Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Interval Observers for Uncertain Nonlinear Systems. Application to bioreactors.

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Abstract: This paper is about state estimation for nonlinear uncertain continuous-time systems in a bounded-error context. We introduce a method for designing a guaranteed interval observer which is based on (*i*) the application of Müller's theorem for bracketing the solution of ordinary differential equations in a way which ensures the positivity of the observation error, (*ii*) the analysis of the monotonicity of each component of system field vectors with respect to both the uncertain parameters and state variables and (*iii*) the choice of a tailor-made observation gain matrix in order to ensure that the observation error converges to the interior of a bounded box. The set of state estimation obtained in this context is a guaranteed approximation of the real solution set in the sense that no solution can be lost.

Keywords: Bounded-error estimation; continuous-time systems; interval analysis; non-linear systems; state estimation; uncertain systems.

1. INTRODUCTION

A great number of industrial or natural processes are usually modeled by a set of complex and uncertain differential equations. Let us consider the following type of uncertain nonlinear systems

$$\left\{\begin{array}{ll}\dot{\mathbf{x}}(t) & = \mathbf{A}\mathbf{x}(t)+\boldsymbol{\psi}(\mathbf{x},\mathbf{p},\mathbf{u}(t))\\ \mathbf{y}(t) & = \mathbf{C}\mathbf{x}(t)\\ \mathbf{x}(t_0) & \in [\mathbf{x}_0]\wedge\mathbf{p}\in[\mathbf{p}]\end{array}\right\} \quad (1)$$

where $\mathbf{A}$ is a matrix of dimension $n\times n$, $t\in[t_0,t_{n_T}]$, $\psi\in C^{k-1}(\mathbb{D}\times\mathbb{U}\times[\mathbf{p}])$, $\mathbb{D}\times\mathbb{U}\times[\mathbf{p}]\subseteq\mathbb{R}^{n+n_u+n_p}$ is an open set; n, n_u, m and n_p are the dimension of respectively the state vector $\mathbf{x}$, the input vector $\mathbf{u}$, the output vector $\mathbf{y}$ and the uncertain parameter vector $\mathbf{p}$. The choice of the type of systems (1) is not restrictive since one can find reservible mappings for the state vector in order to transform a given system to a system of type (1) [Isidori, 1995]. Measurements are subject to an unknown but bounded, with known bound, additive error

$$\mathbb{Y}(t)=[\mathbf{y}_m(t)-\mathbf{b},\quad \mathbf{y}_m(t)+\mathbf{b}] \quad (2)$$

where $\mathbf{b}$ is the vector of maximal measurement error. In addition, we will assume that system (1) satisfies the following hypotheses

Hypothesis 1. The matrix pair $(\mathbf{A},\mathbf{C})$ is observable.

Hypothesis 2. There is a positive matrix $\mathbf{K}$ such as the non-diagonal elements of the matrix $\mathbf{A}-\mathbf{K}\mathbf{C}$ are non-negative.

Hypothesis 3. All the components $\psi_i(.)$ of the vector function $\boldsymbol{\psi}(.)$ are Lipschitz with respect to the state vector for any $\mathbf{p}\in[\mathbf{p}]$. Moreover, there exist two known functions $\underline{\boldsymbol{\psi}}(.)$ and $\overline{\boldsymbol{\psi}}(.)$ built according to the bounds of $[\mathbf{p}]$ and a known number $M<+\infty$ such that:

$$\left\{\begin{array}{c}\forall\mathbf{p}\in[\mathbf{p}],\ \forall(\mathbf{u}(t),\mathbf{x})\in\mathbb{U}\times\mathbb{D},\\ \underline{\boldsymbol{\psi}}(\mathbf{x},\mathbf{p},\mathbf{u}(t))\leq\boldsymbol{\psi}(\mathbf{x},\mathbf{p},\mathbf{u}(t))\leq\overline{\boldsymbol{\psi}}(\mathbf{x},\mathbf{p},\mathbf{u}(t)),\\ \|\overline{\boldsymbol{\psi}}(\mathbf{x},\mathbf{p},\mathbf{u}(t))-\underline{\boldsymbol{\psi}}(\mathbf{x},\mathbf{p},\mathbf{u}(t))\|\leq M.\end{array}\right. \quad (3)$$

In order to apply advanced concepts of control and diagnosis to practical applications, the knowledge of system state variables is often required. In practice, the direct measurement of some state variables $\mathbf{x}$ is impossible for many reasons (physical, economical . . .). But it is possible to estimate them under some conditions of observability [Hermann and J.Krener, 1977] by means of state observers. The estimated state vector is usually given by the following model:

$$\dot{\hat{\mathbf{x}}}(t)=\mathbf{A}\hat{\mathbf{x}}(t)+\boldsymbol{\psi}(\hat{\mathbf{x}},\mathbf{p},\mathbf{u}(t))+\mathbf{K}(\hat{\mathbf{x}})(\mathbf{y}^*_m(t)-\mathbf{y}(t)) \quad (4)$$

where $\mathbf{y}^*_m(t)\in\mathbb{Y}(t)$. Obviously, the observer contains two terms: the first one is only a duplication of the state equation of (1) and the second is a corrective term intended to make the observation error converge towards zero

$$\lim_{t\to\infty}\|\hat{\mathbf{x}}(t)-\mathbf{x}(t)\|=0\quad\forall\mathbf{x}_0\neq\hat{\mathbf{x}}_0(t) \quad (5)$$

For linear systems, one can find two standard types of observers according to the deterministic or random context, respectively the Luenberger observer and the Kalman filter. On the other hand, for nonlinear continuous-time systems there exist many kinds of observers to be used depending on the mathematical structure of the process model and the available information. For instance, the extended Luenberger observer and the extended Kalman filter [Misawa and Hedrick, 1989], the high gain observer [Gauthier et al., 1992], the sliding mode observer [Slotine et al., 1986], ... All these nonlinear approaches are more or less robust with respect to the disturbances and the measurement noise, however they often provide unsat-

isfactory estimations in the presence of uncertainties in model parameters [Dochain, 2003]. To circumvent this kind of problems, several methods have been developed recently in the set membership framework. Their principle consists in computing sets guaranteed to contain all the state vectors consistent with all uncertainties. These two approaches are founded on the reachability analysis of uncertain systems based on special geometrical forms, such as ellipsoids [Chernousko, 2005], zonotopes [Combastel, 2005] ... to represent the state flow generated by uncertain dynamics. In this paper we will use boxes (interval vector) for characterizing the reached state space. There are mainly two types of set membership approaches:
(*i*) The first approach addresses the case of continuous-time state estimation from discrete-time measurement and was introduced in [Jaulin et al., 2001, Raïssi et al., 2004, 2005, Kieffer et al., 2006]. It relies on a two-stage methodology: a *prediction* stage, which consists in reconstructing the state vector by using validated numerical integration methods in order to derive a guaranteed numerical evaluation of the solution of the ordinary differential equations at measurement time steps [Nedialkov, 1999]; a *correction* stage, which consists in studying the consistency between the feasible domain of the measured output and the one of the simulated output in order to eliminate the inconsistent part of the estimated state vector computed in prediction stage. This approach will not be considered in this paper.
(*ii*) The second approach addresses the case of continuous-time state estimation from continuous-time measurement, and was proposed for the first time in [Gouzé et al., 2000]. The main idea consists in building a closed loop interval observer which takes into account model's parameters uncertainties. It computes a lower and an upper bounds for the estimated state vector, i.e., respectively $\underline{\hat{\mathbf{x}}}(t)$ and $\hat{\overline{\mathbf{x}}}(t)$ which bracket all the possible state vectors generated by the uncertain system and which are consistent with the measured data

$$\left\{\begin{array}{c} \forall \mathbf{x}(t_0) \in [\underline{\hat{\mathbf{x}}}(t_0), \hat{\overline{\mathbf{x}}}(t_0)], \quad \forall t > t_0 \\ \underline{\hat{\mathbf{x}}}(t) \leq \mathbf{x}(t) \leq \hat{\overline{\mathbf{x}}}(t) \\ \lim\limits_{t\to\infty} \parallel \hat{\overline{\mathbf{x}}}(t) - \underline{\hat{\mathbf{x}}}(t) \parallel = w(\mathbf{p}, \mathbf{b}) \end{array}\right. \tag{6}$$

Where $w(.)$ depends only on the width of the boxes characterizing the uncertain parameters and the feasible domain for measurements. Initially, this type of observers was used with nonlinear systems when the dynamics of the observation error was cooperative [Gouzé et al., 2000] and when the uncertain parameters were exclusively related to the input or the output of the system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}(t) + \boldsymbol{\psi}(\mathbf{p}, \mathbf{u}(t)) \tag{7}$$

More recently, this type of observers were generalized to the case of systems where the nonlinear term $\boldsymbol{\psi}$ depends also of the state vector [Rapaport and Gouzé, 2003, Moisan and Bernard, 2005]:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}(t) + \boldsymbol{\psi}(\mathbf{x}, \mathbf{p}, \mathbf{u}(t)) \tag{8}$$

In order to design this interval observer, the difficulty resides in the derivation of a guaranteed tight enclosure for the function ψ when both model parameter and state vectors are taken as uncertain quantities. A manner to circumvent this difficulty is based on the concept of observers bundle which consists to run simultaneously a broad set of interval observers and select over each lapse of time the best one [Gouzé, 2004, Moisan and Bernard, 2005].

In this work, our main contribution consists in the introduction of a method for building the closed loop interval observer in a guaranteed way for a large class of non linear systems not necessarily cooperative. Our method uses the classical Müller's existence theorem [Walter, 1997, Kieffer et al., 2006], the theory of quasi-monotone dynamical systems, mainly developed by Smith after the seminal work of Müller, Kamke and Krasnoselskij (see [Smith, 1995] and the references therein) and interval methods for initial value problems (IVPs) for ordinary differential equations (ODEs). However, the observer gain matrix is still choosen in the same way as in [Rapaport and Gouzé, 2003]. This paper is organized as follows. In section 2, we introduce the main idea of validated methods for IVPs for ODEs. We recall the Müller's theorems in section 3. In section 4, we introduce a methodology for building guaranteed interval observers for uncertain systems. Finally, two examples are given in section 5 to illustrate the application of our approach.

2. INTERVAL METHODS FOR IVP IN ODE

Interval analysis was initially developed to account for the quantization errors introduced by the floating point representation of real numbers with computers and was extended to validated numerics [Jaulin et al., 2001]. A real interval $[a] = [\underline{a},\ \bar{a}]$ is a connected and closed subset of $\mathbb{R}$. The set of all real intervals of $\mathbb{R}$ is denoted by $\mathbb{IR}$. Real arithmetic operations are extended to intervals. Consider an operator $\circ \in \{+, -, *, \div\}$ and $[a]$ and $[b]$ two intervals. Then: $[a] \circ [b] = \{u \circ v \mid u \in [a], v \in [b]\}$.
Consider $\boldsymbol{\psi} : \mathbb{R}^n \longmapsto \mathbb{R}^m$; the range of this function over an interval vector $[\mathbf{a}]$ is given by:

$$\boldsymbol{\psi}([\mathbf{a}]) = \{\boldsymbol{\psi}(\mathbf{u}) \mid \mathbf{u} \in [\mathbf{a}]\} \tag{9}$$

The interval function $[\boldsymbol{\psi}] : \mathbb{IR}^n \longmapsto \mathbb{IR}^m$ is an inclusion function for $\boldsymbol{\psi}$ if $\forall [\mathbf{a}] \in \mathbb{IR}^n$, $\boldsymbol{\psi}([\mathbf{a}]) \subseteq [\boldsymbol{\psi}]([\mathbf{a}])$. An inclusion function of $\boldsymbol{\psi}$ can be obtained by replacing each occurrence of a real variable by the corresponding interval and each standard function by its interval counterpart: the resulting function is called the natural inclusion function. The performance of this inclusion function depends on the formal expression for $\boldsymbol{\psi}$.
Consider now the following differential equation:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, t),\ \mathbf{x}(t_0) \in \mathbb{X}_0 \subset \mathbb{D}, \tag{10}$$

with $t_0 \geq 0$. The function $\mathbf{f}$ is assumed to be at least $k-$times continuously differentiable in a domain $\mathbb{D} \subseteq \mathbb{R}^n$. The objective is to compute interval vectors $[\mathbf{x}_j], j = 1, \ldots, n$, that are guaranteed to contain the solution of (10) at $t_1, t_2, \ldots, t_{n_T}$. Effective methods for solving such a problem are based on Taylor expansions, see [Nedialkov, 1999] and the references therein. These methods are usually one-step methods which proceed with two phases: (i) they first verify existence and uniqueness of the solution using the fixed point theorem and the Picard-Lindelöf operator [Nedialkov, 1999], compute an a priori enclosure $[\tilde{\mathbf{x}}_j]$ such that $\mathbf{x}(t) \in [\tilde{\mathbf{x}}_j]$ for all $t \in [t_j, t_{j+1}]$ and adapt integration step size if necessary; (ii) then they compute a tighter enclosure $[\mathbf{x}_{j+1}]$ of the solution of (10) at t_{j+1} as

$$[\mathbf{x}_{j+1}] = [\mathbf{x}_j] + \sum_{i=1}^{k-1} h_j^i \mathbf{f}^{[i]}(t_j, [\mathbf{x}_j]) + h_j^k \mathbf{f}^{[k]}([t_j, t_{j+1}], [\tilde{\mathbf{x}}_j]) \tag{11}$$

which corresponds to a Taylor expansion of order k where $[\tilde{\mathbf{x}}_j]$ is used to compute the remainder term. The coefficients $\mathbf{f}^{[i]}$ are the Taylor coefficients of the solution $\mathbf{x}(t)$ which can be computed either numerically by automatic differentiation or analytically via formal methods. The enclosures thus obtained are said *validated* which is in contrast with conventional numerical integration techniques which derive approximations with unknown global error and where the accumulation of both truncation and round-off errors may cause the computed solution to deviate widely from the real one. However, the *wrapping* effect – i.e., the overestimation of the solution due to the bracketing of a set of any shape by a box – makes the explicit scheme (11) width-increasing and thus not suitable for numerical implementation. Therefore, one must use mean value forms, matrices preconditioning and linear transform [Nedialkov, 1999] to yield practical results. Finally, there are open source softwares available which implement most of the above techniques. In this paper, all computations are performed with the VNODE software [Nedialkov, 1999].

3. THE MÜLLER'S THEOREM

In this section, we introduce an approach for bracketing an uncertain dynamical systems when both the initial state and parameter vectors are defined by boxes. The main idea consists in building a lower and an upper dynamical system which involve no uncertainty and enclose in a guaranteed way all the dynamics generated by the original uncertain system. This approach relies on comparison theorems for differential inequalities [Smith, 1995], and in particular the work of Müller [Walter, 1997].

3.1 Müller theorem

Theorem 4. [Walter, 1997, Kieffer et al., 2006] Consider the dynamical system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, \mathbf{p}, \mathbf{u}(t)), \tag{12}$$

where function $\mathbf{f}$ is continuous over a domain $\mathbb{T}$ defined by

$$\mathbb{T} : \begin{cases} \omega(t) \leq \mathbf{x}(t) \leq \Omega(t) \\ \underline{\mathbf{p}} \leq \mathbf{p} \leq \overline{\mathbf{p}} \\ t_0 \leq t \leq t_{n_T} \end{cases} \tag{13}$$

Functions $\omega_i(t)$ and $\Omega_i(t)$ are continuous over $[t_0, t_{n_T}]$ for all i and satisfy the following properties

(1) $\omega(t_0) = \underline{\mathbf{x}}_0$ and $\Omega(t_0) = \overline{\mathbf{x}}_0$
(2) the left derivatives $D^-\omega_i(t)$ and $D^-\Omega_i(t)$ and the right derivatives $D^+\omega_i(t)$ and $D^+\Omega_i(t)$ of $\omega_i(t)$ and $\Omega_i(t)$ are such that

$$\forall i,\ D^{\pm}\omega_i(t) \leq \min_{\underline{\mathbb{T}}(t)} f_i(\mathbf{x}, \mathbf{p}, t) \tag{14}$$

$$\forall i,\ D^{\pm}\Omega_i(t) \geq \max_{\overline{\mathbb{T}}(t)} f_i(\mathbf{x}, \mathbf{p}, t) \tag{15}$$

where $\underline{\mathbb{T}}(t)$ is the subset of $\mathbb{T}(t)$ defined by

$$\underline{\mathbb{T}}_i : \begin{cases} x_i = \omega_i(t) \\ \omega_j(t) \leq x_j \leq \Omega_j(t),\ j \neq i \\ \underline{\mathbf{p}} \leq \mathbf{p} \leq \overline{\mathbf{p}} \end{cases} \tag{16}$$

and where $\overline{\mathbb{T}}(t)$ is the subset of $\mathbb{T}(t)$ defined by

$$\overline{\mathbb{T}}_i : \begin{cases} x_i = \Omega_i(t) \\ \omega_j(t) \leq x_j \leq \Omega_j(t),\ j \neq i \\ \underline{\mathbf{p}} \leq \mathbf{p} \leq \overline{\mathbf{p}} \end{cases} \tag{17}$$

Then for all $\mathbf{x}_0 \in [\underline{\mathbf{x}}_0, \overline{\mathbf{x}}_0]$, $\mathbf{p} \in [\underline{\mathbf{p}}, \overline{\mathbf{p}}]$, system (1) admits a solution $\mathbf{x}(t)$ that stays in the domain

$$\mathbb{X} : \begin{cases} t_0 \leq t \leq t_{n_T} \\ \omega(t) \leq \mathbf{x}(t) \leq \Omega(t) \end{cases} \tag{18}$$

and takes the value $\mathbf{x}_0$ at t_0. If, in addition, for all $\mathbf{p} \in [\underline{\mathbf{p}}_0, \overline{\mathbf{p}}_0]$, function $\mathbf{f}(\mathbf{x}, \mathbf{p}, t)$ is Lipschitzian with respect to $\mathbf{x}$ over $\mathbb{D}$ then this solution is unique for any given $\mathbf{p}$.

Finally, an enclosure for the solution of (12) is given by

$$\forall t \in [t_0, t_{n_T}], \quad [\mathbf{x}](t) = [\omega(t), \Omega(t)] \tag{19}$$

The main difficulty is to obtain suitable bracketing functions $\omega(t)$ and $\Omega(t)$ in the general case. However, when the components of $\mathbf{f}$ are monotonic with respect to each parameter or each state vector component, it is quite easy to define these systems [Kieffer et al., 2006], while avoiding possible divergence that may occur when both upper and lower components of the parameter/state vector appear simultaneously in the same expression of the components of the bracketing systems [Ramdani et al., 2006].

Rule 1 - Use of monotonicity property [Kieffer et al., 2006] In order to build the upper system, i.e. the one which yields the maximal solution $\Omega(t)$, one can replace in the formal expression of f_i, x_i by Ω_i, x_j $(j \neq i)$ by Ω_j if $\frac{\partial f_i}{\partial x_j} \geq 0$ or by ω_j if $\frac{\partial f_i}{\partial x_j} \leq 0$ and p_r by $\overline{p}_r$ if $\frac{\partial f_i}{\partial p_r} \geq 0$ or by $\underline{p}_r$ if $\frac{\partial f_i}{\partial p_r} \leq 0$. The components of the lower system, i.e. the one which yields the minimal solution $\omega(t)$ are derived by reversing monotonicity conditions.

Obviously $\omega(t)$ and $\Omega(t)$ are in general, solutions of a system of coupled differential equations, i.e.

$$\begin{cases} \dot{\omega}(t) = \underline{\mathbf{f}}(\omega, \Omega, \underline{\mathbf{p}}, \overline{\mathbf{p}}, t) \\ \dot{\Omega}(t) = \overline{\mathbf{f}}(\omega, \Omega, \underline{\mathbf{p}}, \overline{\mathbf{p}}, t) \\ \omega(t_0) = \underline{\mathbf{x}}_0 \wedge \Omega(t_0) = \overline{\mathbf{x}}_0 \end{cases} \tag{20}$$

which involves no *uncertain* quantity. Therefore interval Taylor models such as the one introduced in the previous section can be used for efficiently solving (20). Indeed when these methods are used for solving differential equations with no uncertainty, they are usually able to curb the pessimism induced by the wrapping effect, even over long integration time.

Remark : Althoug interval Taylor models can be used for solving in an efficient way the system (20), there is no guaranty that the size of the enclosure $[\omega(t), \Omega(t)]$ will not diverge. However, when the system under study is cooperative, a property to be defined later, the size of this enclosure seldom diverges.

3.2 Case of cooperative systems

Cooperative systems are systems whose state variables act positively the ones on the others. In fact, the *cooperativity* property is satisfied by a large number of real systems. Most diffusion-reaction systems are cooperative, most models written in biology, chemistry or economics are cooperative or can be rewritten in order to satisfy this property [Smith, 1995].
Definition: The dynamical system (12) is *cooperative* over $\mathbb{D}$, if all the off-diagonal terms of its Jacobian matrix are non negative over $\mathbb{D}$, i.e.

$$\forall i \neq j,\ t \geq 0,\ \mathbf{x} \in \mathbb{D},\ \frac{\partial f_i(\mathbf{x},t)}{\partial x_j} \geq 0. \tag{21}$$

An interesting property of cooperative linear systems is as follows

Property 1. The linear non-autonomous system,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}(t);\quad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{22}$$

where $\mathbf{A}$ is a cooperative matrix and $\mathbf{B}(t) \geq 0\ \forall t \geq 0$ has the following property: If $\mathbf{x}_0 \geq 0$ then $\mathbf{x}(t) \geq 0$ for $t > t_0$.

When function $\mathbf{f}$ in (12) is cooperative, functions $\omega(t)$ and $\Omega(t)$ appear no longuer coupled in (20). Hence the upper system and the lower one can be computed and analyzed independantly.

4. GUARANTEED INTERVAL OBSERVER

After having introduced in the two preceding sections the main tools and methods used for enclosing the state flows generated by an uncertain system, we introduce in this section a methodology for building guaranteed interval observers. The main idea for designing an interval observer for the uncertain system (1) is to build two point observers which estimate in real time the lower and the upper bound of the state vector. The dynamics of the lower and the upper observer are interconnected according to the following set of differential equations :

$$\begin{cases} \dot{\Omega}(t) = (\mathbf{A}-\mathbf{KC})\Omega(t) + \overline{\boldsymbol{\psi}}(\omega,\Omega,\underline{\mathbf{p}},\overline{\mathbf{p}},\mathbf{u}(t)) + \mathbf{K}\overline{\mathbf{y}}_m(t) \\ \dot{\omega}(t) = (\mathbf{A}-\mathbf{KC})\omega(t) + \underline{\boldsymbol{\psi}}(\omega,\Omega,\underline{\mathbf{p}},\overline{\mathbf{p}},\mathbf{u}(t)) + \mathbf{K}\underline{\mathbf{y}}_m(t) \end{cases} \tag{23}$$

In order to obtain validated numerical computations of the interval obserer, we will use Interval methods for IVPs for ODEs, to solve the set of differential equations (23). We will now show that the observation error remains positive and under which conditions on the gain matrix $\mathbf{K}$ this error converges.

4.1 The positivity of the observation error

Thanks to rule 1, it is possible to derive the upper and lower functions $\underline{\boldsymbol{\psi}}$ and $\overline{\boldsymbol{\psi}}$ which satisfy the following double inequalities

$$\begin{gathered} \forall t \geq t_0,\ \forall \mathbf{x}(t_0) \in [\mathbf{x}_0],\ \forall \mathbf{p} \in [\mathbf{p}] \\ \underline{\boldsymbol{\psi}}(\omega,\Omega,\underline{\mathbf{p}},\overline{\mathbf{p}},\mathbf{u}(t)) \leq \boldsymbol{\psi}(\mathbf{x},\mathbf{p},\mathbf{u}(t)) \leq \overline{\boldsymbol{\psi}}(\omega,\Omega,\underline{\mathbf{p}},\overline{\mathbf{p}},\mathbf{u}(t)) \end{gathered} \tag{24}$$

From (1) and (23) one can write the dynamics of the upper observation error $\overline{\mathbf{e}}(t) = (\Omega(t) - \mathbf{x}(t))$ as follows

$$\dot{\overline{\mathbf{e}}}(t) = (\mathbf{A}-\mathbf{KC})\overline{\mathbf{e}}(t) + \overline{\phi}(\mathbf{z}) \tag{25}$$

were $\overline{\phi}(\mathbf{z}) = \overline{\boldsymbol{\psi}}(\omega,\Omega,\underline{\mathbf{p}},\overline{\mathbf{p}},\mathbf{u}(t)) - \boldsymbol{\psi}(\mathbf{x},\mathbf{p},\mathbf{u}(t)) + \mathbf{Kb}$ and $\mathbf{z} = (\omega,\Omega,\underline{\mathbf{p}},\overline{\mathbf{p}},\mathbf{x},\mathbf{p},\mathbf{u},\mathbf{b})^T$. So according to the double inequality (24) and hypothesis 1 on the gain matrix $\mathbf{K}$, we can write

$$\forall t \geq t_0,\ \forall \mathbf{x}(t_0) \in [\mathbf{x}_0],\ \forall \mathbf{p} \in [\mathbf{p}],\ \overline{\phi}(\mathbf{z}) \geq 0. \tag{26}$$

In this context, if the observation gain matrix $\mathbf{K}$ ensures the cooperativity of the matrix $[\mathbf{A}-\mathbf{KC}]$ (hypothesis 2), then from *property 1* the upper observation error remains positive, i.e.

$$\text{if } \forall \mathbf{x}(t_0) \in [\mathbf{x}_0],\ \forall \mathbf{p} \in [\mathbf{p}],\ \overline{\mathbf{e}}(t_0) \geq 0 \Rightarrow \forall t \geq t_0,\ \overline{e}(t) \geq 0$$

In the same way, we can characterize the dynamics of the lower observation error $\underline{\mathbf{e}}(t) = (\mathbf{x}(t) - \omega(t))$ as follows

$$\dot{\underline{\mathbf{e}}}(t) = (\mathbf{A}-\mathbf{KC})\underline{\mathbf{e}}(t) + \underline{\phi}(\mathbf{z}) \tag{27}$$

and

$$\forall t \geq t_0,\ \forall \mathbf{x}(t_0) \in [\mathbf{x}_0],\ \forall \mathbf{p} \in [\mathbf{p}],\ \underline{\phi}(\mathbf{z}) \geq 0. \tag{28}$$

where $\underline{\phi}(\mathbf{z}) = \boldsymbol{\psi}(\omega,\Omega,\mathbf{p},\mathbf{u}(t)) - \underline{\boldsymbol{\psi}}(\omega,\Omega,\underline{\mathbf{p}},\overline{\mathbf{p}},\mathbf{u}(t)) + \mathbf{Kb}$, finally

$$\text{if } \forall \mathbf{x}(t_0) \in [\mathbf{x}_0],\ \forall \mathbf{p} \in [\mathbf{p}], \underline{\mathbf{e}}(t_0) \geq 0 \Rightarrow \forall t \geq t_0,\ \underline{e}(t) \geq 0$$

4.2 The convergence of the observation error

In the preceding subsection, we have shown that the observation error is positive. This ensures that all the state trajectories consistent with parameter and initial state vectors uncertainties are enclosed by the observer in a guaranteed way. It remains to consider the problem of the convergence of the observation error. To do so, we will analyze the differential equation which governes the dynamics of the error $\mathbf{e}(t) = \overline{\mathbf{e}}(t) + \underline{\mathbf{e}}(t)$. Note that this error remains always positive. We have

$$\dot{\mathbf{e}}(t) = (\mathbf{A}-\mathbf{KC})\mathbf{e}(t) + \boldsymbol{\Gamma}(\mathbf{z}), \tag{29}$$

where

$\boldsymbol{\Gamma}(\mathbf{z}) = \overline{\psi}(\omega,\Omega,\underline{\mathbf{p}},\overline{\mathbf{p}},\mathbf{u}(t)) - \underline{\psi}(\omega,\Omega,\underline{\mathbf{p}},\overline{\mathbf{p}},\mathbf{u}(t)) + 2\mathbf{Kb}$.

The material we will use in the sequel is very similar to the one used in [Rapaport and Gouzé, 2003] where readers can find further details. Consider the change of variables

$$\begin{aligned} \overline{\boldsymbol{\xi}} &= \mathbf{V}_{\{\lambda_i\}}\mathbf{P}^{-1}\boldsymbol{\Omega}, \\ \underline{\boldsymbol{\xi}} &= \mathbf{V}_{\{\lambda_i\}}\mathbf{P}^{-1}\boldsymbol{\omega}, \end{aligned} \tag{30}$$

where, $\mathbf{P}$ is the transformation matrix which makes it possible to obtain the companion observability form of the matrix $\mathbf{A}-\mathbf{KC}$ for any $\mathbf{K}$ and $\mathbf{V}$ is the Vandermonde matrix of eigenvalues of $\mathbf{A}-\mathbf{KC}$ which allows diagonalisation of the matrix $\mathbf{P}^{-1}(\mathbf{A}-\mathbf{KC})\mathbf{P}$

$$\mathbf{V}_{\{\lambda_i\}}\mathbf{P}^{-1}(\mathbf{A}-\mathbf{KC})\mathbf{P}\mathbf{V}^{-1}_{\{\lambda_i\}} = \boldsymbol{\Delta}_{\{\lambda_i\}} \tag{31}$$

where $\boldsymbol{\Delta}_{\{\lambda_i\}}$ is the diagonal matrix of eigenvalues λ_i. Moreover, consider the function

$$\boldsymbol{\gamma}(\mathbf{x},\mathbf{p},\mathbf{u}(t)) = \mathbf{P}^{-1}\boldsymbol{\psi}(\mathbf{x},\mathbf{p},\mathbf{u}(t)) \tag{32}$$

Then the observation error $\mathbf{s} = \overline{\boldsymbol{\xi}} - \underline{\boldsymbol{\xi}}$ in the new state base is

$$\dot{\mathbf{s}} = \boldsymbol{\Delta}_{\{\lambda_i\}}\mathbf{s} + \mathbf{V}_{\{\lambda_i\}}(\overline{\boldsymbol{\gamma}}(\omega,\Omega,.,.) - \underline{\boldsymbol{\gamma}}(\omega,\Omega,.,.) + 2\mathbf{P}^{-1}\mathbf{Kb})$$

Note that since function ψ is assumed Lipschitz with respect to vector state with constant L then $\boldsymbol{\gamma}$ is also Lipschitz with constant $\|\mathbf{P}^{-1}\|L$ (see [Rapaport and Gouzé, 2003]). Now, denote λ the smallest eigenvalue of $\mathbf{A}-\mathbf{KC}$ in absolute value. We obtain

$$\begin{aligned} \frac{d}{dt}\|\mathbf{s}\| \leq\ & -|\lambda|\|\mathbf{s}\| \\ & +(\sum_{i=1}^{n}\lambda_j^{i-1}(\overline{\gamma}_i(\omega,\Omega,.,.) - \underline{\gamma}_i(\omega,\Omega,,.,.))\sqrt{n} \\ & +2\|\mathbf{V}\|\|\mathbf{P}^{-1}\|\|\mathbf{Kb}\| \end{aligned}$$

hence

$$\begin{aligned} \frac{d}{dt}\|\mathbf{s}\| \leq\ & -|\lambda|\|\mathbf{s}\| \\ & +\|\mathbf{P}^{-1}\|(M + L\|\mathbf{P}\mathbf{V}^{-1}\|\|\mathbf{s}\|)(\sum_{i=1}^{m}|\lambda|^{i-1})\sqrt{n} \\ & +2\|\mathbf{V}\|\|\mathbf{P}^{-1}\|\|\mathbf{Kb}\| \end{aligned}$$

and finally

$$\frac{d}{dt}\|\mathbf{s}\| \leq \big(-|\lambda| + \mathrm{cond}(\mathbf{P})L\|\mathbf{V}^{-1}\|(\sum_{i=1}^{m}|\lambda|^{i-1})\sqrt{n}\big)\|\mathbf{s}\| + \|\mathbf{P}^{-1}\|\big(M(\sum_{i=1}^{m}|\lambda|^{i-1})\sqrt{n} + 2\|\mathbf{V}\|\|\mathbf{Kb}\|\big)$$

Consequently, an adequate choice of the matrix $\mathbf{K}$ will ensure the convergence of the observation error to a value which depends on the expression of the function ψ and the widths of uncertain variables (see [Rapaport and Gouzé, 2003]). Finally, we can also say that the observation error in the original base $\mathbf{e}(t)$ converges towards the ball:

$$\mathbb{B} = \big(0,\ \frac{D}{\big(|\lambda| - \mathrm{cond}(\mathbf{P})L\|\mathbf{V}^{-1}\|(\sum_{i=1}^{m}|\lambda|^{i-1})\sqrt{n}\big)}\big) \quad (33)$$

where

$$D = \mathrm{cond}(\mathbf{P})\big(\|\mathbf{V}^{-1}\|M(\sum_{i=1}^{m}|\lambda|^{i-1})\sqrt{n} + 2\mathrm{cond}(\mathbf{V})\|\mathbf{Kb}\|\big)$$

5. APPLICATION

To illustrate this approach of designing interval observers, we will present two examples taken from the biology domain. A bioreactor is a reactor in which microorganisms grow by consuming a substrate. The microorganisms and the substrate are assumed to be present at low concentrations in the reactor, so that a constant volume assumption is realistic. In the sequel we will denote by x_1 and x_2 the concentrations of microorganisms and substrate, respectively. The issue is to estimate the state variables from the measurement of some outputs.

5.1 Example 1

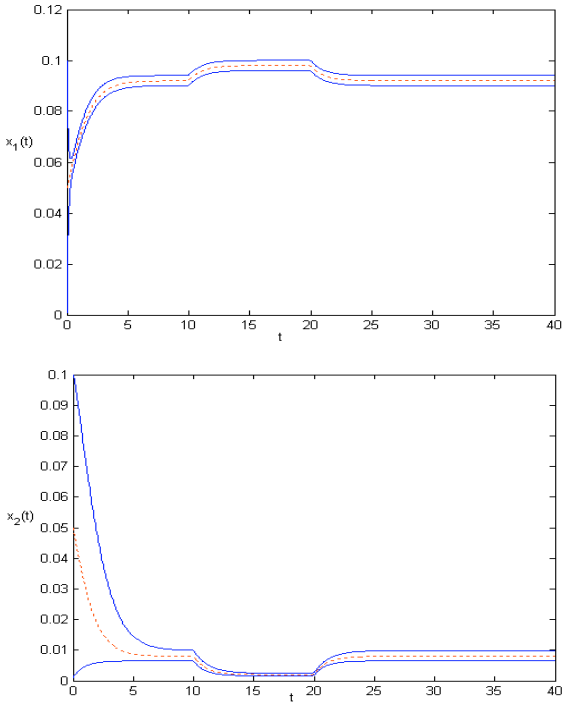
In this example we assume that the growth rate is given by the Contois model [Contois, 1959], we get the following standard equations for the bioreactor

$$\begin{cases} \dot{x}_1 = \dfrac{a_1 x_1 x_2}{a_2 x_1 + x_2} - u x_1 \\ \dot{x}_2 = -\dfrac{a_3 a_1 x_1 x_2}{a_2 x_1 + x_2} - u x_2 + u a_4 \end{cases} \quad (34)$$

where the partial unknown parameter vector is $\mathbf{p} = [a_1\ a_4]^T = [0.9, 1.1] \times [0.09, 0.11]$, the other values $a_2 = a_3 = 1$. The initial state of this system is uncertain $x_1 = [0.001, 0.1]$ and $x_2 = [0.001, 0.1]$. System input are taken as $u(t) = 0.08$ for $t \leq 10$, $u(t) = 0.02$ for $10 \leq t \leq 20$ and again $u(t) = 0.08$ for $t \geq 20$. Model output is taken as $y(t) = x_1(t)$ and the maximal measurement error is $b = \pm 2\% y_m(\infty)$. System (34) is not cooperative, therefore by using the methods introduced in sections 3 and 4, and a gain vector $\mathbf{K}^T = (k_1, k_2)$ one can derive the following guaranteed interval observer

$$\begin{cases} \dot{\Omega}_1 = -u\Omega_1 + \dfrac{\overline{a}_1 \Omega_1 \Omega_2}{a_2\Omega_1 + \Omega_2} + k_1(\overline{y}_m - \Omega_1) \\ \dot{\Omega}_2 = -u\Omega_2 - \dfrac{a_3 \underline{a}_1 \omega_1 \Omega_2}{a_2\omega_1 + \Omega_2} + u\overline{a}_4 + k_2(\overline{y}_m - \Omega_1) \\ \dot{\omega}_1 = -u\omega_1 + \dfrac{\underline{a}_1 \omega_1 \omega_2}{a_2\omega_1 + \omega_2} + k_1(\underline{y}_m - \omega_1) \\ \dot{\omega}_2 = -u\omega_2 - \dfrac{a_3 \overline{a}_1 \Omega_1 \omega_2}{a_2\Omega_1 + \omega_2} + u\underline{a}_4 + k_2(\underline{y}_m - \omega_1) \end{cases} \quad (35)$$

In this case for any $k_1 > 0$ and for $k_2 = 0$, we can ensure both the positivity and the convergence of the observation error. Figure 1 shows the enclosures of state



(a)

(b)

Fig. 1. (a) Estimated microorganisms concentration; (b) Estimated substrate concentration.

variables of (34) as obtained with the guaranteed interval observer (35). The blue curves on Fig. 1(a) and Fig. 1(b) show the lowers and uppers estimated bounds for the state components x_1 and x_2, respectively. In fact, for all uncertainties in (34) the interval observer estimates a box for the state vector with a width very close to the maximal measurement error and which contains in a guaranteed way the true value of the state vector.

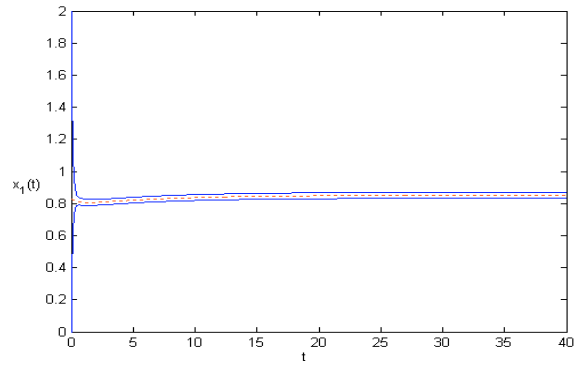
5.2 Example 2

Now, we consider the Monod model for the growth rate, so the bioreactor is defined by the following equation:

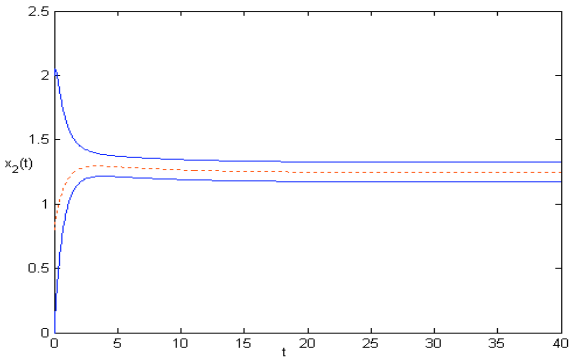
$$\begin{cases} \dot{x}_1 = \mu_0 \dfrac{x_2}{x_2 + k_s} x_1 - \alpha u x_1 \\ \dot{x}_2 = -k\mu_0 \dfrac{x_2}{x_2 + k_s} x_1 + u(s_{in} - x_2) \end{cases} \quad (36)$$

where u is the system input, which is taken constant $u = 0.36$, the parameters of this system are $k = 10.53$, $\mu_0 = [1.15, 1.25]$, $k_s = [7.05, 7.15] mmol/l$, $\alpha = 0.5$ and $s_{in} = 5.7$. The initial state is considered uncertain $\mathbf{x} = [0, 2] \times [0, 2]$. System output is $y(t) = x_1(t)$ and the maximum measurement error is $b = \pm 2\% y_m(\infty)$. In the same way as is example 1, this system is not cooperative, then by using the methods introduced in sections 3 and 4, we obtain,

$$\begin{cases} \dot{\Omega}_1 = \overline{\mu}_0 \dfrac{\Omega_2}{\Omega_2 + \underline{k}_s} \Omega_1 - \alpha u \Omega_1 + k_1(\overline{y}_m - \Omega_1) \\ \dot{\Omega}_2 = -k\underline{\mu}_0 \dfrac{\overline{\Omega}_2}{\Omega_2 + \overline{k}_s} \omega_1 + u(s_{in} - \Omega_2) + k_2(\overline{y}_m - \Omega_1) \\ \dot{\omega}_1 = \underline{\mu}_0 \dfrac{\omega_2}{\omega_2 + \overline{k}_s} \omega_1 - \alpha u \omega_1 + k_1(\underline{y}_m - \omega_1) \\ \dot{\omega}_2 = -k\overline{\mu}_0 \dfrac{\omega_2}{\omega_2 + \underline{k}_s} \Omega_1 + u(s_{in} - \omega_2) + k_2(\underline{y}_m - \omega_1) \end{cases} \quad (37)$$



(a)



(b)

Fig. 2. (a) Estimated microorganisms concentration; (b) Estimated substrate concentration.

As in example 1 any observation gain vector $\mathbf{K}$, such as $k_1 > 0$ and $k_2 = 0$ ensures both the positivity and the convergence of the observation error. The guaranteed observer makes it possible to characterize the whole state variables consistent with the measured data, the uncertainty bounds as well as the uncertainty bounds on the parameters. The blue curves in full line on the figures Fig. 2(a) and Fig. 2(b) show guaranteed enclosures for these estimated state components.

6. CONCLUSION

In this article we presented a set membership approach dedicated to the state observation problem of uncertain nonlinear continuous-time systems. The advantage of this approach is its robustness with respect to model parameter uncertainties and measurement errors. In a future work, we will use this type of interval observers with actual data.

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