REJECTION OF UNMEASUREABLE EXTENDED CONSTANT DISTURBANCES USING MODEL PREDICTIVE CONTROL

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Abstract: This paper considers the Model Predictive Control (MPC) set point tracking/regulation problem for a discrete LTI system, which is subject to a class of unbounded disturbances/tracking signals called extended constant signals. The main contribution is a formulation of the system's plant equations under which, for output regulation, **no knowledge of the structure or magnitude of disturbances is needed in order to achieve set point regulation for this class of signals.** The result is of interest since it implies that **no disturbance observer is necessary in order to solve the set point tracking/regulation problem when full-state feedback is available.** The results are experimentally verified.

Keywords: model predictive control, disturbance rejection, output feedback/regulation, observers, tracking

1. INTRODUCTION

One of the most attractive features of MPC is its ability to directly deal with constraints. There is a vast literature on MPC; some recent survey articles and texts on the topic include (Rossiter, 2003), (Maciejowski, 2002), (Goodwin *et al.*, 2001), (Lee and Kouvaritakis, 2001), (Lee and Kouvaritakis, 2000), (Mayne *et al.*, 2000), (Camacho and Bordons, 1999), (Lee and Kouvaritakis, 1999), (Rawlings, 1999), and some representative research results include (Clarke, 1994), (Lee *et al.*, 1994), (Muske and Rawlings, 1993), (Soeterboek, 1992), (Bitmead *et al.*, 1990), (Garcia *et al.*, 1989), (Keerthi and Gilbert, 1988). This paper focuses on the MPC tracking and regulation problem, for a plant with unknown, unmeasurable disturbances, as modeled by

$$x[k+1] = Ax[k] + Bu[k] + Ew[k]$$
$$y[k] = Cx[k] + Du[k] + Fw[k],$$

where u, w and y denote the control signals, disturbances and outputs respectively, and where E and F are not necessarily assumed to be known. $w[k]$ is assumed to satisfy the property $\lim_{k\to\infty}(w[k+1] - w[k]) = 0$. Signals which satisfy this property are called *extended constant signals* (Davison and Scherzinger, 1987) and include constant signals as a special case; other examples include the unbounded signals $w[k] = k^{\theta}$, $0 \leq \theta < 1$ and $w[k] = \log(k)$, $\forall k = 1, 2, 3, \cdots$. In this MPC problem, it is assumed that constraints are imposed on the inputs or states of the system. For the case of constant disturbances, the conventional approach to apply MPC is to use an observer based estimate of the disturbance; e.g. see (Rossiter, 2003), (Maciejowski, 2002), (Goodwin *et al.*, 2001), (Rawlings, 1999). In particular, a common way to deal with unmeasured disturbances is to use Kalman filtering in which the disturbances are included in the system dynamics, e.g. see (Rossiter, 2003), (Maciejowski, 2002) and (Goodwin *et al.*, 2001). This typically involves a knowledge of the E and F matrices. For example, for the case of constant disturbances, which are a subset of extended constant disturbances, the following comments are made in recent texts dating as late as 2003 :

- In (Rossiter, 2003), pg. 21-22 "Of course the disturbance is unknown, as is the state $x[k]$, so it must be estimated. ... An observer can be constructed for the model under the usual assumption of observability, to give an estimate of both the state x and the disturbance *d*." and "Again the overall process model should be augmented to include the disturbance dynamics as follows ..."

- In (Maciejowski, 2002), pg. 56-57, "It is instructive to see how the 'constant output disturbance' assumption we made in the previous section can be handled using observer theory. We can do this by augmenting the model of the plant, so that it includes a model of the output disturbance."
- In (Goodwin *et al.*, 2001), pg. 756-757 "one could use a Kalman Filter to estimate the current state from observations of the output. Disturbances can also be included in this strategy by including the appropriate noise shaping filters in a composite model."

As well, when considering MPC which is based on state estimation, the use of disturbance models for the state estimator is assumed: e.g.

- In (Rawlings, 1999) "disturbances are obviously uncontrollable and are *required* only in the state estimator."

In this paper **no knowledge of the disturbance matrices E and F, nor of the structure or magnitude of the disturbance, nor an estimate of the unmeasurable extended constant disturbance** is needed in order to guarantee asymptotic regulation of the output to a specified extended constant tracking signal. Furthermore, no bounds of any type are required on the allowable disturbances. This is an advantage over conventional approaches, which typically assume that the disturbances are constant and that a knowledge of the disturbance matrices E and F, in a given problem, is available.

2. LTI SYSTEMS AND THE RSP

Consider the LTI discrete time system which has the input $u[k] \in \Re^m$, the output $y[k] \in \Re^r$, state $x[k] \in \Re^n$, an unmeasurable extended constant disturbance signal $w[k] \in \Re^q$ and the extended constant tracking signal $y_{ref}[k] \in \Re^r$:

$$\begin{aligned} x[k+1] &= Ax[k] + Bu[k] + Ew[k] \\ y[k] &= Cx[k] + Du[k] + Fw[k] \\ e[k] &= y[k] - y_{ref}[k], \end{aligned} \tag{1}$$

where $e[k]$ is the error in the system. The following existence result for a solution to the robust servomechanism problem (RSP) for (1), assuming no constraints are applied, is given:

Lemma 2.1. (Davison and Scherzinger, 1987), (Davison, 1996) There exists a solution to the RSP for (1) for extended constant tracking/disturbance signals iff the following conditions are all satisfied:

(1) (C, A, B) is stabilizable and detectable,

(2) $\text{rank}\begin{bmatrix} A-I & B \\ C & D \end{bmatrix} = n + r$ and

(3) the output $y[k]$ is measurable.

Assume now that the existence conditions of lemma 2.1 all hold and that the control signal of (1) is bounded. Dropping the time sample notation $[k]$, assume that the upper and lower bound constraints are imposed as follows: Let $u_i^{min} < u_i^{max}, i = 1, 2, \cdots, m$ and define $U := \{u \in \Re^m | u_i^{min} \leq u_i \leq u_i^{max}, \forall i = 1, 2, \cdots, m\}$. Let ∂U be the boundary of U, U^C be the center of U and U^o be the interior of U. The following results can be obtained from (Miller and Davison, 1993) to provide conditions for the feasibility of (y_{ref}, w). Define

$$T := (0 \ I_m) \begin{pmatrix} A-I & B \\ C & D \end{pmatrix}^{\dagger} \begin{pmatrix} 0 \\ I_r \end{pmatrix} \tag{2}$$

$$\mathcal{E} := -(0 \ I_m) \begin{pmatrix} A-I & B \\ C & D \end{pmatrix}^{\dagger} \begin{pmatrix} E \\ F \end{pmatrix} \tag{3}$$

where $(\cdot)^{\dagger} = (\cdot)'[(\cdot)(\cdot)']^{-1}$ is the pseudo-inverse; then we say that the extended constant signals (y_{ref}, w) are *feasible w.r.t. (1)* if

$$(T, \mathcal{E}) \begin{pmatrix} y_{ref} \\ w \end{pmatrix} \in U^o. \tag{4}$$

If $m = r$, and condition (4) does not hold, then this implies that there exists no control input subject to the constraints $u \in U$, so that asymptotic tracking and regulation occurs w.r.t. (y_{ref}, w) (Miller and Davison, 1993).

In what follows we will assume that the conditions of lemma 2.1 hold, the control signal is bounded, and that (y_{ref}, w) are feasible.

2.1 Equivalent Representation

In order to solve the RSP using MPC methods, the following equivalent representation of (1) is made. Let $\delta[k] := x[k] - x[k-1]$ and $v[k] := u[k] - u[k-1]$; then the following equivalent representation of (1) is obtained

$$\begin{bmatrix} \delta[k+1] \\ e[k] \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & I \end{bmatrix} \begin{bmatrix} \delta[k] \\ e[k-1] \end{bmatrix} + \begin{bmatrix} B \\ D \end{bmatrix} v[k] + \begin{bmatrix} E \\ F \end{bmatrix} w^*[k] - \begin{bmatrix} 0 \\ I \end{bmatrix} y^*_{ref}[k] \tag{5}$$

where $w^*[k] = w[k] - w[k-1]$ and $y^*_{ref}[k] = y_{ref}[k] - y_{ref}[k-1]$. (Davison, 1996) Stabilizing properties of the equivalent representation are described in (Davison and Davison, 2002)(Davison and Davison, 2003),(Davison and Scherzinger, 1987).

Now, define the performance index J

$$J = \sum_{i=k}^{\infty} e[i-1]'Qe[i-1] + v[i]'Rv[i] \tag{6}$$

for the system (5), where $Q > 0$ and $R > 0$. In this case, assuming no control signal constraints are applied, it follows from properties of the solution to the equivalent representation, that there always exists a solution to the problem of minimizing (6) such that the resultant closed loop system obtained for (5) is stable, i.e. the system $\left\{ \begin{pmatrix} 0 & I \end{pmatrix}, \begin{pmatrix} A & 0 \\ C & I \end{pmatrix} \right\}$ of (5) is detectable and the system $\left\{ \begin{pmatrix} A & 0 \\ C & I \end{pmatrix}, \begin{pmatrix} B \\ D \end{pmatrix} \right\}$ of (5) is stabilizable, which implies that the optimal controller which minimizes (6) is a stabilizing controller.

3. MPC FORMULATION

The use of the difference equations from the equivalent representation in (5) to model the plant dynamics is the key to extended constant disturbance rejection when using an open loop control law formulation, in which no knowledge of the structure or magnitide of the disturbance is assumed. Using (5) enables the open loop formulation to reject extended constant disturbances, which can not be similarly rejected when the plant model (1) is used, and when $w[k]$ or an estimate of $w[k]$ is not known.

The MPC cost can be defined as

$$J[k] = \sum_{i=k}^{k+P-1} e[i-1]'Qe[i-1] + v[i]'Rv[i] \quad (7)$$

for the system (5), where the weighting matrices satisfy $Q > 0$ and $R > 0$. The structure of (7) is a finite horizon version of its RSP counterpart (6). When P is large enough and when there are no active constraints, then the cost in (7) will closely approximate that of the RSP, and the properties of the RSP will apply to the resulting MPC controller. This follows from results pertaining to the length of the horizon time (Keerthi and Gilbert, 1988).

The cost function (7) itself is not new to MPC. It's importance to the tracking/disturbance rejection MPC problem is that, for a specified constant tracking and extended constant disturbance signal, there is no requirement to determine the steady-state value of the control signal, as is often the case when the traditional approach of MPC is used with constant disturbances, e.g. see (Goodwin *et al.*, 2001), (Rawlings, 1999), (Muske and Rawlings, 1993) and (Bitmead *et al.*, 1990), which use a norm of the difference between the control signal and its steady-state value in the performance index. This is important because, again, such a steady-state calculation requires a knowledge or estimate of the disturbance signal.

3.1 The QP subproblem in MPC

In order to solve the MPC problem at each sampling interval, the cost in (7) is restated as a constrained quadratic programming (QP) problem with the control vector $U[k] = [u[k]' \; u[k+1]' \cdots u[k+P-1]']'$ and the output vector $Y_{ref}[k] = [y_{ref}[k]' \; y_{ref}[k+1]' \cdots y_{ref}[k+P-1]']'$ where

$$J[k] = \tfrac{1}{2}U[k]'HU[k] + f[k]'U[k] + c[k], \quad (8)$$

and any constraints on the inputs or states of the system can be redefined in a standard vector format as $AU[k] \leq B$. The optimal solution $U^*[k]$ to this constrained minimization problem depends completely on H, $f[k]$ and the constraints on $U[k]$. The constant $c[k]$ can be dropped as the minimization over $U[k]$ is independent of this constant. State feedback of the plant at time k is built into the cost (8) through the $f[k]$ vector. Using the difference model (5) to build H and $f[k]$ allows for the resulting open loop minimization to naturally reject step disturbances.

The method used to convert (7) to (8) using the information provided by (5) is standard, e.g. see (Goodwin *et al.*, 2001). The vector $f[k]$ must be calculated at each sample time as it depends on the values of y_{ref}, $y[k]$, $\delta[k]$, $u[k]$ and $v[k]$, all of which are defined at sample time k. In this case, the vector $f[k]$ can be rewritten at time k as $f[k] = F_u u[k] + F_v v[k] + F_{y_{ref}} Y_{ref}[k] + F_y y[k] + F_\delta \delta[k]$, where F_u, F_v, $F_{y_{ref}}$, F_y and F_δ are constant matrices.

4. OBSERVER DESIGN

The equivalent representation framework for the QP subproblem in MPC which is described in section 3.1 leads to a natural choice of related observer design. Since only the difference in state $\delta[k]$ is needed for the calculation of $f[k]$, and since an extended constant disturbance does not enter into this equation, only an observer for the state difference is required to be designed. An observer for $\delta[k]$ of (5) is given by:

$$\hat{\delta}[k+1] = (A - \Lambda C)\hat{\delta}[k] + Bv[k] + \Lambda(e[k] - e[k-1] - Dv[k] + y^*_{ref}[k]) \quad (9)$$

where Λ is any gain matrix which stabilizes the system matrix $(A - \Lambda C)$. Since $\lim_{k\to\infty} w^*[k] = 0$ in (5) from (Callier and Desoer, 1991), this observer has the property that $\lim_{k\to\infty} \left(\delta[k] - \hat{\delta}[k] \right) = 0$, $\forall w \in \Re^q, \forall y_{ref} \in \Re^r, \forall x[0] \in \Re^n$.

It is to be noted that the above observer design can be done using Kalman Filtering and *that no knowledge of the disturbance vectors E and F in (1) is required.* MPC can be implemented by using the observer (9) to estimate the state.

4.1 MPC error due to the observer

Recall that the QP subproblem which is solved in (8) only requires state feedback with respect to the calculation of the $f[k]$ vector. In this term, the state difference $\delta[k]$ at sample time k is propagated into the $f[k]$ vector. With full state measurement, $\delta[k]$ is known, but if this information is not available then an estimate of $f[k]$, denoted $\hat{f}_k$, can be calculated by replacing $\delta[k]$ with its observer estimate $\hat{\delta}[k]$ given by (9). Defining the observer error as $\tilde{\delta}[k] := \delta[k] - \hat{\delta}[k]$ and noting that $\hat{\delta}[k] = \hat{\delta}[k] - \delta[k] + \delta[k] = \delta[k] - \tilde{\delta}[k]$ leads to

$$\hat{f}_k = F_u u[k] + F_v v[k] + F_{y_{ref}} Y_{ref}[k] + F_y y[k] + F_\delta(\delta[k] - \tilde{\delta}[k]) \quad (10)$$

$$\hat{f}_k = f[k] - F_\delta \tilde{\delta}[k]. \quad (11)$$

The error which is introduced into the MPC system due to the use of $\hat{f}_k$ is thus proportional to the state difference observer error for those portions of the curve which are unconstrained. This error is exponentially decaying. Given the MPC cost (8), with no constraints, the true optimal control vector U_k^* is calculated to be $U_k^* = -H^{-1} f[k]$. Computing the optimal control estimate, denoted by $\hat{U}_k^*$, found using an observer it can be seen that

$$\hat{U}_k^* = -H^{-1}\hat{f}_k = -H^{-1}\left\{ f[k] - F_\delta \tilde{\delta} \right\}. \quad (12)$$

Thus the error that is found by using the observer estimate of the system difference is directly proportional to the state difference observer error $\tilde{\delta}[k]$, which is exponentially decaying.

$$U_k^* - \hat{U}_k^* = -H^{-1} F_\delta \tilde{\delta}[k]. \quad (13)$$

The formulation for the optimal control with constraints leads to similar results, e.g. see (Bryson Jr. and Ho, 1975).

5. EXPERIMENTAL VERIFICATION

In evaluating the performance of the proposed MPC controller two example systems are used. The first is a SISO three pole diagonal state-space model which is marginally stable, and the second is a two input/output mass-spring-damper system.

Simple three pole example The simple three pole example is constructed in order to illustrate the behavior of the extended constant tracking problem. The following continuous time LTI state-space model is constructed:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, E = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \quad D = [0], \quad F = [0].$$

The above continuous time-time plant is approximated by a discrete time system with a sampling interval of 1 sec. The horizon used for the MPC cost is $P = 100$, and the weighting matrices used are $Q = 1$ and $R = 1$.

In the simulation, it is desired that the output should track the extended constant signal $y_{ref}[k] = \sqrt{k}$, with no disturbances present.

Mass spring damper system The mass spring damper system represents a system with two forces as inputs and two positions as outputs. This system is sampled with a sampling interval of 0.1 seconds using a 10 second MPC horizon window, leading to $P = 100$. The horizon cost weighting matrices used are $Q = I_2$ and $R = 100I_2$. The continuous system is defined by

$$A = \begin{bmatrix} -0.06 & 1.09 & 0 & 0 & 0 & 0 \\ -1.09 & -0.06 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.05 & 0.9987 \\ 0 & 0 & 0 & 0 & -0.9987 & -0.05 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.616 & 0.616 \\ -0.0338 & -0.0338 \\ 1.0 & -1.0 \\ -101 & -101 \\ 0 & 0 \\ -.7075 & 0.7075 \end{bmatrix}, \ E = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 3 \\ 4 \end{bmatrix},$$

$$C = \begin{bmatrix} -0.0338 & -0.6163 & 101 & -101 & -0.7075 & 0 \\ -0.0338 & -0.6163 & 1.0 & -101 & 0.7075 & 0 \end{bmatrix},$$

$$D = 0, \ F = 0$$

Two sets of simulations are performed. In the first set of simulations, it is desired that regulation be achieved despite an extended constant disturbance of $w[k] = \sqrt{k}$, first with no input constraints, then with input constraints. In the second set of simulations there is no disturbance present, and it is desired that the system should track the extended constant reference signals $y_1 = 0$, $y_2 = \sqrt{k}$. Again, this is performed both with and without input constraints.

5.1 Results obtained for examples

Simple three pole example Figure 1 shows the effect of tracking the extended constant reference signal $y_{ref} = \sqrt{k}$ both with and without input constraints, using full-state feedback only. The input constraint used is $u[k] \leq 0.2$ which is about half the magnitude of the peak control effort that would otherwise be used. This constraint is evident for approximately 20 seconds. Its affect on the output is only evident during the period of time when the constraint is active, after that the signal quickly matches the desired reference trajectory.

Mass spring damper system Figure 2 shows the effect of disturbance rejection of an extended constant signal using observer feedback for the

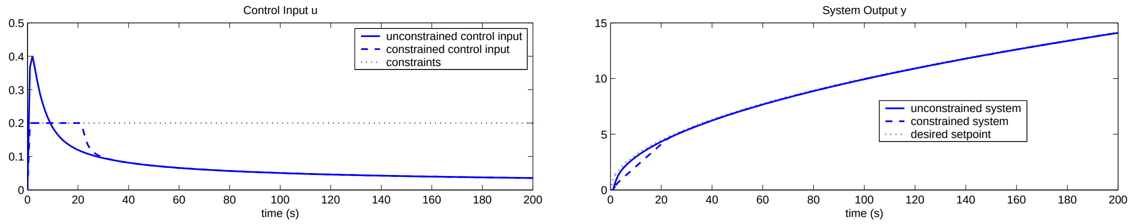

Fig. 1. Tracking of an extended constant output trajectory, with and without constraints.

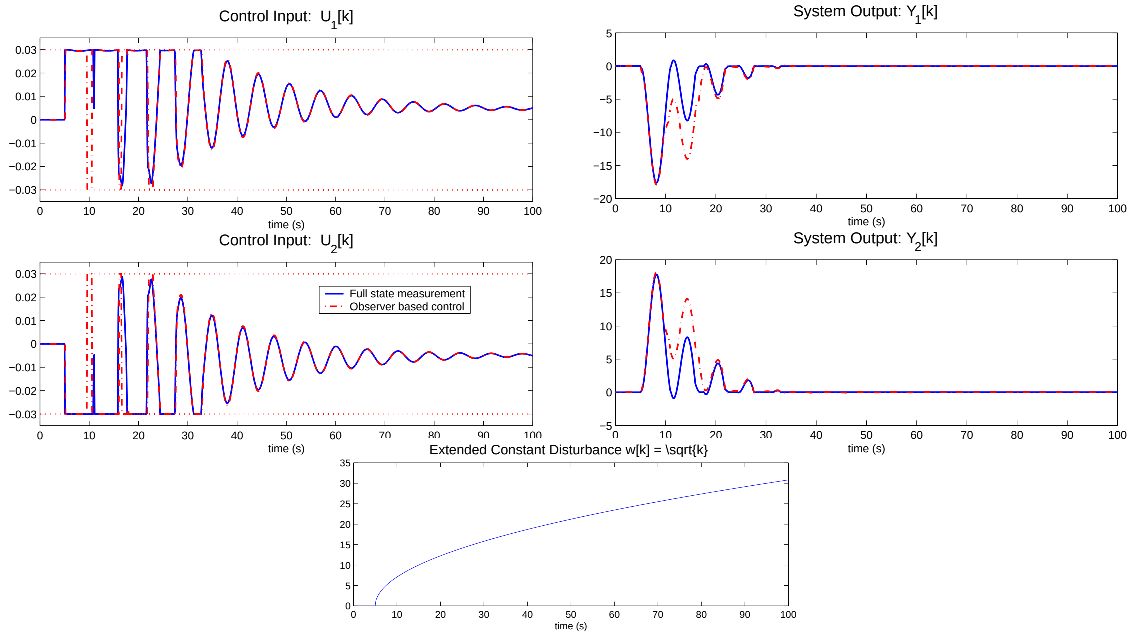

Fig. 2. Rejection of an extended constant disturbance using both full state and output feedback.

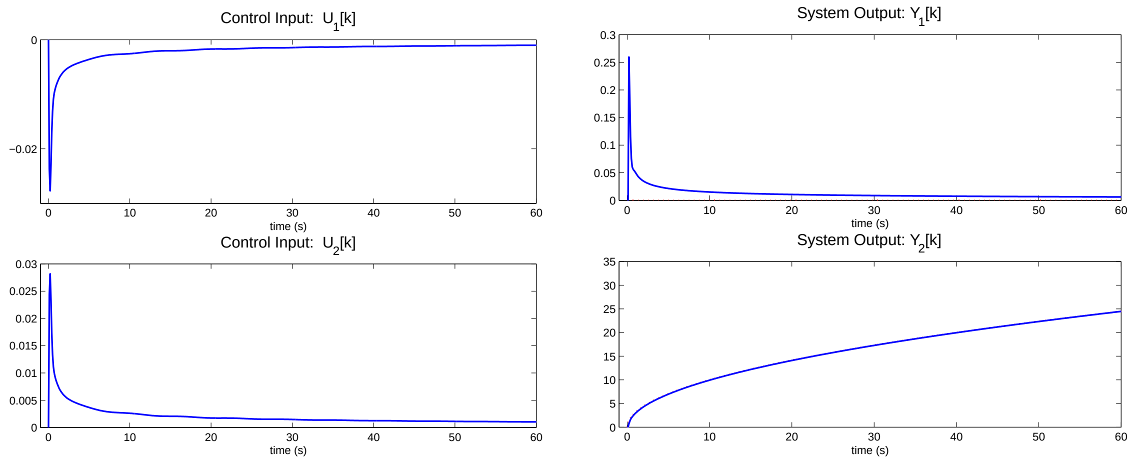

Fig. 3. Tracking of an extended constant reference signal, using no input constraints

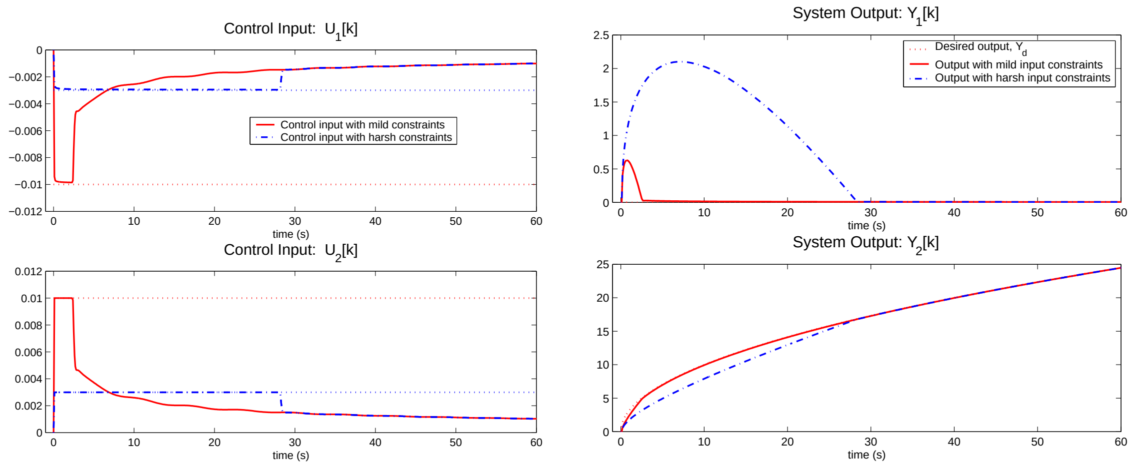

Fig. 4. Tracking of an extended constant reference signal with both mild and harsh constraints.

case when input constraints of $|u_i[k]| \leq 0.03$ are imposed on the system.

Figures 3 and 4 illustrate tracking of an extended constant reference signal $y_2[k] = \sqrt{k}$ both with and without input constraints, using full-state feedback. Figure 3 shows the case where no input constraints are present. Figure 4 shows the case where both relatively mild and relatively harsh input constraints are imposed. The "mild" constraints used are $|u_i[k]| \leq 0.01$ and the "harsh" constraints used are $|u_i[k]| \leq 0.005$. When no constraints are present, the peak magnitudes of the system are just under 0.03. Again the effects of the constraints are most noticeable in the duration in which they are active.

6. CONCLUSIONS

In the MPC tracking/regulation problem for a LTI system subject to unknown, unmeasurable extended constant disturbances, it is shown that:

(1) On using the formulation of the plant equations given in (5), and on assuming that the state is measurable, it is possible to minimize the finite horizon MPC cost (8) **without having any knowledge or estimation of the disturbance magnitude or structure and without using any observer.**

(2) In the case, when the state is not unmeasureable, it is shown that the observer defined in (9) can be used. This observer has the desirable property that again **it is not necessary to have any knowledge of the magnitude or structure of the disturbance.** Since the observer need only estimate the state and not the disturbance, the order of the observer is smaller than that normally used in practice, where an estimate of both the state and disturbance is generally made.

Experimental verification shows that MPC controllers based on the above approach work well.

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