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Robust Regulation of Infinite-Dimensional Systems with Infinite-Dimensional Exosystems

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Abstract: In this paper robust regulation problem for infinite-dimensional systems with infinite-dimensional exosystems is discussed. The exosystems considered in this paper have infinite number of eigenvalues on the imaginary axis and thus include periodic signals. It is shown that there exists a feedback controller which robustly regulates the class of signals generated by the exosystem and strongly stabilizes the closed-loop system. As far as the authors know, the result is new even for finite-dimensional systems.

Keywords: Infinite-Dimensional Systems, Infinite-Dimensional Exosystems, Robust Regulation, Strong Stabilization

1. INTRODUCTION

One of the cornerstones of the classical automatic control theory of finite-dimensional linear systems is the Internal Model Principle (IMP) due to Francis and Wonham [1976], and Davison [1976]. Roughly stated, this principle asserts that any error feedback controller which achieves closed loop stability also achieves robust (i.e. structurally stable) output regulation (i.e. asymptotic tracking/rejection of a class of exosystem-generated signals) if and only if the controller incorporates a suitably reduplicated model of the dynamic structure of the exogenous reference/disturbance signals which the controller is required to process.

Regulation problem for infinite-dimensional systems has been studied by Schumacher [1983] and more recently by Byrnes et al. [2000]. They formulate the regulation problem with aid of Sylvester's equation but do not discuss the robustness of the controllers. Although the systems considered are infinite-dimensional, the reference and perturbation signals are generated by finite-dimensional exosystems.

In his Ph.D-thesis Bhat [1976] investigated regulation and robustness of infinite-dimensional systems with finite-dimensional exosystem. The approach is based on extending Francis's and Wonham's regulation and structural stability results to distributed parameter systems.

Robust regulation problem for infinite-dimensional systems sense of Davison [1976] has been introduced by Pohjolainen [1982] and Hämäläinen and Pohjolainen [2000]. In these papers robustness is a property of the selected controller but the papers do not discuss why a given controller is robust. Immonen in his recent PhD-thesis Immonen [2006] and Immonen and Pohjolainen [2006] were able to derive conditions for robustness for infinite-dimensional systems with infinite-dimensional reference and perturbation signals. However, these conditions seem sometimes to be difficult to check and good existence conditions were still missing.

In this paper we discuss the state space generalization of the Internal Model Principle for infinite-dimensional systems. The presentation is based on the concept of the steady state behavior of the system under infinite-dimensional exosystem generated signals. This approach leads us naturally to infinite-dimensional Sylvester equation, and a constrained infinite-dimensional Sylvester equation, which adds a constraint to Sylvester's equation for regulation. Then it is shown that feedback structure enables robustness, as the regulation equation is contained in the Sylvester's equation and as the system reaches its steady state this equation is automatically satisfied. Finally it will be shown that if the controller contains a p-copy internal model of the exosystem, then Sylvester's equations imply robust regulation.

It is shown that the smoothness of the reference and disturbance signals that can be regulated, is determined by the high-frequency behaviour of the plant transfer function.

Due to the fact that the exosystem has infinite number of eigenvalues on the imaginary axis, the closed-loop system cannot be exponentially stabilized. Instead strong stabilization must be used.

The presentation generalizes partly Huang's simple derivation Huang [2004] on robust regulation for linear finite-dimensional systems, and partly that of Francis and Wonham [1976], to infinite-dimensional systems and infinite-dimensional signals.

2. PROBLEM FORMULATION

2.1 The Reference and Disturbance Signals

We assume that the disturbance signals and the reference signal are of the form

$$\sum_{n=-\infty}^{\infty} a_n \mathrm{e}^{\mathrm{i}\omega_n t}, \qquad \omega_n \in \mathbb{R}. \tag{1}$$

More precise conditions for the coefficients a_n will be given later. It is convenient to assume that the signals (1) are generated by the exosystem

$$\dot{v} = Sv, \qquad v(0) = v_0 \in \mathcal{D}(S), \tag{2}$$

Here $S : \mathcal{D}(S) \subset W \to W$ is assumed to be a generator of C_0-semigroup on a Hilbert-space W.

In this paper we assume that the operator S in (2) is given by

$$\begin{aligned} Sv &= \sum_{n=-\infty}^{\infty} \mathrm{i}\omega_n \langle v, \phi_n \rangle \phi_n, \\ \mathcal{D}(S) &= \Big\{ v \in W \;\Big|\; \sum_{n=-\infty}^{\infty} \omega_n^2 |\langle v, \phi_n\rangle|^2 < \infty \Big\}, \end{aligned} \tag{3}$$

where $(\phi_n)_{n\in\mathbb{Z}}$ is an orthonormal basis of W and the sequence $(\mathrm{i}\omega_n)_{n\in\mathbb{Z}}$ has no finite accumulation points. Then $v(t)$ is given by

$$v(t) = T_S(t)v_0 = \sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega_n t}\langle v_0, \phi_n\rangle \phi_n, \qquad v_0 \in W. \tag{4}$$

Clearly $T_S(t)$ is invertible for every $t \geq 0$ and we have $T_S(t)^{-1} = T_S(-t)$. In fact $T_S(t)$ is a C_0-group.

2.2 The Plant

The plant P is described by the equations

$$\dot{x} = Ax + Bu + F_s v, \qquad x(0) \in \mathcal{D}(A) \tag{5a}$$
$$y = Cx + Du + F_m v, \tag{5b}$$

where the state $x(t) \in X$, the input $u(t) \in U$, the output $y(t) \in Y$ and the signal $v(t) \in W$ is given by (4). The spaces X, U, Y are Banach spaces with $U = \mathbb{C}^m$ and $Y = \mathbb{C}^p$. The system operator $A : \mathcal{D}(A) \subset X \to X$ is the generator of a C_0-semigroup $T(t)$, all the other operators are bounded: $B \in \mathcal{L}(U, X)$, $C \in \mathcal{L}(X, Y)$, $D \in \mathcal{L}(U, Y)$, $F_s \in \mathcal{L}(W, X)$, and $F_m \in \mathcal{L}(W, Y)$.

We assume that the transfer function of the plant, $P(s) = C(sI - A)^{-1}B + D \in \mathcal{L}(U, Y)$, satisfies $\operatorname{rank} P(s) = p$ for $s \in \sigma(S)$. We assume also that A is exponentially stabilizable and exponentially detectable.

We assume that the reference signal $r : [0, \infty) \to Y$ is given by $r = F_r v$ where $F_r \in \mathcal{L}(W, Y)$ Combining the plant equations (5) and the tracking error $e = y - r = y - F_r v$ we get the standard form

$$\dot{x} = Ax + Bu + Ev, \qquad x(0) \in \mathcal{D}(A) \tag{6a}$$
$$e = Cx + Du + Fv, \tag{6b}$$

where $E = F_s$ and $F = F_m - F_r$.

The reference signal $F_r v$ satisfies

$$F_r v(t) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega_n t}\langle v_0, \phi_n\rangle F_r \phi_n. \tag{7}$$

Then the coefficients a_n in (1) are given by $a_n = \langle v_0, \phi_n\rangle F_r \phi_n$. Hence the smoothness of the reference signals can be controlled by conditions placed on the sequence $(F_r\phi_n)_{n\in\mathbb{Z}}$. Similar considerations hold for the disturbance signals. Later in Section 7 conditions, depending on the behaviour of the transfer function $P(\mathrm{i}\omega_n)$ as $n \to \infty$, are given for the sequences $(F_r\phi_n)_{n\in\mathbb{Z}}$, $(F_m\phi_n)_{n\in\mathbb{Z}}$ and $(F_s\phi_n)_{n\in\mathbb{Z}}$.

2.3 The Controller

The controller is defined by the equations

$$\dot{z} = \mathcal{G}_1 z + \mathcal{G}_2 e, \qquad z(0) \in \mathcal{D}(\mathcal{G}_1) \tag{8a}$$
$$u = Kz, \tag{8b}$$

where $\mathcal{G}_1 : \mathcal{D}(\mathcal{G}_1) \subset Z \to Z$ generates a C_0-semigroup on the Banach space Z, $\mathcal{G}_2 \in \mathcal{L}(Y, Z)$ and $K \in \mathcal{L}(Z, U)$.

2.4 The Closed-Loop System

Let $X_e = X \times Z$ be the extended state-space, consisting of the plant and controller states, and let $x_e(t) = (x(t), z(t)) \in X_e$ be the extended state. Combining the equations (5) and (8) we get the closed-loop system

$$\dot{x}_e = A_e x_e + B_e v, \qquad x_e(0) \in \mathcal{D}(A_e) \tag{9a}$$
$$e = C_e x_e + D_e v, \tag{9b}$$

where $C_e = [C \;\, DK] \in \mathcal{L}(X_e, Y)$, $D_e = F \in \mathcal{L}(W, Y)$, and $A_e : \mathcal{D}(A_e) = \mathcal{D}(A) \times \mathcal{D}(\mathcal{G}_1) \subset X_e \to X_e$ and $B_e \in \mathcal{L}(W, X_e)$ are given by

$$A_e = \begin{bmatrix} A & BK \\ \mathcal{G}_2 C & \mathcal{G}_1 + \mathcal{G}_2 DK \end{bmatrix}, \qquad B_e = \begin{bmatrix} E \\ \mathcal{G}_2 F \end{bmatrix}.$$

3. THE OUTPUT REGULATION PROBLEM

Definition 1. We define The Output Regulation Problem (ORP) as follows: Design a controller (8) such that

(i) The closed-loop system operator A_e genearates a strongly stable C_0-semigroup.
(ii) For all initial states $x_e(0)$ and $v(0) \in \mathcal{D}(S)$
$$\lim_{t\to\infty} e(t) = 0.$$

Next we prove some general results that do not depend on the exact choice of the controller parameters. The most important of these is Theorem 4 which shows that the solution of the ORP is equivalent to the existence of a solution to a certain constrained Sylvester equation. For this we need to make the following Assumption:

Assumption 2. We assume that $B_e\phi_n \in \mathcal{R}(\mathrm{i}\omega_n I - A_e)$ for $n \in \mathbb{Z}$ and

$$\sum_{n=-\infty}^{\infty} \|R(\mathrm{i}\omega_n; A_e)B_e\phi_n\|^2 < \infty. \tag{10}$$

Later in the paper we choose the controller parameters in such a way that Assumption 2 holds.

Lemma 3. Assume that A_e generates a strongly stable C_0-semigroup and that Assumption 2 holds.

(a) There exists a unique operator $\Sigma_{\mathrm{ss}} \in \mathcal{L}(W, X_e)$ given by

$$\Sigma_{\mathrm{ss}} w = \sum_{n=-\infty}^{\infty} \langle w, \phi_n\rangle R(\mathrm{i}\omega_n; A_e)B_e\phi_n, \qquad w \in W, \tag{11}$$

that satifies $\Sigma_{ss}(\mathcal{D}(S)) \subset \mathcal{D}(A_e)$ and the Sylvester equation

$$\Sigma_{ss} S - A_e \Sigma_{ss} = B_e \quad \text{on } \mathcal{D}(S). \tag{12}$$

(b) For $v(0) \in \mathcal{D}(S)$ the error signal e satisfies

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty} (C_e\Sigma_{ss} + D_e)v(t), \tag{13}$$

where $v(t) = T_S(t)v(0)$.

Proof.

(a) Using the Cauchy-Schwarz inequality we have $\|\Sigma_{ss} w\|^2 \le \|w\|^2 \sum_{n=-\infty}^{\infty} \|R(\mathrm{i}\omega_n; A_e)B_e\phi_n\|^2$ for every $w \in W$ and hence Σ_{ss} is bounded. Let $s \in \rho(A_e)$. Using the resolvent equation we get for every $w \in \mathcal{D}(S)$

$$\begin{aligned} &-R(s;A_e)\Sigma_{ss}(S - sI)w \\ &= -\sum_{n=-\infty}^{\infty} \langle Sw - sw, \phi_n\rangle R(s;A_e)R(\mathrm{i}\omega_n;A_e)B_e\phi_n \\ &= \Sigma_{ss} w - R(s;A_e)B_e w. \end{aligned}$$

Therefore on $\mathcal{D}(S)$ we have $\Sigma_{ss} = -R(s;A_e)\Sigma_{ss}(S - sI) = R(s;A_e)B_e$, which shows that $\Sigma_{ss}(\mathcal{D}(S)) \subset \mathcal{D}(A_e)$. Multiplying by $sI - A_e$ we get $(sI - A_e)\Sigma_{ss} = -\Sigma_{ss}(S - sI) + B_e$ which implies (12).

Finally we show that the solution of (12) is unique. Suppose Σ satisfies (12). Then for every $n \in \mathbb{Z}$ we have $B_e\phi_n = \Sigma S\phi_n - A_e\Sigma\phi_n = (\mathrm{i}\omega_n - A_e)\Sigma\phi_n$. Hence $\Sigma\phi_n = R(\mathrm{i}\omega_n;A_e)B_e\phi_n$ and $\Sigma w = \sum_{n=-\infty}^{\infty}\langle w,\phi_n\rangle\Sigma\phi_n = \Sigma_{ss} w$ for every $w \in W$.

(b) The solution of (9a) is given by

$$x_e(t) = T_{A_e}(t)x_e(0) + \int_0^t T_{A_e}(t-s)B_e v(s)\,\mathrm{d}s. \tag{14}$$

Since Σ_{ss} satisfies (12), $v(0) \in \mathcal{D}(S)$ and $\Sigma_{ss} v(s) \in \mathcal{D}(A_e)$ we have $T_{A_e}(t-s)B_e v(s) = \frac{\mathrm{d}}{\mathrm{d}s}T_{A_e}(t-s)\Sigma_{ss} v(s)$. Substituting this into (14) gives the equation $x_e(t) = T_{A_e}(t)(x_e(0) - \Sigma_{ss} v(0)) + \Sigma_{ss} v(t)$. Therefore the error signal e is given by $e(t) = C_e T_{A_e}(t)(x_e(0) - \Sigma_{ss} v(0)) + (C_e\Sigma_{ss} + D_e)v(t)$. Because $T_{A_e}(t)$ is strongly stable and C_e is bounded, taking the limit of this as $t \to \infty$ gives (13).

Theorem 4. Assume that A_e generates a strongly stable C_0-semigroup and that Assumption 2 holds. Then the following are equivalent.

(a) The controller (8) solves the ORP.

(b) There exists a unique operator $\Sigma_{ss} \in \mathcal{L}(W, X_e)$ that satisfies $\Sigma_{ss}(\mathcal{D}(S)) \subset \mathcal{D}(A_e)$ and the constrained Sylvester equation

$$\Sigma_{ss} S - A_e\Sigma_{ss} = B_e, \text{ on } \mathcal{D}(S) \tag{15a}$$

$$C_e\Sigma_{ss} + D_e = 0. \tag{15b}$$

(c) There exist bounded operators $\Pi \in \mathcal{L}(W,X)$ and $\Gamma \in \mathcal{L}(W,Z)$ such that $\Pi(\mathcal{D}(S)) \subset \mathcal{D}(A)$, $\Gamma(\mathcal{D}(S)) \subset \mathcal{D}(\mathcal{G}_1)$ and Π and Γ satisfy the regulator equations

$$\Pi S = A\Pi + BK\Gamma + E, \tag{16a}$$

$$\Gamma S = \mathcal{G}_1\Gamma + \mathcal{G}_2(C\Pi + DK\Gamma + F), \tag{16b}$$

$$0 = C\Pi + DK\Gamma + F, \tag{16c}$$

where the first two equations hold on $\mathcal{D}(S)$.

Proof. (a) $\Longrightarrow$ (b). Since A_e generates a strongly stable semigroup and Assumption 2 holds, it follows from Lemma 3(a) that there is a unique operator Σ_{ss} satisfying $\Sigma_{ss}(\mathcal{D}(S)) \subset \mathcal{D}(A_e)$ and (15a). It follows from Lemma 3(b) that the error signal satisfies (13). Since the controller solves the ORP, $\lim_{t\to\infty} e(t) = 0$ and hence $\lim_{t\to\infty}(C_e\Sigma_{ss} + D_e)v(t) = 0$. Choosing in particular $v(0) = \phi_n$ and using $T_S(t)\phi_n = \mathrm{e}^{\mathrm{i}\omega_n t}\phi_n$, we get $\|(C_e\Sigma_{ss} + D_e)\phi_n\| = 0$ for every $n \in \mathbb{Z}$, and since the vectors ϕ_n form an orthonormal basis of W we have $C_e\Sigma_{ss} + D_e = 0$. Therefore Σ_{ss} satisfies also the regulation constraint (15b).

(b) $\Longrightarrow$ (a). Because the operator A_e generates a strongly stable semigroup and the operator Σ_{ss} satisfies(15a), it follows from Lemma 3(b) that for $v(0) \in \mathcal{D}(S)$ the error signal satisfies (13). Substituting (15b) into (13) gives $\lim_{t\to\infty} e(t) = 0$ and output regulation is achieved.

(b) $\Longleftrightarrow$ (c). If Σ_{ss} satisfies equations (15) we can define the operators $\Pi \in \mathcal{L}(W,X)$ and $\Gamma \in \mathcal{L}(W,Z)$ by decomposing Σ_{ss} as $\Sigma_{ss} = \left[\begin{smallmatrix}\Pi\\ \Gamma\end{smallmatrix}\right]$. Then (15) in component form is given by

$$\begin{bmatrix}\Pi\\ \Gamma\end{bmatrix} S - \begin{bmatrix} A & BK \\ \mathcal{G}_2 C & \mathcal{G}_1 + \mathcal{G}_2 DK\end{bmatrix}\begin{bmatrix}\Pi\\ \Gamma\end{bmatrix} = \begin{bmatrix} E \\ \mathcal{G}_2 F\end{bmatrix}$$

$$[C \;\; DK]\begin{bmatrix}\Pi\\ \Gamma\end{bmatrix} + D_e = 0,$$

and these are clearly equivalent to equations (16). Conversely, if $\Pi \in \mathcal{L}(W,X)$ and $\Gamma \in \mathcal{L}(W,Z)$ satisfy (16), we can define Σ_{ss} as above and then Σ_{ss} satisfies (15).

Now it follows from Theorem 4 that the controller solves the ORP provided that the controller parameters are chosen in such a way that the closed loop is strongly stable, Assumption 2 holds and the constrained Sylvester equations (15) are satisfied. Closed loop stabilization will be done in Section 6 and Assumption 2 and equations (15) are shown to hold, with suitable constraints on the operators E and F, in Section 7.

4. ROBUST REGULATION

In this section we give a definition of a robust controller. Assume that

- The system parameters (A,B,C,D,E,F) are perturbed to $(A_p,B_p,C_p,D_p,E_p,F_p)$.
- The strong stability of the closed-loop system and Assumption 2 is conserved under these perturbations.

The purpose is to find a *robust* feedback controller that also regulates the error e of the perturbed system to zero. As the closed-loop system remains stable and Assumption 2 continues to hold, the ORP under perturbed parameters is solvable if the constrained Sylvester equations (15), or equivalently equations (16), are satisfied for the perturbed parameters. The two perturbed Sylvester equations (16a) and (16b) are automatically satisfied since the closed-loop system is strongly stable and Assumption 2 holds. The regulation constraint (16c) does not necessarily hold.

The regulation constraint (16c) is also a part of the second Sylvester equation (16b). Select the controller parameters $(\mathcal{G}_1, \mathcal{G}_2)$ so that (16b) is equivalent to

$$\Gamma_p S = \mathcal{G}_1\Gamma_p \tag{17a}$$

$$\mathcal{G}_2(C_p\Pi_p + D_pK\Gamma_p + F_p) = 0 \tag{17b}$$

and further $\mathcal{G}_2$ so that (17b) implies

$$C_p\Pi_p + D_pK\Gamma_p + F_p = 0. \tag{17c}$$

Thus the regulation constraint is satisfied also for the perturbed parameters and the controller solves the ORP. Therefore we make the following definition.

Definition 5. The controller $(\mathcal{G}_1, \mathcal{G}_2)$ is *robust* if (16b) decomposes into (17a) and (17c).

5. INTERNAL MODEL

Definition 6. The controller (8) has an internal model of the exosystem S if the state-space Z can be decomposed as $Z = Z_1 \times Z_2$ and the operators $\mathcal{G}_1$ and $\mathcal{G}_2$ are of the form

$$\mathcal{G}_1 = \begin{bmatrix} R_1 & R_2 \\ 0 & G_1 \end{bmatrix}, \qquad \mathcal{G}_2 = \begin{bmatrix} R_3 \\ G_2 \end{bmatrix}, \tag{18}$$

where $R_1 : \mathcal{D}(R_1) \subset Z_1 \to Z_1$, $R_2 \in \mathcal{L}(Z_2, Z_1)$, $R_3 \in \mathcal{L}(Y, Z_1)$, $G_1 : \mathcal{D}(G_1) \subset Z_2 \to Z_2$, $G_2 \in \mathcal{L}(Y, Z_2)$ and G_1 and G_2 satisfy

$$\mathcal{N}(G_2) = \{0\}, \tag{19a}$$

$$\mathcal{R}(G_2) \cap \mathcal{R}(G_1 - sI) = \{0\}, \qquad \forall s \in \sigma(S), \tag{19b}$$

and the pair (G_1, G_2) is approximately controllable.

Next we show that a controller with an internal model is robust in the sense of Definition 5, i.e., that (16b) decomposes into (17a) and (17c).

Theorem 7. A controller with an internal model is robust.

Proof. Let $H = C\Pi + DK\Gamma + F$ and partition Γ as $\Gamma = \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix}$ Substituting (18) into (16b) we get

$$\begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} S = \begin{bmatrix} R_1 & R_2 \\ 0 & G_1 \end{bmatrix} \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} + \begin{bmatrix} R_3 \\ G_2 \end{bmatrix} H$$

the second equation of which gives $\Gamma_2 S = G_1\Gamma_2 + G_2H$. Applying both sides of this to the basis vector ϕ_n we get $(\mathrm{i}\omega_n I - G_1)\Gamma_2\phi_n = G_2H\phi_n$ for each $n \in \mathbb{Z}$. Now equation (19b) implies that $G_2H\phi_n = 0$ and equation (19a) that $H\phi_n = 0$. Since the sequence $(\phi_n)_{n\in\mathbb{Z}}$ is an orthonormal basis of W, we get $H = 0$ and (17c) holds. Finally substituting (17c) into (16b) gives (17a).

6. AN OBSERVER-BASED CONTROLLER

In this section a choice of the controller parameters $\mathcal{G}_1$, $\mathcal{G}_2$ and K will be given which stabilizes the closed-loop system operator A_e and has an internal model, and therefore is robust.

Let $Z = Z_1 \times Z_2 = X \times W^p$ and $K = [K_1\ K_2]$ where $K_1 \in \mathcal{L}(X, U)$, $K_2 \in \mathcal{L}(W^p, U)$. We choose R_1, R_2 and R_3 in (18) as follows: $R_1 = A + BK_1 + L(C + DK_1)$, $R_2 = (B + LD)K_2$, and $R_3 = -L$, where the operator $L \in \mathcal{L}(Y, X)$ will be determined later. Then the equations (18) take the form

$$\mathcal{G}_1 = \begin{bmatrix} A + BK_1 + L(C + DK_1) & (B + LD)K_2 \\ 0 & G_1 \end{bmatrix}, \qquad \mathcal{G}_2 = \begin{bmatrix} -L \\ G_2 \end{bmatrix}, \tag{20}$$

The operators G_1 and G_2 are chosen as follows: $G_1 = \operatorname{diag}(S, \ldots, S) : \mathcal{D}(G_1) \subset W^p \to W^p$, $\mathcal{D}(G_1) = \mathcal{D}(S)^p$, and $G_2y = (g_1y_1, \ldots, g_py_p)$, $y \in Y$, for $g_i \in \mathcal{D}(S)$ satisfying the conditions

$$\langle g_i, \phi_n \rangle \neq 0, \qquad i = 1, \ldots, p, \quad n \in \mathbb{Z}. \tag{21}$$

We also define $G_{2n} = \operatorname{diag}(\langle g_1, \phi_n \rangle, \ldots, \langle g_p, \phi_n \rangle) \in \mathbb{C}^{p\times p}$, which is clearly nonsingular.

Define $\psi_{nj} = (0, \ldots, \phi_n, \ldots, 0)$ for $1 \le j \le p$ where ϕ_n is in the jth place. It is easily seen that the sequences $(\psi_{nj})_{n\in\mathbb{Z}}$, $1 \le j \le p$, form an orthonormal basis of W^p. Then $v \in W^p$ can be written as $v = (v_1, \ldots, v_p) = \sum_{n=-\infty}^{\infty}\sum_{j=1}^{p}\langle v, \psi_{nj}\rangle\psi_{nj}$. Similarly $G_1v = \sum_{n=-\infty}^{\infty} \mathrm{i}\omega_n \sum_{j=1}^{p}\langle v, \psi_{nj}\rangle\psi_{nj}$ with $v \in \mathcal{D}(G_1) \iff \sum_{n=-\infty}^{\infty}\omega_n^2\sum_{j=1}^{p}|\langle v, \psi_{nj}\rangle|^2 < \infty$.

Lemma 8. The controller (20) has an internal model and therefore is robust.

Proof. First, let us show that G_1 and G_2 satisfy equations (19). For $y = (y_1, \ldots, y_p)$ we have $G_2y = 0 \iff (g_1y_1, \ldots, g_py_p) = 0 \iff y = 0$, since $g_i \neq 0$. Hence (19a) is satisfied. Now let $w \in \mathcal{R}(G_2) \cap \mathcal{R}(G_1 - \mathrm{i}\omega_n I)$ for some $n \in \mathbb{Z}$. Then $w = G_2y = (G_1 - \mathrm{i}\omega_n I)v$ for some $y \in Y$ and $v \in W^p$. Thus for $i = 1, \ldots, p$

$$g_iy_i = (S - \mathrm{i}\omega_n I)v_i = \sum_{k=-\infty}^{\infty}(\mathrm{i}\omega_k - \mathrm{i}\omega_n)\langle v_i, \phi_k\rangle\phi_k,$$

which implies $y_i\langle g_i, \phi_n\rangle = (\mathrm{i}\omega_n - \mathrm{i}\omega_n)\langle v_i, \phi_n\rangle = 0$. Since $\langle g_i, \phi_n\rangle \neq 0$ we have $y_i = 0$ and therefore $y = 0$. Hence $w = 0$ and (19b) is satisfied.

Finally, let us show that the pair (G_1, G_2) is approximately controllable. We can write G_2 as $G_2y = \sum_{k=1}^{p} y_k\tilde{g}_k$, where $\tilde{g}_k = (0, \ldots, g_k, \ldots, 0) \in W^p$. The approximate controllability now follows easily from the nonsingularity of the matrix G_{2n}.

The next Theorem gives a choice of the controller parameters K and L such that the closed-loop system operator A_e is strongly stable.

Theorem 9. The controller (20) strongly stabilizes the closed-loop system operator A_e, provided that K and L are chosen as follows: Choose L so that $A{+}LC$ is exponentially stable. Let $K_1 = K_{11} + K_{12}$ and choose K_{11} so that $A + BK_{11}$ is exponentially stable. Choose $K_{12} = K_2H$ where H is the solution of the Sylvester equation $G_1H - H(A{+}BK_{11}) = G_2(C{+}DK_{11})$. Finally choose $K_2 = -B_1^*$ where $B_1 = HB + G_2D$.

Proof. The closed-loop system operator is given by

$$A_e = \begin{bmatrix} A & BK_1 & BK_2 \\ -LC & A + BK_1 + LC & BK_2 \\ G_2C & G_2DK_1 & G_1 + G_2DK_2 \end{bmatrix}.$$

Now applying to A_e the similarity transformation

$$T = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ -I & I & 0 \end{bmatrix}, \qquad T^{-1} = \begin{bmatrix} I & 0 & 0 \\ I & 0 & I \\ 0 & I & 0 \end{bmatrix} \tag{22}$$

and letting $\tilde{A}_e = TA_eT^{-1}$ we get

$$\tilde{A}_e = \begin{bmatrix} A + BK_1 & BK_2 & BK_1 \\ G_2(C + DK_1) & G_1 + G_2DK_2 & G_2DK_1 \\ 0 & 0 & A + LC \end{bmatrix}. \tag{23}$$

Clearly A_e is strongly stable if and only if $\tilde{A}_e$ is strongly stable. Since A is exponentially detectable, we can find an operator $L \in \mathcal{L}(Y, X)$ such that $A + LC$ is exponentially stable. Therefore $\tilde{A}_e$ is strongly stable if and only if the opearator

$$\tilde{A}_{e1} = \begin{bmatrix} A + BK_1 & BK_2 \\ G_2(C + DK_1) & G_1 + G_2DK_2 \end{bmatrix}$$

is strongly stable. Now let $K_1 = K_{11} + K_{12}$. Then

$$\tilde{A}_{e1} = \begin{bmatrix} A + BK_{11} & 0 \\ G_2(C + DK_{11}) & G_1 \end{bmatrix} + \begin{bmatrix} B \\ G_2D \end{bmatrix} [K_{12} \;\; K_2].$$

Since A is exponentially stabilizable, we can choose K_{11} in such a way that $A_K = A + BK_{11}$ is exponentially stable. We also define $C_K = C + DK_{11}$. Next we apply to $\tilde{A}_{e1}$ the similarity transformation $T_H = [\begin{smallmatrix} I & 0 \\ H & I \end{smallmatrix}]$, $T_H^{-1} = [\begin{smallmatrix} I & 0 \\ -H & I \end{smallmatrix}]$, where $H \in \mathcal{L}(X, W^p)$ is to be determined. We get

$$\begin{bmatrix} I & 0 \\ H & I \end{bmatrix} \begin{bmatrix} A_K & 0 \\ G_2C_K & G_1 \end{bmatrix} \begin{bmatrix} I & 0 \\ -H & I \end{bmatrix} = \begin{bmatrix} A_K & 0 \\ HA_K + G_2C_K - G_1H & G_1 \end{bmatrix}$$

Next we choose H to be the solution of the Sylvester equation

$$G_1H - HA_K = G_2C_K. \tag{24}$$

Note that in order to (24) make sense, we must have $H(X) \subset \mathcal{D}(G_1)$. It is not too difficult to show that the solution of (24) is given by

$$Hx = \sum_{n=-\infty}^{\infty} \sum_{j=1}^{p} \langle G_2C_KR(\mathrm{i}\omega_n; A_K)x, \psi_{nj} \rangle \psi_{nj}, \tag{25}$$

for $x \in X$. Now choosing $K_{12} = K_2H$ we get

$$\begin{bmatrix} I & 0 \\ H & I \end{bmatrix} \begin{bmatrix} B \\ G_2D \end{bmatrix} [K_{12} \;\; K_2] \begin{bmatrix} I & 0 \\ -H & I \end{bmatrix} = \begin{bmatrix} B \\ HB + G_2D \end{bmatrix} [0 \;\; K_2]$$

and so

$$T\tilde{A}_{e1}T_H^{-1} = \begin{bmatrix} A_K & BK_2 \\ 0 & G_1 + (HB + G_2D)K_2 \end{bmatrix}.$$

It is easily seen that if A_K is exponentially stable and $G_1 + (HB + G_2D)K_2$ is strongly stable, then $T_H\tilde{A}_{e1}T_H^{-1}$ is strongly stable. Since A_K is exponentially stable, it remains to find a K_2 so that $G_1 + (HB + G_2D)K_2$ is strongly stable, i.e., we must strongly stabilize the pair $(G_1, HB + G_2D) = (G_1, B_1)$.

Since H is given by (25) in the Appendix, we get for $u \in U$

$$B_1u = HBu + G_2Du = \sum_{n=-\infty}^{\infty} \sum_{j=1}^{p} \langle G_2P_K(\mathrm{i}\omega_n)u, \psi_{nj} \rangle \psi_{nj}$$

where $P_K(s) = C_KR(s; A_K)B + D$. Now $[B_1u]_j = \sum_{k=1}^{m} b_{jk}u_k$ where $b_{jk} = \sum_{n=-\infty}^{\infty} [G_{2n}P_K(\mathrm{i}\omega_n)]_{jk}\phi_n$. In particular $\langle b_{jk}, \phi_n \rangle = [G_{2n}P_K(\mathrm{i}\omega_n)]_{jk}$. Hence B_1 can be written as $B_1u = \sum_{k=1}^{m} \tilde{b}_ku_k$, where $\tilde{b}_k = (b_{1k}, \ldots, b_{pk})$. Since A_K is exponentially stable, $P_K(\mathrm{i}\omega_n)$ is bounded with respect to n and hence B_1 is bounded. Since S generates a contraction semigroup and has compact resolvent, (G_1, B_1) can be strongly stabilized with the feedback $K_2 = -B_1^*$ provided that the pair (G_1, B_1) is approximately controllable. This holds iff rank $B_p = p$, where

$$B_p = \begin{bmatrix} \langle \tilde{b}_1, \psi_{n1} \rangle & \ldots & \langle \tilde{b}_m, \psi_{n1} \rangle \\ \vdots & & \vdots \\ \langle \tilde{b}_1, \psi_{np} \rangle & \ldots & \langle \tilde{b}_m, \psi_{np} \rangle \end{bmatrix} = G_{2n}P_K(\mathrm{i}\omega_n).$$

Since G_{2n} is nonsingular, rank $B_p = p$ iff rank $P_K(\mathrm{i}\omega_n) = p$ iff rank $P(\mathrm{i}\omega_n) = p$. Since the last condition holds by assumption, the pair (G_1, B_1) is approximately controllable.

Finally, a straightforward computation shows that

$$K_2v = B_1^*v = \sum_{n=-\infty}^{\infty} \sum_{j=1}^{p} \langle v, \psi_{nj} \rangle [G_2P_K(\mathrm{i}\omega_n)]^* \psi_{nj} \tag{26}$$

for $v \in W^p$.

7. CONDITIONS FOR ASSUMPTION 2 TO HOLD

In this section we give conditions on the operators E and F for Assumption 2 to hold and show that the constrained Sylvester equations (15) are satisfied, provided that the controller parameters K and L are chosen as in Theorem 9.

An easy computation shows that $G_2^*v = (\langle v_1, g_1 \rangle, \ldots, \langle v_p, g_p \rangle)$. We will also need the following

$$B_1^*\psi_{nj} = [G_{2n}P_K(\mathrm{i}\omega_n)]^*e_j. \tag{27}$$

To simplify notation we define

$$R_n = R(\mathrm{i}\omega_n; A_K), L_n = R(\mathrm{i}\omega_n; A + LC), P_n = P_K(\mathrm{i}\omega_n).$$

We also denote by A^+ the pseudoinverse $A^*(AA^*)^{-1}$ of the matrix A.

Theorem 10. Let K and L be as in Theorem 9. Then $B_e\phi_n \in \mathcal{R}(\mathrm{i}\omega_nI - A_e)$ for $n \in \mathbb{Z}$ and the series $\sum_{n=-\infty}^{\infty} \|R(\mathrm{i}\omega_n; A_e)B_e\phi_n\|^2$ converges if and only if the series

$$\sum_{n=-\infty}^{\infty} \left(\|\alpha_n\|^2 + \|\beta_n\|^2 + \|\beta_n + \eta_n\|^2 \right) \tag{28}$$

converges, where $\eta_n = -L_n(E\phi_n + LF\phi_n)$, $\xi_n = F\phi_n + C_KR_nE\phi_n$, and

$$\beta_n = R_nB((I - P_n^+P_n)K_1\eta_n - P_n^+\xi_n) + R_nE\phi_n$$
$$\alpha_n = G_{2n}^{-1}(P_n^+)^*(P_n^+P_nK_1\eta_n + P_n^+\xi_n + K_2H\beta_n).$$

Proof. Let us find $x_{en} = (x_{n1}, x_{n2}, x_{n3}) = R(\mathrm{i}\omega_n; A_e)B_e\phi_n$ for $n \in \mathbb{Z}$ by solving the equation

$$(\mathrm{i}\omega_nI - A_e)x_{en} = B_e\phi_n. \tag{29}$$

Applying the similarity transformation (22) to (29) gives the equation

$$(\mathrm{i}\omega_nI - \tilde{A}_e)Tx_{en} = TB_e\phi_n, \tag{30}$$

where $\tilde{A}_e$ is given by (23). Letting $Tx_{en} = (x_{n1}, x_{n3}, x_{n2} - x_{n1}) = (z_{n1}, z_{n2}, z_{n3})$ we get from (30) the equations

$$(\mathrm{i}\omega_nI - A - BK_1)z_{n1} - BK_2z_{n2} - BK_1z_{n3} = E\phi_n \tag{31a}$$

$$-G_2(C + DK_1)z_{n1} + (\mathrm{i}\omega_nI - G_1 - G_2DK_2)z_{n2} - G_2DK_1z_{n3} = G_2F\phi_n \tag{31b}$$

$$(\mathrm{i}\omega_nI - A - LC)z_{n3} = -E\phi_n - LF\phi_n. \tag{31c}$$

Solving z_{n3} from (31c) we get

$$z_{n3} = -L_n(E\phi_n + LF\phi_n). \tag{32}$$

Note that $z_{n3} \in \mathcal{D}(A)$. Since G_1 and G_2 satisfy equations (19) we get from (31b) the two equations

$$(C + DK_1)z_{n1} + DK_2z_{n2} + DK_1z_{n3} + F\phi_n = 0 \tag{33a}$$

$$(\mathrm{i}\omega_nI - G_1)z_{n2} = 0. \tag{33b}$$

Setting $K_1 = K_{11} + K_{12} = K_{11} + K_2H$ in (31a) and (33a) gives the equations

$$(\mathrm{i}\omega_nI - A_K)z_{n1} - BK_2(Hz_{n1} + z_{n2}) - BK_1z_{n3} = E\phi_n \tag{34a}$$

$$C_Kz_{n1} + DK_2(Hz_{n1} + z_{n2}) + DK_1z_{n3} = -F\phi_n. \tag{34b}$$

Solving z_{n1} from (34a) gives

$$z_{n1} = R_n B(K_2(Hz_{n1} + z_{n2}) + K_1 z_{n3}) + R_n E\phi_n. \quad (35)$$

Substituting this into (34b) we get

$$P_n K_2(Hz_{n1} + z_{n2}) = -P_n K_1 z_{n3} - F\phi_n - C_K R_n E\phi_n.$$

Since P_n has full row rank, the pseudo-inverse $P_n^+ = P_n^*(P_n P_n^*)^{-1}$ exists and is a right inverse of P_n. Hence we can solve for $K_2(Hz_{n1} + z_{n2})$

$$K_2(Hz_{n1} + z_{n2}) = -P_n^+ P_n K_1 z_{n3} - P_n^+ \xi_n, \quad (36)$$

where $\xi_n = F\phi_n + C_K R_n E\phi_n$. Substituting (36) into (35) we get

$$x_{n1} = z_{n1} = R_n B((I - P_n^+ P_n)K_1 z_{n3} - P_n^+ \xi_n) + R_n E\phi_n$$

and $x_{n2} = x_{n1} + z_{n3}$. Clearly $x_{n1} \in \mathcal{D}(A)$ and since $z_{n3} \in \mathcal{D}(A)$ we also have $x_{n2} \in \mathcal{D}(A)$. It is seen from (33b) that z_{n2} is an eigenvector of G_1 and therefore can be written as $z_{n2} = \sum_{j=1}^{p} \alpha_{nj}\psi_{nj}$ for some $\alpha_{nj} \in \mathbb{C}$. Now we get an equation for z_{2n} from (36)

$$K_2 z_{2n} = -P_n^+ P_n K_1 z_{n3} - P_n^+ \xi_n - K_2 H z_{n1}. \quad (37)$$

It follows from (27) that $K_2 z_{n2} = -[G_{2n} P_K(\mathrm{i}\omega_n)]^* \alpha_n$, where $\alpha_n = (\alpha_{n1}, \ldots, \alpha_{np})$. Substituting this into (37) we get the following equation for α_n

$$P_n^* G_{2n} \alpha_n = P_n^+ P_n K_1 z_{n3} + P_n^+ \xi_n + K_2 H z_{n1}. \quad (38)$$

Equation (38) has a solution if the right-hand side is in $\mathcal{R}(P_n^*)$. It is seen from (26) that every term of the series is in $\mathcal{R}(P_n^*)$. Since $\mathcal{R}(P_n^*)$ is a closed subspace, $K_2 H z_{1n} = -B_1^* H z_{1n}$ is in $\mathcal{R}(P_n^*)$. Because $\mathcal{R}(P_n^*) = \mathcal{R}(P_n^+)$, the term $P_n^+ \xi_n$ is also in $\mathcal{R}(P_n^*)$. Therefore (38) has unique solution, which can be given in terms of any left inverse of P_n^*. Since P_n^* has a full column rank, it has the left inverse $(P_n^*)^+ = (P_n^+)^*$. Hence $\alpha_n = G_{2n}^{-1}(P_n^+)^*(P_n^+ P_n K_1 z_{n3} + P_n^+ \xi_n + K_2 H z_{n1})$ In particular, $x_{n3} = z_{n2} \in \mathcal{D}(G_1)$. Therefore $x_{en} \in \mathcal{D}(A_e)$ and $B_e \phi_n \in \mathcal{R}(\mathrm{i}\omega_n I - A_e)$.

Since $x_{n3} = z_{n2}$, $\|z_{n2}\|^2 = \|\alpha_n\|^2$ and $\|R(\mathrm{i}\omega_n; A_e) B_e \phi_n\|^2 = \|x_{n1}\|^2 + \|x_{n2}\|^2 + \|x_{n3}\|^2$ the result follows by setting $\eta_n = z_{n3}$ and $\beta_n = x_{n1}$.

Since the expressions α_n and β_n in the series (28) depend on P_n^+, the convergence of the series depends on the behaviour of P_n^+ as $n \to \infty$. If $\lim_{n\to\infty} P_n = 0$, then $E\phi_n$ and $F\phi_n$ must decrease faster than P_n^+ increases, hence increasing the smoothness of the reference and disturbance signals.

Finally, let us show that the constrained Sylvester equations (15) are satisfied.

Theorem 11. Let K and L be as in Theorem 9. Then the constrained Sylvester equations (15) are satisfied.

Proof. Clearly it is sufficient to show that the equations

$$\Sigma_{\mathrm{ss}} S\phi_n - A_e \Sigma_{\mathrm{ss}} \phi_n = B_e \phi_n$$
$$C_e \Sigma_{\mathrm{ss}} \phi_n + D_e \phi_n = 0$$

are satisfied for $n \in \mathbb{Z}$. Since the closed-loop system is strongly stable, the first equation is satisfied and we have $\Sigma_{\mathrm{ss}}\phi_n = (\mathrm{i}\omega_n I - A_e)^{-1} B_e \phi_n = x_{en}$. Now it follows from (33a) that the second equation is also satisfied.

8. CONCLUSIONS

Robust regulation problem for infinite-dimensional systems with infinite-dimensional exosystems has been discussed. A feedback controller which robustly regulates the class of signals generated by the exosystem and strongly stabilizes the closed-loop system has been constructed. Strong stability is used because the exosystem has infinite number of eigenvalues on the imaginary axis, and hence there is little hope to achieve exponential stability. As far as the authors know, the result is new even for finite-dimensional systems.

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