered reference input. Let a denote the impulse response response of A. Then, for all $t \in [0, r]$, (74) gives

$$x(t) = \int_r^\infty a(t - \tau)y_{df}(\tau)d\tau. \quad (75)$$

Since A is anticausal and strictly proper, there exists $b, \lambda > 0$ such that, for all $\theta \leq 0$,

$$|a(\theta)| \leq be^{\theta/\lambda}. \quad (76)$$

Substitution of (76) into (75) gives

$$|x(t)| \leq b \int_r^\infty e^{(t-\tau)/\lambda}|y_{df}(\tau)|d\tau. \quad (77)$$

In (77), λ is the effective time constant of the filter A, and is typically much smaller than the ILC interval r. In a robot application, for example, r might be a few seconds and λ a few milliseconds. At a time t where $r - t >> \lambda$ the exponential term in (77) is approximately zero for all $\tau \in [r, \infty)$, giving $|x(t)| \approx 0$. Hence the truncation error is negligible at times far away from r. However (77) shows that the truncation error sharply increases as t approaches r.

A simple remedy to the problem of truncation error is suggested by (77). Suppose the ILC must accurately track a reference $y_d(t)$ for $t \in [0, t_f]$. If we extend $y_d(t)$ beyond t_f to time $r >> t_f + \lambda$, and apply ILC on the interval $[0, r]$, then (77) shows that the truncation error is negligible in $[0, t_f]$. We call $[t_f, r]$ the *runoff* or *follow-through* region. It need only be a few time constants λ in duration to nearly eliminate truncation errors in $[0, t_f]$. Note that the robust convergence result (66) applies on the entire interval $[0, r]$ (including the runoff region), not just on $[0, t_f]$.

6. CONCLUSION

We have investigated the problem of designing optimal robust ILC algorithms for LTI processes. The optimal learning operators were found to be noncausal. A proposed finite-time implementation of the ILC was proved to preserve the robustness of the infinite-time design, and a simple method was proposed for attenuating truncation artifacts.

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