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Switched Feedback Control for a class of First-order Nonholonomic Driftless Systems

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Abstract: This paper is concerned with stabilizing feedback control for a class of nonholonomic driftless systems, whose controllability Lie algebra rank condition is satisfied by up to first-order Lie brackets. We propose a switched feedback law which drives all the initial states to the origin with bounded control inputs (as opposed unbounded, division-by-zero-type discontinuous control). The discontinuity of the feedback law takes place on a subspace defined by the 'parallelism' condition for the base and the fiber vectors in $\mathbb{R}^3$ (or simply $\boldsymbol{q} \times \boldsymbol{\phi} = 0$). We also show that the complement of this discontinuity region is homotopic to $SO(3)$ which is also isomorphic to $\mathbb{S}^2 \times \mathbb{S}$. The proposed control law is examined by numerical simulations.

1. INTRODUCTION

In the last two decades, control of nonholonomic driftless systems has been an attractive issue of nonlinear control theory. This is partially because a lot of important mechanical systems with nonintegrable kinematic constraints, such as non-slip rolling constraints or conservation of angular momentum, are modeled as driftless systems. Another reason is that driftless systems cannot be asymptotically stabilized by any continuous state feedback according to Brockett's necessary condition (Brockett [1983]), that motivates us to pursuit non-standard control tools such as discontinuous or time-varying (periodic) feedback control.

Among the subclasses of driftless systems, intensive works have been contributed for *chained form* systems (Murray et al. [1994]) and their feedback equivalents (Pomet [1992], M.Sampei et al. [1995]). By virtue of the fact that their controllability Lie algebra have particularly simple structure (constructed by iteration of Lie brackets with a special vector-field, called *generator*), there are wide variety of controllers proposed for this class; in essence, the clue to stabilization problem has been already established.

On the other hand, this paper focuses on a class of driftless systems with 3 inputs and 6 states, whose controllability Lie algebra rank condition is satisfied by up to first-order Lie brackets. We call them *first-order systems* according to (Murray et al. [1994]), though they are also called differently (Khaneja and Brockett [1999]). Attitude control of 3-D spacecraft using shape changes (Sreenath [1992]) is known as a typical example of first-order systems. The author also proposed a new actual example of this class, called *trident snake robot* (Ishikawa [2004]), which is a wheeled planar mobile robot with three snake-like branches.

First-order systems are never feedback equivalent to conventional chained systems since there is a structural difference between them. Although the number of studies is relatively small, there have been several challenges to feedback control for non-chained structure systems, such as time-varying feedback control(Khaneja and Brockett [1999]), step-by-step feedback algorithm(Bloch et al. [2000], Iwatani et al. [2002]);

In this paper, we suggest a simple discontinuous feedback control law for this class of systems which brings all the initial states to the origin. We emphasize that this is a complete *static* state feedback law, in the sense that (i) there is no exception of the control law in which the control input is not assigned; (ii) neither 'step counter' or 'time variable' is used in the control law on the contrary to the existing method. Moreover, the discontinuity of the control law is of sliding-mode type switching, so it is bounded and safer than division-by-zero-type discontinuity (as shown by Tsuchiya et al. [2002]).

This paper is organized as follows. In section 2, we prepare basic notations and the system model. The proposed control law is given in section 3.1 followed by the proofs to show that all the initial states in the state space are brought to the origin in finite time. In section 4, we also discuss geometric interpretation of the control law and its possible variations. The proposed control law is examined by numerical simulations in section 5.

2. PRELIMINARIES

For a pair of arbitrary spatial vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$, $\times$ denotes their cross product defined by

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix} \tag{1}$$

In the rest of paper, we frequently make use of the formula of scalar triple product:

$$\boldsymbol{a}^T(\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}^T(\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c}^T(\boldsymbol{a} \times \boldsymbol{b}) \tag{2}$$

For a pair of subsets X and Y, $X \setminus Y := \{x | x \in X, x \notin Y\}$ denotes the set subtraction. $X \cong Y$ denotes homeomorphism.

For a natural number k, $SO(k)$ denotes the k-dimensional special orthogonal group and $\mathbb{S}^k$ denotes k-dimensional unit sphere. $\|\cdot\|$ indicates Euclidean norm.

2.1 First-order systems with 3 inputs

Consider the following driftless nonholonomic system

$$\Sigma_3^6: \quad \begin{aligned} \dot{\boldsymbol{q}} &= \boldsymbol{u} \\ \dot{\boldsymbol{\phi}} &= \boldsymbol{q} \times \boldsymbol{u} \end{aligned} \tag{3}$$

where $\boldsymbol{q} \in \mathbb{R}^3$ is called *base* vector and $\boldsymbol{\phi} \in \mathbb{R}^3$ is called *fiber* vector. The whole state vector is $\boldsymbol{z} = (\boldsymbol{q}^T, \boldsymbol{\phi}^T)^T \in \mathbb{R}^6$. $\boldsymbol{u} \in \mathbb{R}^3$ denotes the control input. Its basic behavior can be interpreted as follows; the base vector $\boldsymbol{q}$ is directly driven by the control input $\boldsymbol{u}$, while $\dot{\boldsymbol{\phi}}$ is given by cross product of the base vector $\boldsymbol{q}$ and its velocity $\dot{\boldsymbol{q}}$, i.e., $\dot{\boldsymbol{\phi}}$ is always perpendicular to the plane including $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$. Our purpose is to find the control input $\boldsymbol{u}$ which brings the state $\boldsymbol{z}$ from given initial state $\boldsymbol{z}(0)$ to the origin.

If we rewrite the state equation (3) as a vector-field form $\dot{\boldsymbol{z}} = g_1(\boldsymbol{z})u_1 + g_2(\boldsymbol{z})u_2 + g_3(\boldsymbol{z})u_3$, its controllability distribution is given by

$$\bar{\mathcal{G}} = C^\infty \operatorname{span}\{\ g_1, g_2, g_3,\ [g_1, g_2],\ [g_1, g_3],\ [g_2, g_3]\ \} \tag{4}$$

which has full rank at each point $\boldsymbol{z} \in \mathbb{R}^6$. This ensures its local accessibility according to Chow's theorem (see e.g., Nijmeijer and van der Schaft [1990]). Moreover, $\bar{\mathcal{G}}$ is said to be nilpotent of order 2 because all the higher order Lie brackets (such as $[g_1, [g_1, g_2]]$) are zero. As mentioned above, the 3-D spacecraft with shape controls and the trident snake robot can be modeled as Σ_3^6 under nilpotent approximation(see Hermes [1991], Bellaiche et al. [1992], Struemper [1998]).

3. SWITCHED FEEDBACK CONTROL LAW

3.1 Construction of Controlled Invariant Manifolds

In this section, we propose a switched feedback law for the system Σ_3^6. Let us begin with defining an important vector

$$\boldsymbol{n} := \boldsymbol{\phi} \times \boldsymbol{q} \tag{5}$$

which vanishes when $\boldsymbol{q}$ and $\boldsymbol{\phi}$ are parallel. Next, suppose a vector $\boldsymbol{p}$ defined by

$$\boldsymbol{p} := \boldsymbol{q} \times \boldsymbol{n}, \tag{6}$$

which is perpendicular to both $\boldsymbol{n}$ and $\boldsymbol{q}$, and belongs to the plane spanned by $\boldsymbol{q}$ and $\boldsymbol{\phi}$ (see Fig. 1).

As long as $\boldsymbol{n} \neq 0$, we observe that $(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{n})$ forms an orthogonal coordinate frame of $\mathbb{R}^3$. For any vector $\boldsymbol{x} \in \mathbb{R}^3$, let $\boldsymbol{x}_0$ denote its normalized vector

$$\boldsymbol{x}_0 := \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}, \quad \text{if } \boldsymbol{x} \neq 0$$

where $\|\cdot\|$ indicates Euclidean norm. Note that $\boldsymbol{x}_0$ is not defined if $\boldsymbol{x} = 0$, while it is bounded and continuous otherwise. Let us define $\boldsymbol{q}_0, \boldsymbol{\phi}_0, \boldsymbol{n}_0$ and $\boldsymbol{p}_0$ in this manner.

Now, suppose the following subsets of the state space:

$$D := \{(\boldsymbol{q}, \boldsymbol{\phi}) \in \mathbb{R}^6 \mid \boldsymbol{\phi} \times \boldsymbol{q} = 0,\ \boldsymbol{\phi} \neq 0\} \tag{7}$$

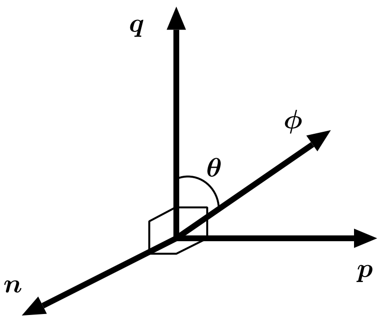

Fig. 1. The base, fiber vectors with their normals in $\mathbb{R}^3$

$$S_0 := \{0 \in \mathbb{R}^6\} \tag{8}$$

$$S_1 := \{(\boldsymbol{q}, \boldsymbol{\phi}) \in \mathbb{R}^6 \mid \boldsymbol{\phi} = 0\} \tag{9}$$

$$S_2 := \{(\boldsymbol{q}, \boldsymbol{\phi}) \in \mathbb{R}^6 \mid \boldsymbol{q}^T \boldsymbol{\phi} = 0\} \cap S_3 \tag{10}$$

$$\begin{aligned} S_3 &:= \mathbb{R}^6 \setminus D \\ &= \{(\boldsymbol{q}, \boldsymbol{\phi}) \in \mathbb{R}^6 \mid \boldsymbol{q} \times \boldsymbol{\phi} \neq 0 \text{ or } \boldsymbol{\phi} = 0\} \end{aligned} \tag{11}$$

Note that $S_0 \subset S_1 \subset S_2 \subset S_3$. S_2 is the subset on which $\boldsymbol{q}$ and $\boldsymbol{\phi}$ are orthogonal to each other, while D is the subset on which $\boldsymbol{q}$ and $\boldsymbol{\phi}$ are parallel. Basic idea of our method is to make all S_i's invariant under the proposed control law, and let the trajectory $\boldsymbol{z}$ go through the subsets

$$D \to S_3 \to S_2 \to S_1 \to S_0$$

in sequence.

3.2 Feedback Control Law

We suggest the following feedback control law

$$\boldsymbol{u} = \alpha_q(\boldsymbol{z})\,\boldsymbol{q}_0 + \alpha_p(\boldsymbol{z})\,\boldsymbol{p}_0 + \alpha_n(\boldsymbol{z})\,\boldsymbol{n}_0 \tag{12}$$

where α_q, α_p and α_n are bounded scalar-valued functions chosen as follows:

Case 0: If $\boldsymbol{z} \in S_0$:

$$\alpha_q(\boldsymbol{z}) = \alpha_p(\boldsymbol{z}) = \alpha_n(\boldsymbol{z}) = 0. \tag{13}$$

Case 1: If $\boldsymbol{z} \in S_1$:

$$\alpha_q(\boldsymbol{z}) = -k_q \tag{14}$$

$$\alpha_p(\boldsymbol{z}) = 0 \tag{15}$$

$$\alpha_n(\boldsymbol{z}) = 0 \tag{16}$$

Case 2: If $\boldsymbol{z} \in S_2$:

$$\alpha_q(\boldsymbol{z}) = k_q \operatorname{sgn}(\max\{\|\boldsymbol{\phi}\|, \epsilon\} - \|\boldsymbol{q}\|) \tag{17}$$

$$\alpha_p(\boldsymbol{z}) = 0 \tag{18}$$

$$\alpha_n(\boldsymbol{z}) = -k_n \tag{19}$$

Case 3: If $\boldsymbol{z} \in S_3$:

$$\alpha_q(\boldsymbol{z}) = 0 \tag{20}$$

$$\alpha_p(\boldsymbol{z}) = -k_p \operatorname{sgn}\left(\frac{\boldsymbol{\phi}^T \boldsymbol{q}}{\boldsymbol{\phi}^T \boldsymbol{p}_0}\right) \tag{21}$$

$$\alpha_n(\boldsymbol{z}) = 0 \tag{22}$$

Case 4: If $\boldsymbol{z} \in D$:

$$\boldsymbol{u} = \boldsymbol{c}_0 \tag{23}$$

where $\boldsymbol{c}_0 \in \mathbb{R}^3$ is any vector satisfying $\boldsymbol{\phi} \times \boldsymbol{c}_0 \neq 0$, $\|\boldsymbol{c}_0\| = 1$.

k_q, k_n, k_p and ϵ are positive constant design parameters.

3.3 Finite-time Reaching to the Origin

Now we are ready to show that the proposed control law achieves finite-time reaching to the origin for every initial state.

Lemma 1. S_1, S_2 and S_3 are all invariant under the proposed control law. Moreover, if $\boldsymbol{z}(0)$ belongs to $S_{i+1} \setminus S_i$ for some $i \in \{0, 1, 2\}$, then $\boldsymbol{z}(t)$ will reach S_i in finite time. •

Proof:
Since $\boldsymbol{q}^T \boldsymbol{p}_0 = 0$ and $\boldsymbol{q}^T \boldsymbol{n}_0 = 0$, we have

$$\begin{aligned}\frac{d}{dt}\|\boldsymbol{q}\| &= \frac{\boldsymbol{q}^T \dot{\boldsymbol{q}}}{\|\boldsymbol{q}\|} \\ &= \alpha_q(\boldsymbol{z})\,\frac{\boldsymbol{q}^T \boldsymbol{q}_0}{\|\boldsymbol{q}\|} \\ &= \alpha_q(\boldsymbol{z}) \end{aligned} \tag{24}$$

Similarly, since $\boldsymbol{n}^T \boldsymbol{q}_0 = 0$ and $\boldsymbol{n}^T \boldsymbol{p}_0 = 0$, we have

$$\begin{aligned}\frac{d}{dt}\|\boldsymbol{\phi}\| &= \frac{\boldsymbol{\phi}^T \dot{\boldsymbol{\phi}}}{\|\boldsymbol{\phi}\|} \\ &= \frac{\boldsymbol{\phi}^T (\boldsymbol{q} \times \boldsymbol{u})}{\|\boldsymbol{\phi}\|} \\ &= \frac{\boldsymbol{u}^T (\boldsymbol{\phi} \times \boldsymbol{q})}{\|\boldsymbol{\phi}\|} \\ &= \alpha_n(\boldsymbol{z})\frac{\boldsymbol{n}_0^T \boldsymbol{n}}{\|\boldsymbol{\phi}\|} \\ &= \alpha_n(\boldsymbol{z})\|\boldsymbol{q}\| \sin|\theta| \end{aligned} \tag{25}$$

where $|\theta| > 0$ is the angle between $\boldsymbol{q}$ and $\boldsymbol{\phi}$ (we do not care its sign).

$$\begin{aligned}\frac{d}{dt}(\boldsymbol{\phi}^T \boldsymbol{q}) &= \boldsymbol{q}^T \dot{\boldsymbol{\phi}} + \boldsymbol{\phi}^T \dot{\boldsymbol{q}} \\ &= \boldsymbol{q}^T (\boldsymbol{q} \times \boldsymbol{u}) + \boldsymbol{\phi}^T \boldsymbol{u} \\ &= \boldsymbol{\phi}^T (\alpha_q(\boldsymbol{z})\,\boldsymbol{q}_0 + \alpha_p(\boldsymbol{z})\,\boldsymbol{p}_0) \\ &= \alpha_q(\boldsymbol{z})\,(\boldsymbol{\phi}^T \boldsymbol{q}_0) + \alpha_p(\boldsymbol{z})\,(\boldsymbol{\phi}^T \boldsymbol{p}_0) \end{aligned} \tag{26}$$

Here we used $\boldsymbol{q}^T (\boldsymbol{q} \times \boldsymbol{u}) = \boldsymbol{u}^T (\boldsymbol{q} \times \boldsymbol{q}) = 0$.

Case 0: It is obvious that S_0 is invariant under $\boldsymbol{u} = 0$ (i.e., the origin is an equilibrium of the closed-loop system).

Case 1: From (16) and (25), $\frac{d}{dt}\|\boldsymbol{\phi}\| = 0$ is satisfied on S_1 since $\alpha_n(\boldsymbol{z}) = 0$. This implies S_1 is invariant.

Moreover, since

$$\frac{d}{dt}\|\boldsymbol{q}\| = -k_q,$$

$\|\boldsymbol{q}(t)\|$ vanishes at $t = \|\boldsymbol{q}(0)\|/k_q$. Thus $\boldsymbol{z}(t)$ reaches S_0 in finite time.

Case 2:

(*Proof of invariance*) From the definition (10), $\boldsymbol{q}^T \boldsymbol{\phi} = 0$ is satisfied on S_2. Moreover, the points $\{\boldsymbol{q} = 0, \boldsymbol{\phi} \neq 0\}$ are excluded from S_2 because they belongs to D; i.e., either $\boldsymbol{q} \neq 0$ or $\boldsymbol{\phi} = 0$ is satisfied. This leads us to notice that

$$\boldsymbol{\phi}^T \boldsymbol{q}_0 = \frac{\boldsymbol{\phi}^T \boldsymbol{q}}{\|\boldsymbol{q}\|} = 0$$

is satisfied on S_2. Substituting (17)–(19) into (26), we have

$$\frac{d}{dt}(\boldsymbol{\phi}^T \boldsymbol{q}) = -\alpha_q(\boldsymbol{z})\,(\boldsymbol{\phi}^T \boldsymbol{q}_0) = 0$$

which concludes S_2 is invariant.

(*Proof of finite-time reaching*) Next, suppose $\boldsymbol{z}(0) \in S_2 \setminus S_1$, i.e,

$$\|\boldsymbol{q}(0)\| \neq 0,\ \|\boldsymbol{\phi}(0)\| \neq 0,\ \boldsymbol{\phi}(0)^T \boldsymbol{q}(0) = 0$$

Let us show that $\|\boldsymbol{q}\|$ reaches $\max\{\|\boldsymbol{\phi}\|, \epsilon\}$ in finite time. If $\|\boldsymbol{q}(0)\| < \max\{\|\boldsymbol{\phi}\|, \epsilon\}$,

$$\frac{d}{dt}\|\boldsymbol{q}\| = k_q$$

until $\|\boldsymbol{\phi}\| - \|\boldsymbol{q}\| = 0$. Thus $\|\boldsymbol{q}\|$ is increasing, and

$$\|\boldsymbol{q}(t)\| = \|\boldsymbol{q}(0)\| + k_q t.$$

On the other hand, substituting (19) into (25) and considering $\sin|\theta| = 1$, we have

$$\begin{aligned}\frac{d}{dt}\|\boldsymbol{\phi}\| &= -k_n\|\boldsymbol{q}\| \\ &= -k_n(\|\boldsymbol{q}(0)\| + k_q t) \\ \|\boldsymbol{\phi}(t)\| &= \|\boldsymbol{\phi}(0)\| - k_n\|\boldsymbol{q}(0)\|t - \frac{1}{2}k_n k_q t^2\end{aligned}$$

So $\|\boldsymbol{\phi}(t)\| - \|\boldsymbol{q}(t)\| = 0$ is a quadratic equation with respect to t. It is easy to see that there exists a positive root, say r_1. Therefore $\|\boldsymbol{q}\| = \max\{\|\boldsymbol{\phi}\|, \epsilon\}$ is satisfied at the time

$$t = t_1, \quad t_1 := \max\left\{r_1, \frac{\epsilon - \|\boldsymbol{q}(0)\|}{k_q}\right\}$$

It is also true in the case of $\|\boldsymbol{\phi}(0)\| < \|\boldsymbol{q}(0)\|$ by repeating similar argument.

Once $\|\boldsymbol{q}(t_1)\| = \max\{\|\boldsymbol{\phi}(t_1)\|, \epsilon\}$ is satisfied, $\|\boldsymbol{q}(t)\| \geq \epsilon$ will hold afterward. Thus we obtain

$$\begin{aligned}\frac{d}{dt}\|\boldsymbol{\phi}\| &\leq -k_n\epsilon \\ \|\boldsymbol{\phi}(t)\| &\leq \|\boldsymbol{\phi}(t_1)\| - k_n\epsilon(t - t_1),\end{aligned}$$

so it is clear that $\|\boldsymbol{\phi}(t)\|$ reaches $\mathbf{0}$ no later than

$$t = t_2, \quad t_2 := t_1 + \frac{\|\boldsymbol{\phi}(t_1)\|}{k_n\epsilon},$$

i.e., $\boldsymbol{z}(t)$ reaches S_1 in finite time.

Case 3: In order to prove $S_3 = \mathbb{R}^6 \setminus D$ is invariant, it is sufficient to show that any trajectory starting from S_3 will not get close to D. Substituting (20)–(22) into (26), we have

$$\begin{aligned}\frac{d}{dt}\|\boldsymbol{q}\| &= 0, \\ \frac{d}{dt}\|\boldsymbol{\phi}\| &= 0, \\ \frac{d}{dt}(\boldsymbol{\phi}^T \boldsymbol{q}) &= -k_p \operatorname{sgn}\left(\frac{\boldsymbol{\phi}^T \boldsymbol{q}}{\boldsymbol{\phi}^T \boldsymbol{p}_0}\right)(\boldsymbol{\phi}^T \boldsymbol{p}_0) \\ &= -k_p\,|\boldsymbol{\phi}^T \boldsymbol{p}_0|\,\operatorname{sgn}(\boldsymbol{\phi}^T \boldsymbol{q})\end{aligned}$$

Therefore both $\|\boldsymbol{q}\|$ and $\|\boldsymbol{\phi}\|$ are kept constant, while $|\boldsymbol{\phi}^T \boldsymbol{q}|$ is strictly decreasing. This implies $|\cos\theta|$ is decreasing, thus $\boldsymbol{z}$ will not approach D.

Note that $\boldsymbol{z}(0) \in S_3 \setminus S_2$ implies $\|\boldsymbol{q}(0)\| \neq 0$ and $\|\boldsymbol{\phi}(0)\| \neq 0$. Since

$$\frac{d}{dt}(\boldsymbol{q}^T \boldsymbol{\phi}) = k_p \boldsymbol{q}_0^T \boldsymbol{\phi} = -\|\boldsymbol{\phi}\| \operatorname{sgn}(\boldsymbol{q}^T \boldsymbol{\phi})$$

we can see that $\boldsymbol{q}^T\boldsymbol{\phi}$ vanishes in finite time, i.e., $\boldsymbol{z}(t)$ reaches S_2 in finite time.

□

Lemma 2. No trajectory stays on D. •

Proof:
Since **Case 4** of the control law is chosen when $\boldsymbol{z} \in D$, let us compute the derivative of $\boldsymbol{n}$ under (23).

$$\begin{aligned}\frac{d}{dt}\boldsymbol{n} &= \dot{\boldsymbol{q}} \times \boldsymbol{\phi} + \boldsymbol{q} \times \dot{\boldsymbol{\phi}} \\ &= -\boldsymbol{\phi} \times \boldsymbol{c}_0 + \boldsymbol{q} \times (\boldsymbol{q} \times \boldsymbol{c}_0) \\ &= -\boldsymbol{\phi} \times \boldsymbol{c}_0 + \kappa^2\boldsymbol{\phi} \times (\boldsymbol{\phi} \times \boldsymbol{c}_0)\end{aligned}$$

where we introduced $\kappa \in \mathbb{R}$ such that $\boldsymbol{q} = \kappa\boldsymbol{\phi}$ because $\boldsymbol{q}$ is parallel to $\boldsymbol{\phi}$. Considering $\boldsymbol{\phi} \times \boldsymbol{c}_0 \neq 0$ and $(\boldsymbol{\phi} \times \boldsymbol{c}_0) \perp (\boldsymbol{\phi} \times (\boldsymbol{\phi} \times \boldsymbol{c}_0))$, we can conclude that $\frac{d}{dt}\boldsymbol{n} \neq 0$ on D. This implies $\dot{\boldsymbol{z}}$ does not belong to the tangent space of D. Every trajectory starting from D will leave D (thus enter S_3), in infinitesimally short time. □

The observations above are summarized into the following theorem.

Theorem 3. Under the proposed control law (12), any trajectory $\boldsymbol{z}(t)$ starting from $\mathbb{R}^6$ reach the origin in finite time. •

The proof is omitted because it is direct combination of Lemma 1 and Lemma 2. Let us close this section with summarizing the effect of each terms in the control law (12):

- The term $\alpha_q(\boldsymbol{z})\boldsymbol{q}_0$ contributes to control of $\|\boldsymbol{q}\|$.
- The term $\alpha_n(\boldsymbol{z})\boldsymbol{n}_0$ contributes to control of $\|\boldsymbol{\phi}\|$; basically this effect corresponds to the principle of holonomy.
- The term $\alpha_p(\boldsymbol{z})\boldsymbol{p}_0$ is used to change the angle between $\boldsymbol{\phi}$ and $\boldsymbol{q}$.

4. DISCUSSION

4.1 Underlying topology of the control law

The proposed method can be regarded as a natural extension of the sliding-mode controller proposed by Bloch and Drakunov [1996], for Brockett integrator

$$\Sigma_2^3: \quad \begin{aligned}\dot{\boldsymbol{q}} &= \boldsymbol{u} \\ \dot{\phi} &= q_2u_1 - q_1u_2\end{aligned} \tag{27}$$

where $\boldsymbol{q} \in \mathbb{R}^2$, $\phi \in \mathbb{R}$ and $\boldsymbol{z} := (\boldsymbol{q}^T, \phi)^T \in \mathbb{R}^3$. They suggested a control law

$$\boldsymbol{u} = -\alpha_q(\boldsymbol{z})\,\boldsymbol{q} - \alpha_n(\boldsymbol{z})\begin{bmatrix}-q_2 \\ q_1\end{bmatrix} \tag{28}$$

which has exception at $D := \{\boldsymbol{q} = 0, \phi \neq 0\}$; i.e., all the trajectory starting from $\mathbb{R}^3 \setminus D$ *moves towards* the origin $\{0\}$. A topological interpretation of the effect of introducing this D is as follows. If there had been a continuous state feedback, it would have had non-zero value on $\mathbb{R}^3 \setminus \{0\}$. Now, from the viewpoint of *homotopy equivalence*, $\mathbb{R}^3 \setminus \{0\}$ is homotopic to $\mathbb{S}^2$. Namely, the support (complement of the kernel) of the control law would have been homotopic to $\mathbb{S}^2$. On the other hand, once the set of exception D is introduced, we consider that the whole state space is restricted to $\mathbb{R}^3 \setminus D$. Then the support of the control law becomes $(\mathbb{R}^3 \setminus D) \setminus \{0\}$, which is homotopic to $SO(2) \cong \mathbb{S}^1$. Roughly speaking, the introduction of D is a topological operation which remodels $\mathbb{S}^2$ into $\mathbb{S}^1$ (indeed, this is the way to get rid of Brockett's necessary condition; see Coron [1990] for topological generalization of Brockett's condition).

Let us turn to apply this interpretation to Σ_3^6. $\mathbb{R}^6 \setminus D$ is equal to S_3 as we defined in (11), so $(\mathbb{R}^6 \setminus D) \setminus \{0\}$ is equal to $S_3 \setminus \{0\}$.

Theorem 4. $S_3 \setminus \{0\}$ is homotopic to $SO(3)$.

Proof:
To each point $\boldsymbol{z} \in S_3 \setminus \{0\}$, assign the 3×3-matrix defined by

$$G(\boldsymbol{z}) := [\boldsymbol{q}, \boldsymbol{\phi}, \boldsymbol{\phi} \times \boldsymbol{q}]\,. \tag{29}$$

Since all the column vectors of $G(\boldsymbol{z})$ is linearly independent on $S_3 \setminus \{0\}$, we see that $\det G$ is sign definite, namely, G provides one-to-one correspondence between $S_3 \setminus \{0\}$ and $GL_+(3)$ (the set of 3×3-matrix whose determinant is positive, which is clearly homotopic to $SO(3)$). From this correspondence, we can conclude that $S_3 \setminus \{0\}$ is homotopic to $SO(3)$. □

Note that $SO(3)$ is homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$. If there had been a continuous state feedback, it would have had non-zero value on $\mathbb{R}^6 \setminus \{0\}$, which is homotopic to $\mathbb{S}^5$. Therefore, we can say that the topological effect of introducing D is to remodel $\mathbb{S}^5$ into $SO(3) \simeq \mathbb{S}^2 \times \mathbb{S}^1$. A rough visualization of this observation is depicted in Fig. 2. In the case of Σ_2^3, the support of the control law is homotopic to $\mathbb{S}^1$, which is just a 'circle'. While in the case of Σ_3^6, it is $\mathbb{S}^2 \times \mathbb{S}^1$ – a circle is assigned to every point of a sphere.

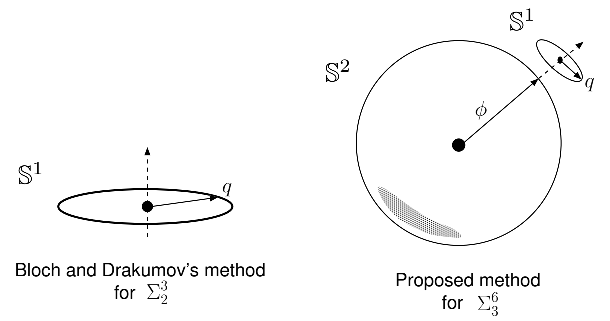

Fig. 2. Topological image of the proposed control law

4.2 Partial continuation of the control law

The feedback control law proposed in Section 3.1 is discontinuous (as a function of state vector $\boldsymbol{z}$) at each point of D, as well as on S_1, S_2 and S_3. However, the meaning of discontinuity is different.

Discontinuity of the control law at the points of D is caused by the nature of $\boldsymbol{n}_0 = \boldsymbol{n}/\|\boldsymbol{n}\|$. Suppose a point $\boldsymbol{z}^* \in D$. Then $\boldsymbol{n}_0$ is not defined at $\boldsymbol{z}^*$ because $\boldsymbol{n}$ vanishes there, while

$$\lim_{\boldsymbol{z} \to \boldsymbol{z}^*} \boldsymbol{n}_0$$

converges to a constant vector, depending on the direction in which $\boldsymbol{z}$ approaches to $\boldsymbol{z}^*$. Thus this discontinuity is of *signum-type* referring to $x/|x| = \operatorname{sgn} x$ in scalar case. Let us say it *essential* discontinuity.

On the other hand, the discontinuity of $\alpha_p(\boldsymbol{z})$, which is found in (21) defined on S_3, is not essential; it is practically avoidable, e.g., by simply replacing the signum function by standard sigmoid function (see Fig.3).

$$\sigma(x) = \frac{1 - \exp(-k_\sigma x)}{1 + \exp(-k_\sigma x)}, \quad k_\sigma > 0 \tag{30}$$

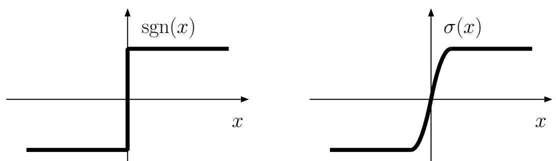

Fig. 3. Signum and Sigmoid functions

Considering the discontinuity of $\alpha_q(\boldsymbol{z})$ and $\alpha_n(\boldsymbol{z})$ on S_1, S_2 recalls exactly the same argument discussed by Bloch and Drakunov [1996], since the behavior of the system Σ_3^6 restricted to S_2 is essentially the same as that of system Σ_2^3. For example, other choices for $\alpha_q(\boldsymbol{z})$ are possible in (17), such as

$$\alpha_q(\boldsymbol{z}) = k_q \operatorname{sgn}(\|\boldsymbol{\phi}\| - \|\boldsymbol{q}\|), \tag{31}$$

$$\text{or} \quad \alpha_q(\boldsymbol{z}) = k_q \operatorname{sgn}(\epsilon - \|\boldsymbol{q}\|), \tag{32}$$

each of which will yield different convergence results (see the aforementioned literature for detail). In any case, we should remember that there is a trade-off between the speed of convergence/reaching and smoothness of the control input.

5. SIMULATION

Let us examine the proposed method by numerical simulations. Figures (4)–(8) show the simulation results for

$$\boldsymbol{z}(0) = (\boldsymbol{q}(0), \boldsymbol{\phi}(0)) = (-0.05, 0.1, -0.1, 5, -1, 1)^T$$

as initial condition. Note that $\boldsymbol{z}(0)$ belongs to D in this case because $\boldsymbol{q}(0)$ and $\boldsymbol{\phi}(0)$ are parallel. For smoothness of the numerical computation, the signum functions are blurred by substituting sigmoid function with $k_\sigma = 1.0 \times 10^3$. Other design parameters are chosen as $k_q = k_p = k_n = 1$, $\epsilon = 0.5$.

Fig. 4 and 5 show the time history of the state vectors $\boldsymbol{q}$ and $\boldsymbol{\phi}$, while Fig. 6 shows the corresponding control input $\boldsymbol{u}$. At the beginning of the simulation, $\boldsymbol{z}$ instantaneously exits from D driven by **Case 4** of the control law, and enters S_3. Then **Case 3** of the control law is chosen, so that $\boldsymbol{\phi}$ and $\boldsymbol{q}$ become orthogonal to each other in finite time. Both $\|\boldsymbol{q}\|$ and $\|\boldsymbol{\phi}\|$ are kept constant (see Fig.7) while $\boldsymbol{\phi}^T\boldsymbol{q}$ decreases and $\|\boldsymbol{n}\|$ increases (see Fig.8). Right after $\boldsymbol{z}(t)$ reaches S_2 at around $t = 0.25$, **Case 2** of the control law is chosen in turn. Since the radius $\|\boldsymbol{q}\|$ is smaller than the required level $\|\boldsymbol{\phi}\|$ at this moment, it increases until $\|\boldsymbol{q}\|$ becomes equal to $\max\{\|\boldsymbol{\phi}\|, \epsilon\}$; it turns to decrease after that. Meanwhile, $\|\boldsymbol{\phi}\|$ decreases monotonically to $\mathbf{0}$ by virtue of the holonomy term $-k_n\boldsymbol{n}_0$. $\boldsymbol{z}(t)$ reaches S_1 at around $t = 2.8$[sec]. Finally, $\boldsymbol{q}$ goes directory to $\mathbf{0}$ driven by **Case 1** of the control law.

In addition, Fig. 9 shows the corresponding trajectory of $\boldsymbol{q}$ and $\boldsymbol{\phi}$ plotted in $\mathbb{R}^3$, to help the readers' geometrical comprehension. We can see that $\boldsymbol{\phi}$ goes almost straightly to the origin, while $\boldsymbol{q}$ leaves a circle-like trajectory (which is mainly the trace of Case 2).

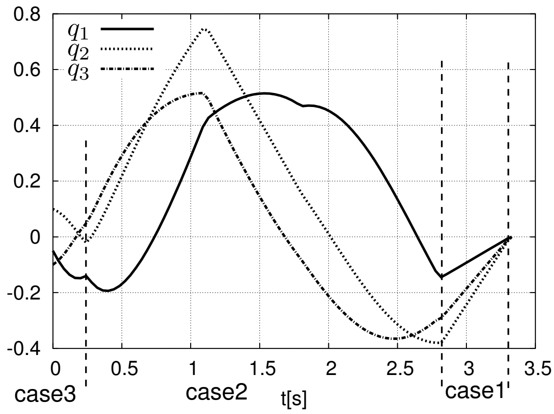

Fig. 4. Time history of $\boldsymbol{q}$

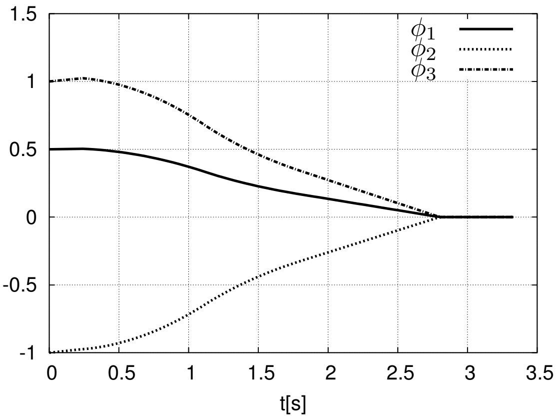

Fig. 5. Time history of $\boldsymbol{\phi}$

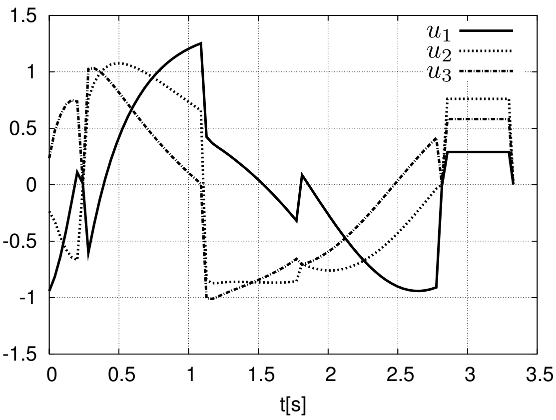

Fig. 6. The control input $\boldsymbol{u}$

6. CONCLUSION

In this paper, we suggested a switched feedback control law for first-order driftless nonholonomic systems with 3-inputs. The proposed method guarantees the boundedness of the control inputs and finite-time reaching to the origin

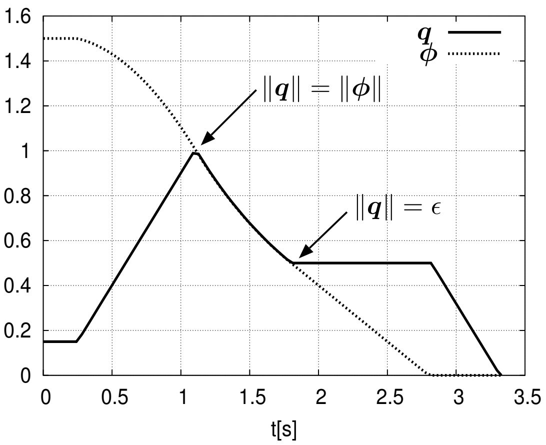

Fig. 7. Time history of $\|\boldsymbol{q}\|, \|\boldsymbol{\phi}\|$

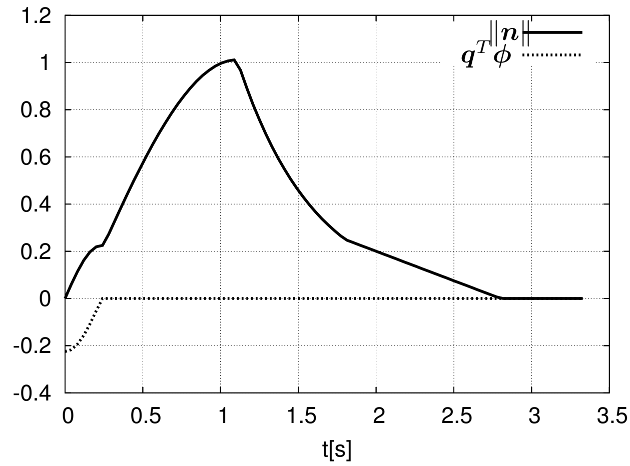

Fig. 8. Time history of $\|\boldsymbol{n}\|, \boldsymbol{\phi}^T\boldsymbol{q}$

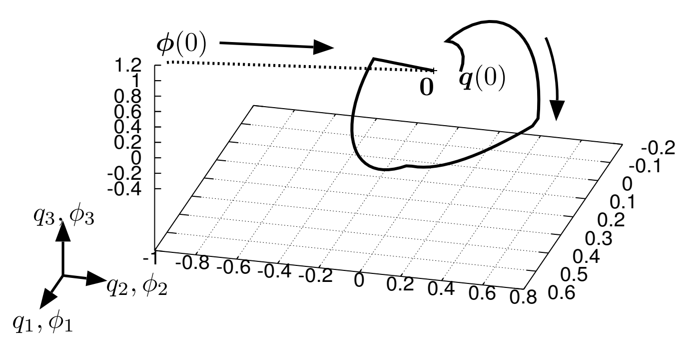

Fig. 9. Spatial trajectory of $\boldsymbol{q}$, $\boldsymbol{\phi}$ on $\mathbb{R}^3$

for all initial states without exception. Its extension to non-nilpotent systems without approximation, such as trident snake robot, will be considered in future works.

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