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Towards the Guaranteed Control of Production Output: a Probabilistic Approach

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Abstract: The paper presents a problem statement of guaranteed control of production output by use of a probabilistic criterion that requires presence of the product output within specified limits with a given probability. Within the problem statement, the existence of the controls is considered in the form of sufficient mutual conditions which the initial problem characteristics are to meet to. Deriving the conditions is based on applying corresponding probabilistic inequalities enabling one to establish a connection between the limits for the production output, the amount of the mean square deviation of the production output, and the required probability of presence of the production output within the specified limits. The probabilistic conditions obtained are also shown to be applicable within a distribution free input/output system identification problem statement with a probabilistic criterion. *The paper has been supported by a grant of the Russian Foundation for Basic Researches (RFBR): project 12-08-01205-a.*

Keywords: distribution density, guaranteed control, probabilistic criterion, production output, system identification, unimodal distribution.

1. PROBLEM STATEMENT

Numerous of papers are devoted to modeling production output control and, correspondingly, to modeling output production, e.g. (Chen et al., 1998, Danziger, 2008, Elmaghraby, 2011, Frohling at al., ( ), Rentz, 2009, Fuping and Min, 2005, Geist et al., (2008), Humphreys et al., 2001, Jahanshahloo and Khodabakhshi, 2003, Jun, 2004, Kraines and Yoshikuni, 2004, Percoco, 2006, Proietti et al., 2007, Wiendahl and Springer, (1988), Zegordi and Nia, (2009), Zofo and Prieto, 2007). Within the present paper, a probabilistic approach to such a problem is considered within the following problem statement.

Let us consider a problem statement of guaranteed control of production output of a manufacturing process within which the output variable v is modeled by use of the conditional mathematical expectation

$$y_{\mathrm{UZ}} = \mathbf{E}\{v/\ \mathbf{U},\mathbf{Z}=\mathbf{Z}_0\} = h(u_1,\ldots,u_n). \tag{1}$$

In expression (1) $\mathbf{U}=(u_1,\ldots,u_n)^T$ represents a vector of control variables, $\mathbf{Z}=(z_1,\ \ldots,\ z_m)^T$ represents a vector of input controlled variables, and $\mathbf{E}\{./.\}$ stands for the conditional the mathematical expectation. The essence of the approach proposed is to apply a hypothesis concerned with the assumption that the probability distribution density of the product output v is to belong to a class $\mathbf{C_{Pr}}$ of admissible probability distribution densities. Meanwhile, the make-up of the class $\mathbf{C_{Pr}}$ may be a priory arbitrary, but formally the class may involve all reasonable probability distribution densities known in the probability theory. Regarding the present problem statement members of the class $\mathbf{C_{Pr}}$ are considered as conditional probability distribution densities

$$p_k(v,y_{\mathrm{UZ}},\sigma^2_{\mathrm{UZ}}),\ k=1,\ldots,N \tag{2}$$

(the number N is, of course, finite but not specifically limited). Parameters $y_{\mathrm{UZ}},\sigma^2_{\mathrm{UZ}}$ of the densities are determined at the each stage (time interval of forming the control) by use of observation data. In densities (2) σ^2_{UZ} is the conditional variance.

The performance index of the production output control is expressed in the form of meeting the following condition of a probabilistic form

$$\mathbf{P}\left\{v\in\left[\delta_*,\ \delta^*\right]\right\}\ge p_*, \tag{3}$$

In inequality (3), p_* stands for the desired probability value, $\left[\delta_*\ ,\ \delta^*\right]$ denotes the required production output interval within which the production output v is to be with a probability being not less than preset probability value p_*. Within the probabilistic problem statement, the probability distribution density of the output variable v belongs to the above introduced class $\mathbf{C_{Pr}}$ of admissible probability distribution densities.

For each one representative $p_k(v,y_{\mathrm{UZ}},\sigma^2_{\mathrm{UZ}}),\ k=1,\ldots,N$ of the class $\mathbf{C_{Pr}}$, the required domain of admissible controls meeting, finally, to probabilistic criterion (3) is determined as follows. For each member of the probability distribution density class $\mathbf{C_{Pr}}$ considered, solution of the algebraic equation with respect to y_{UZ}

$$\int_{\delta_*}^{\delta^*} p_k(v, y_{UZ}, \sigma_{UZ}^2)dv = p_*, k = 1,\ldots,N. \quad (4)$$

is obtained (within assumptions, that σ_{UZ}^2 is constant). Any unimodal probability distribution density (both symmetric and non-symmetric ones) of the class $\mathbf{C_{Pr}}$ provides a pair of roots $y_{1k} < y_{2k}$, $k = 1,\ldots,N$ of equation (4). Each of these pairs of the roots forms corresponding interval $[y_{1k}, y_{2k}]$, $k = 1,\ldots,N$ of admissible values of the conditional mathematical expectation y_{UZ}. Inside each of the intervals, for the corresponding k-th member of the class $\mathbf{C_{Pr}}$ probabilistic criterion (3) is met.

Again, the intersection of these intervals is built,

$$I = \bigcap_{k=1}^{N} [y_{1k}, y_{2k}] \quad (5)$$

Mapping intersection (5) by use of regression (1) into the space of controls forms in this space a domain of guaranteed controls S_{GC}. In other words,

$$I = \left[y_1^{\max}, y_2^{\min} \right]. \quad (6)$$

In (6)

$$y_1^{\max} = \max_{k=1,\ldots,N} \{y_{2k}\}, \quad (7)$$

$$y_2^{\min} = \min_{i=1,\ldots,N} \{y_{2k}\}. \quad (8)$$

Within such a problem statement, the guaranteed controls domain S_{GC} existence is equivalent to the intersection I existence in formula (5). Meanwhile, intersection (5) (defined by (6)-(8)) may be empty. Indeed, let $\Phi_k(y_{UZ})$, $k = 1,\ldots,N$ be a set of functions in the variable y_{UZ} for particular condition probability distribution densities $p_k(v, y_{UZ}, \sigma_{UZ}^2)$ $k = 1,\ldots,N$ under a fixed σ_{UZ}^2:

$$\Phi_k(y_{UZ}) = \int_{\delta_*}^{\delta^*} p_k(v, y_{UZ}, \sigma_{UZ}^2)dv, k = 1,\ldots,N. \quad (9)$$

Figure 1 displays plots of the functions $\Phi_k(y_{UZ})$ in formula (9), obtained for the Gaussian (the black line) and the log-normal (the cyan line) probability distribution densities under the following data: $\delta_* = 10^{-3}$, $\delta^* = 0.5$, $\sigma_{UZ}^2 = 0.01$. One can easily be seen that for, say, the probability level p_*=0.980, the intersection of the corresponding (two) intervals I in formula (5) exists (being non-empty). For the probability level p_*=0.985, the same intersection I in formula (5) is not exist (empty) under non-emptiness of each of the two intervals $[y_{1k}, y_{2k}]$, $k = 1, 2$, and a solution of the problem of the existence of guaranteed controls does not exist. For the probability level p_*=0.990 one of two intervals $[y_{1k}, y_{2k}]$, $k = 1, 2$, is empty, while for the probability level p^*=0.995 both the intervals $[y_{1k}, y_{2k}]$, $k = 1, 2$ are empty.

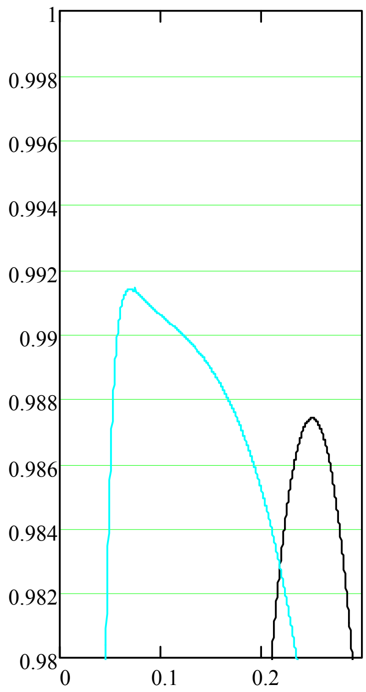
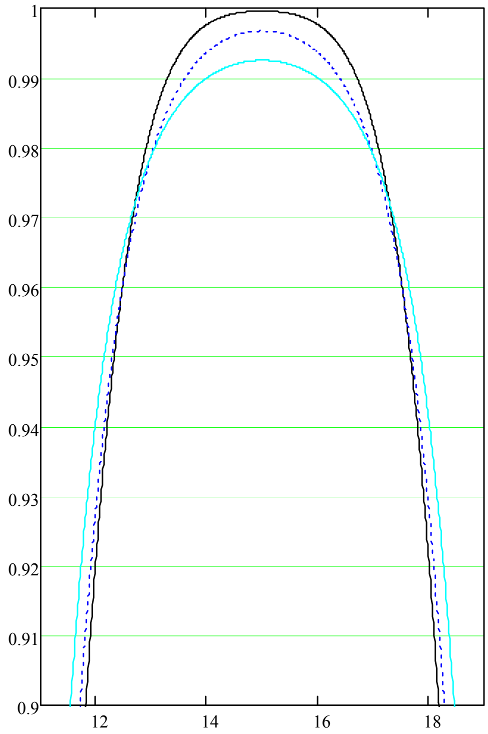


Figure 1: Towards the possible non-existence of the intersection I in formula (5) for the Gaussian (the black line) and log-normal (the cyan line) distributions calculated for $\delta_* = 10^{-3}$, $\delta^* = 0.5$, $\sigma_{UZ}^2 = 0.01$.

Also, let in (5) the intersection I exist. As well as above, Figures 2 and 3 present plots of the functions $\Phi_k(y_{UZ})$, $k = 1, 2, 3$ calculated for the Gaussian (the black line), logistic (the blue dotted line), and Student (the cyan line) distributions for the following data: $\delta_* = 10$, $\delta^* = 20$. In these figures, $\sigma_{UZ}^2 = 2$ for Figure 2, and $\sigma_{UZ}^2 = 5$ for Figure 3. Meanwhile, Figure 2b presents a refined scale representation of changing the density choice in the "neighborhood" of the level of $p_* = 0.980$. These figures evidently demonstrate that for a constant variance the choice of suitable probability distribution densities considerably depends on the required level of the probability p_* in (3), and it is by no means universal or trivial. Here, as the admissible probability distribution densities, those ones are assumed which provide the maximal amount for the root y_{1k} in accordance to expression (7) and the minimal amount of the root y_{2k} in accordance to expression (8).

All these and many others examples evidently demonstrates that the problem of investigation of conditions of existence of intersection (5) is vital and in the present paper it is solved by applying corresponding probabilistic inequalities, e.g. (Lin and Bai, 2011).



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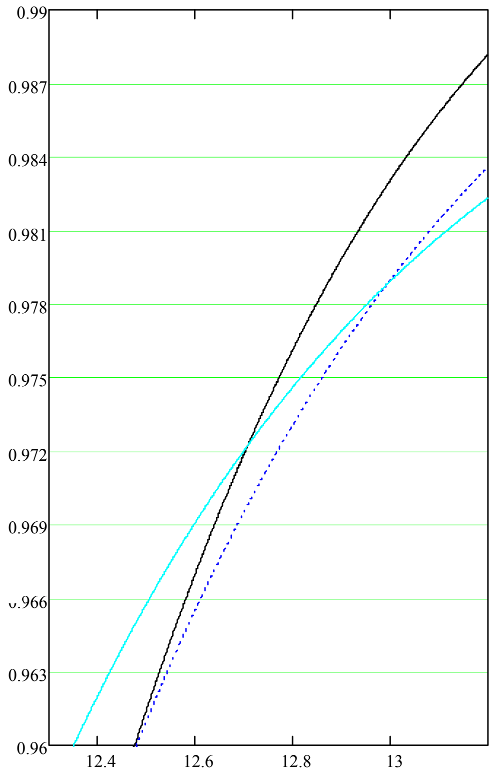

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Figure 2: Forming intersection (5) for the Gaussian (the black line), logistic (the blue dotted line) and Student (the cyan line) distributions for $\delta_* = 10$, $\delta^* = 20$, $\sigma^2_{\mathrm{UZ}} = 2$.

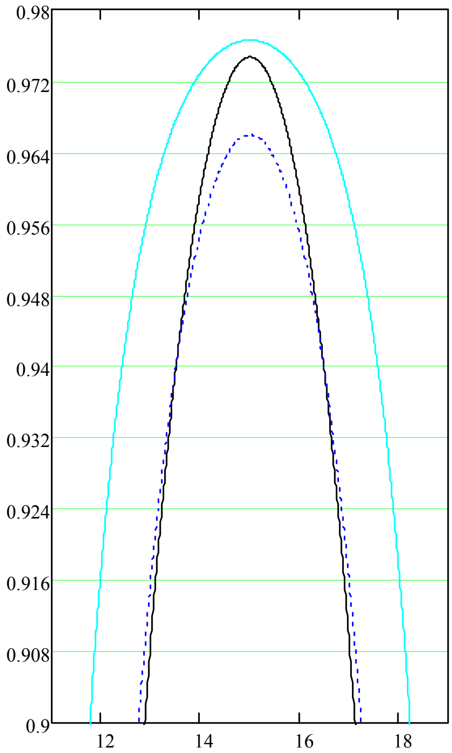

Figure 3: Forming intersection (5) for the Gaussian (the black line), logistic (the blue dotted line) and Student (the cyan line) distributions at $\delta_* = 10$, $\delta^* = 20$, $\sigma^2_{\mathrm{UZ}} = 5$.

2. SUFFICIENT CONDITIONS FOR GENERAL TYPE DENSITIES

In the present Section, sufficient conditions that provides the existence of the intersection I in formula (5) for probability distribution densities of a general type are derived. Within the problem consideration, one may initially note that the length of the intervals $[y_{1k}, y_{2k}]$ in formula (5) and, as a consequence, the existence of their intersection depend on values of two “parameters” of the general problem statement. In other words, these parameters are the desired probability p_* and the variance σ^2_{UZ}. As the above examples evidently demonstrate, increasing the probability p_* and/or increasing the variance σ^2_{UZ} give rise to the non-existence of the intersection I in formula (5). Then, one may naturally assume that there should exist a certain relationship (dependence) between the values of amounts of p_* and σ^2_{UZ}, meanwhile maintaining such a relationship at an appropriate level is to guarantee the existence of the intersection I in formula (5).

First of all, let us consider a particular case when the class $\mathbf{C}_{\mathbf{Pr}}$ of probability distribution densities is formed by all available symmetric unimodal ones. Within the case, applying the Tchebychev inequality by virtue of performance index (3) gives rise to the following relationship

$$1-p_* \le \frac{\sigma_{\mathrm{UZ}}^2}{\left(\left(\delta^* - \delta_*\right)/2\right)^2}. \tag{10}$$

Meanwhile, the present paper problem statement motivates to consider just achieving the equality formula (10) as the main subject of interest. Thus the value

$$\Sigma_{\mathrm{UZ}}^2 = \left(\left(\delta^* - \delta_*\right)/2\right)^2 (1-p_*) \tag{11}$$

defines the maximally admissible value of the variance that may guarantee the existence of the intervals $[y_{1k}, y_{2k}]$ in formula (5) for the desired probability p_* and the desired margins of the interval $\left[\delta_*, \delta^*\right]$. For any variance $\sigma_{\mathrm{UZ}}^2 < \Sigma_{\mathrm{UZ}}^2$ (equality (11)) the existence of the intersection I in formula (5) is guaranteed. Meanwhile the existence is valid not for symmetric probability distribution densities only, but for non-symmetric ones as well. Figure 4 displays a corresponding example involving the Gumbel distribution density (Johnson et al., 1995) $p(v) = \frac{1}{\beta}\exp\left(\frac{v-\alpha}{\beta} - \exp\left(\frac{v-\alpha}{\beta}\right)\right)$, where α and β are the distribution parameters.

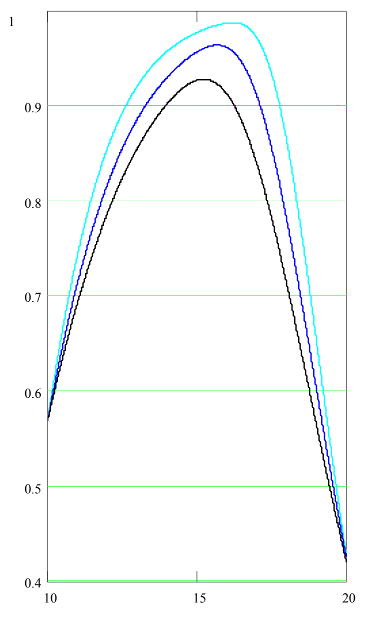

Figure 4: "Embedding" of the intervals $[y_{1k}, y_{2k}]$ in formula (4) under decreasing the variance σ_{UZ}^2 for the Gumbel distribution (at $\delta_* = 10$, $\delta^* = 20$, $\sigma_{\mathrm{UZ}}^2 = 4$. (the cyan line), $\sigma_{\mathrm{UZ}}^2 = 6.25$ (the dotted blue line), $\sigma_{\mathrm{UZ}}^2 = 9$ (the black line)).

Above, general type probability distribution densities (not necessary unimodal ones) were considered. The consideration below is concerned with symmetric and non-symmetric *unimodal* probability distribution densities involved into the class $\mathbf{C}_{\mathbf{Pr}}$. For the present case, by virtue of the Tchebychev inequality the following inequality may be written

$$1-p_* \le \frac{\mathbf{E}(v-\xi)^2}{\left(\min\left\{\xi - \delta_*, \delta^* - \xi\right\}\right)^2}. \tag{12}$$

The inequality is formally even more strict the one presented by formula (10)). In inequality (12), $\mathbf{E}(.)$ is the symbol of the the mathematical expectation, and ξ is an arbitrary point belonging to the interval $\left[\delta_*, \delta^*\right]$. In turn, in complete analogy to the reasoning presented above, one may affirm that reaching the marginal case (equality) in formula (12) is the basic subject of the investigation. Meanwhile, minimization of the function that is formed by the right-hand side of relationship (12) is just to help to reach the equality. In turn, the function $\phi(\xi) = \left(\min\left\{\xi - \delta_*, \delta^* - \xi\right\}\right)^2$ properties and the relationship $\mathbf{E}(v-\xi)^2 = \sigma_{\mathrm{UZ}}^2 + (\mathbf{E}(v) - \xi)^2$, imply the following inequality

$$\frac{\mathbf{E}(v-\xi)^2}{\left(\min\left\{\xi - \delta_*, \delta^* - \xi\right\}\right)^2} \ge \frac{\sigma_{\mathrm{UZ}}^2}{\left(\left(\delta^* - \delta_*\right)/2\right)^2}. \tag{13}$$

Meanwhile, the equality in non-strict inequality (13) is reached under the conditions

$$\xi = \left(\delta_* + \delta^*\right)/2 \text{ and } \mathbf{E}(v) = \xi, \tag{14}$$

and conditions (14) are necessary and sufficient ones.

As a direct consequence from the presented considerations, the relationships

$$\sigma_{\mathrm{UZ}}^2 = \frac{1}{4}\left(\delta^* - \delta_*\right)^2 (1-p_*), \tag{15}$$

determine the sufficient conditions to provide the existence of the intersection I in formula (5) for arbitrary unimodal probability distribution density as a member of the class $\mathbf{C}_{\mathbf{Pr}}$. (The sufficiency is guaranteed by condition (15), since meeting it ensures that *all* intervals $[y_{1k}, y_{2k}]$ will involve the center of the interval $\left[\delta_*, \delta^*\right]$, the point $\left(\delta_* + \delta^*\right)/2$).

Also, if condition (15) is met, the center point of the interval $\left[\delta_*, \delta^*\right]$ always belongs each interval $[y_{1k}, y_{2k}]$. Thus, the condition of the unimodality may be disregarded. Meanwhile, it is natural that for a multimodal probability distribution density two roots of equation (4) are to be chose only, namely those ones being nearest to the point $\left(\delta_* + \delta^*\right)/2$, one of which located to the left, and the second one, to the right of this point.

The example presented of the preceding Section (Figure 1) evidently demonstrates that a violation of condition (15) gives rise to the case, when the center the interval $\left[\delta_*, \delta^*\right]$ may not belong to all intervals $[y_{1k}, y_{2k}]$ in formula (5). Under the data of this example, condition (15) implies the following (upper) bound for the variance: $\sigma_{\mathrm{UZ}}^2 \le \left(\left(\delta^* - \delta_*\right)^2 (1-p_*)\right)/4 = 0.00934$ under p_*=0.985, and

$\sigma_{\mathrm{UZ}}^2 \le \left(\left(\delta^* - \delta_*\right)^2 (1-p_*)\right)/4 = 0.000311$ under p_*=0.995, what contradicts to the considered amount of the mean square deviation $\sigma_{\mathrm{UZ}}^2 = 0.01$.

Thus, condition (15) derived by no means limits (even taking into account the unimodality condition) forming the class of probability distribution densities which may be involved in deriving the required intersection in formula (5) to form the domain of admissible guaranteed controls, since a numerical solution of equations (4) is not hard for any given specific probability distribution density $p_k(v, y_{\mathrm{UZ}}, \sigma_{\mathrm{UZ}}^2),\ k = 1,\ldots,N$. (One may be noted within this context that an analytical solution of equations (4) is of course not mandatory).

3. SUFFICIENT CONDITIONS FOR UNIMODAL DISTRIBUTIONS

Condition (15) obtained in the preceding Section is universal within the present problem statement, but its applicability to all (general) types of probability distribution densities (multimodal ones) may be exhaustive, if the class $\mathbf{C}_{\mathbf{Pr}}$ is restricted by involvement of unimodal densities only. For unimodal distributions, condition (15) may be specified using the above presented general approach under applying some generalizations relating to probabilistic inequalities (Lin and Bai, 2011). Specifically, one may apply the Gauss inequality (in its generalized form):

$$\mathbf{P}\left\{\left|\zeta - x_0\right| \ge \gamma\rho\right\} \le \frac{4}{9\gamma^2} \quad \forall\ \tau > 0 . \tag{16}$$

In (16), $\rho^2 = \mathbf{E}(\zeta - x_0)^2$, x_0 represents any arbitrary real number, and ζ stands for a unimodally distributed (arbitrary) random value. Worthwhile to note that in the conventional Gauss inequality the real above number x_0 is assumed to be the mode of the probability distribution of the random value ζ. Inequality (16) was in the fullness of time derived by Soviet mathematicians D.F. Vysochanskij and Yu.I. Petunin and referred with their names.

Again, similarly to expression (12), η is considered as an arbitrary point belonging to the interval $\left[\delta_*, \delta^*\right]$. Setting in inequality (16) $\zeta \overset{\Delta}{=} v,\ x_0 \overset{\Delta}{=} \eta$, and $\gamma = \dfrac{\min\left\{\eta - \delta_*, \delta^* - \eta\right\}}{\sqrt{\mathbf{E}(v-\eta)^2}}$, enables one to obtain directly from (16) the following inequality

$$1 - p_* \le \frac{4}{9} \frac{\mathbf{E}(v-\eta)^2}{\left(\min\left\{\eta - \delta_*, \delta^* - \eta\right\}\right)^2}. \tag{17}$$

Meanwhile, analogously to the considerations of the preceding Section, one may affirm that reaching the marginal case (equality) in formula (17) is the basic subject of the investigation. Namely, to reach the equality in non-strict inequality (17), since just this condition is the main point of interest, it is enough to minimize the function (in η)

$$\varphi(\eta) = \frac{\mathbf{E}(v-\eta)^2}{\left(\min\left\{\eta - \delta_*, \delta^* - \eta\right\}\right)^2}. \tag{18}$$

Again, following to the considerations of the preceding Section, the minimum of function (in η) (18) is easily seen to be reached under the conditions $\eta = \left(\delta_* + \delta^*\right)/2$ and $\mathbf{E}(v) = \eta$. Meanwhile, these conditions are necessary and sufficient ones.

As a direct consequence, relationship (17) and the above considerations give rise to the following sufficient condition for the maximally admissible value of the variance σ_{UZ}^2 in the event of unimodal distributions:

$$\sigma_{\mathrm{UZ}}^2 = \frac{9}{16}\left(\delta^* - \delta_*\right)^2 (1 - p_*). \tag{19}$$

One again, appeal to the numerical conditions displayed in Figure 1 shows that in the event of mismatching condition (19), the center of the interval $\left[\delta_*, \delta^*\right]$ will not mandatory be available in all intervals $[y_{1k}, y_{2k}],\ k = 1,\ldots,N$ in formula (5). Condition (19) implies the following upper bound for the variance: $\sigma_{\mathrm{UZ}}^2 \le 0.002101$ under p_*=0.985, and $\sigma_{\mathrm{UZ}}^2 \le 0.0007$ under p_*=0.995, what also contradicts to the considered amount $\sigma_{\mathrm{UZ}}^2 = 0.01$.

4. SYSTEM IDENTIFICATION WITH A PROBABILISTIC CRITERION: APPLICATION OF THE RESULTS

In the section, probabilistic conditions (15) and (19) are shown to be applicable within a distribution free input/output system identification problem statement with a probabilistic criterion.

For a single output discrete time ($t = 1, 2, \ldots$) system process $y(t)$, given a model structure $\hat{y}(t;\theta)$, the identification error $e(t;\theta) = y(t) - \hat{y}(t;\theta)$ may be considered subject to the following

$$\mathbf{P}\{|e(t;\theta)| \le \varepsilon\} \ge p_* \quad \forall t \text{ for a certain required } \varepsilon > 0 . \qquad (20)$$

Condition (20) is, thus, a probabilistic identification criterion with respect to θ standing for the parameter vector subject to identification by virtue of observation of input and output processes of the system. Within the problem statement, all random processes are assumed to be stationary and joint stationary in the strict sense. Obviously, criterion (20) is completely equivalent to performance index (3) with $\delta_* = -\varepsilon$ and $\delta^* = \varepsilon$. Thus, conditions (15) or (19) may be directly applied to obtain the parameter vector θ as one providing the corresponding value of the variance of the identification error $\mathbf{var}(e(t;\theta))$ by use of observation of the input and output processes of the system subject to identification. Namely, $\mathbf{var}(e(t;\theta)) \le \varepsilon^2(1-p_*)$ (for the hypothesis of general type probabilistic distribution of the identification error $e(t;\theta)$) and $\mathbf{var}(e(t;\theta)) \le (9/4)\varepsilon^2(1-p_*)$ (for the hypothesis of a unimodal probabilistic distribution of the identification error $e(t;\theta)$).

5. CONCLUSIONS

Conditions (15) and (19) obtained provide, thus, a possibility of maximal restriction of selection of admissible probability distribution densities within a given class $\mathbf{C_{Pr}}$. Specifically, one may select not more than two (in the case of applying non-symmetric distributions) or one (in the case of applying symmetric distributions only) admissible probability distribution densities of the production output and corresponding to these (this) representatives (representative) domain of admissible guaranteed controls. Thus, the class $\mathbf{C_{Pr}}$ may be preliminary selected as broad as required by use of, say, any probability distribution density from, for instance, the books of Johnson et al. (1994, 1995).

Besides that, conditions (15) and (19) may evidently be rewritten in the forms, determining the appropriate maximal/minimal admissible values $p_*^{\max}$ and $d_{\min}$ of the probability p_* and length $d = \delta^* - \delta_*$ of the interval $\left[\delta_*, \delta^*\right]$ correspondingly: $p_*^{\max} = \psi_1(\sigma_{UZ}^2, d)$, $d_{\min} = \psi_2(\sigma_{UZ}^2, p_*)$. Meeting one of such a variable under given two resting ones provides also the sufficient condition of existence of intersection (5).

Also, conditions (15) and (19) obtained are also shown to be applicable within a distribution free input/output system identification problem statement with a probabilistic criterion.

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