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Stability Analysis of Impulsive Positive Systems

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Abstract: This paper is concerned with the problems of stability for a class of impulsive positive systems. An impulsive positive system model is introduced for the first time and a necessary and sufficient condition guaranteeing the positivity of this kind of system is proposed. Several sufficient criteria of global exponential stability and global asymptotical stability for impulsive positive systems are established respectively by using a linear copositive Lyapunov function. Two numerical examples are given to illustrate the effectiveness and applicability of the proposed results.

Keywords: Impulsive positive system; Positivity; Exponentially stable; Asymptotically stable; Linear copositive Lyapunov function.

1. INTRODUCTION

Many dynamic systems in physics, biology, engineering, chemistry, economics and information science have impulsive dynamical behaviors due to abrupt jumps at certain instants during the dynamical processes. These complex dynamic behaviors can be modeled by impulsive systems, see Bainov et al. (1989) and Haddad et al. (2006). Impulsive dynamical systems can be viewed as a certain class of hybrid systems and consist of three elements; namely, continuous evolution, which is typically described by ordinary differential equations; impulse effects, which is described by a difference equation and also referred to as state jumps; and a rule for determining when the states of the system are to be jump. Impulsive systems have been studied extensively over the past two decades and plenty of results have been reported, see Guan et al. (2005), Zhang et al. (2005), Nersesov et al. (2007), Liu et al. (2010) and Belkoura et al. (2010).

On the other hand, in many practical systems, there is such a special kind of system, namely, the positive system, whose state variables and outputs are always positive (at least nonnegative) whenever the initial states and the inputs are positive, see Kaczorek (2002). Positive systems play a key role in many and diverse areas such as economics (Johnson 1974), biology (Jacquez et al. 1993), communication networks (Shorten et al. 2006) and synchronisation/consensus problems (Jadbabaie et al. 2003). Positivity of the system state for all times will bring about many new issues, which cannot be solved in general by using the well-established methods for general linear systems. The main reason is that the states of positive systems are defined on cones rather than in the whole space. Therefore, many interesting issues of this kind of system have been investigated, and some valuable results have been established, see Leenheer et al. (2002), Kaczorek (2005), Benzaouia et al. (2010), Rami et al. (2011), Feng et al. (2011), Li et al. (2012) and Zhao et al. (2013).

Note that all the positive systems are modeled without impulsive effects in these literature. However, in practice, some dynamical positive system models are required to consider impulsive effects due to the various jumping parameters and changing environmental factors at some instants. For example, in dynamic portfolio management, the dynamical behavior of stock value for a particular investor can be described by a positive system. More specifically, we can see that when a certain amount of stock is purchased or sold at certain instants, the stock value changes instantaneously to a new value. This situation can be described by impulsive effects exactly.

A positive system can be called an impulsive positive system, if the positive system is modeled with impulsive effect, which cannot be well described by using pure continuous or pure discrete models. Therefore, it is important and, in fact, necessary to study impulsive positive systems. To the best our knowledge, no results about impulsive positive systems have been reported up to now. This has motivated our research.

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In this paper, we will first introduce an impulsive positive system model, and give a criterion (necessary and sufficient) to ensure positivity of this kind of system. A sufficient criterion of stability analysis of impulsive positive systems is obtained by virtue of a linear copositive Lyapunov function. The theoretical result is proposed in terms of linear programming (LP), which can easily be solved by using certain available software. Finally, simulations are presented to demonstrate the effectiveness of the proposed approach.

The remainder of this paper is organized as follows. In Section 2, some preliminaries and definitions are given so as to formulate the main problems. The main results are presented in Section 3, where a necessary and sufficient condition is proposed for guaranteeing positivity of impulsive positive systems, and a sufficient criterion of the stability is established for impulsive positive systems. Several numerical examples are presented in Section 4 to illustrate the effectiveness of the proposed theoretical results. The conclusion is finally drawn in Section 5.

Notations: The notations used in this paper are fairly standard. Let N denote the set of positive integers, i.e., $N = \{1, 2, \ldots\}$; $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$ denote the n-dimensional Euclidean space and the space of $n \times n$-dimensional matrices with real entries; $\mathbb{R}^n_+$ stands for the nonnegative orthant in $\mathbb{R}^n$. The superscript "T" stands for matrix transposition, the notation $\|\bullet\|$ refers to the vector norm. For a matrix $A \in \mathbb{R}^{n\times n}$, $A \succ 0$ ($\succeq 0$) means that all elements of the matrix A are positive (nonnegative); $\rho(A) \triangleq \max\{|\lambda| : \lambda \in \sigma(A)\}$ denotes the spectral radius of A, where $\sigma(A)$ is the spectrum of A. For a vector $v \in \mathbb{R}^n$, $v \succ 0$ ($\succeq 0$, $\prec 0$, $\preceq 0$) means that all elements of the vector v are positive (nonnegative, negative, nonpositive), $max\{v\}$ and $min\{v\}$ denote the maximum and the minimum element of the vector, respectively. $\mathbf{M}$ denotes the set of Metzler matrices whose off-diagonal entries are nonnegative. I denotes an identity matrix with appropriate dimension. Matrices, if not explicitly stated, are assumed to have compatible dimensions for algebraic operations.

2. PROBLEM FORMULATION AND PRELIMINARIES

Consider the following impulsive positive system

$$\begin{cases} \dot{x}(t) = Ax(t),\ t \neq t_k, \\ \Delta x = Bx(t),\ t = t_k,\ k \in N, \\ x(t_0) = x_0 \succeq 0,\ t_0 \geqslant 0, \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n_+$ is the state variable, A and B are the known $n \times n$ matrices, where $B \neq 0$, $\Delta x = x(t_k^+) - x(t_k^-)$, where $\lim_{h\to 0^+} x(t_k + h) = x(t_k^+)$, $\lim_{h\to 0^-} x(t_k + h) = x(t_k^-)$.

Consider a discrete time set $\{t_k\}_{k=1}^{\infty}$ of impulsive jump instants which satisfy $0 < t_1 < t_2 < \ldots < t_{k-1} < t_k < \ldots$ and $\lim_{k\to\infty} t_k \to \infty$, where $t_1 > t_0$ and $t_k - t_{k-1}$ is finite. Without loss of generality, we assume $x(t_k^+) = x(t_k)$, which implies that the solution of system (1) is right-continuous at t_k.

Remark 1. (1) When $t \neq t_k$, the system (1) is a continuous-time linear system for $t \in [t_{k-1}, t_k)$.

(2) When $t = t_k$, the state variable is changed from $x(t_k^-)$ to $x(t_k) = x(t_k^-) + \Delta x|_{t=t_k}$ instantly.

Remark 2. Compared with the general impulsive system, the impulsive positive system (1) must possess positivity, i.e., $x(t) \in \mathbb{R}^n_+$ for all $t \geqslant t_0$, which will increase complexity for stability analysis of such system.

Before proceeding, we need to introduce some definitions which will be used in what follows for the system (1).

Definition 3. (Farina et al. 2000) The system (1) is said to be an impulsive positive system if for any initial condition $x_0 \succeq 0$, we have $x(t) \succeq 0$ for $\forall t \geqslant 0$.

Definition 4. The system (1) is said to be stable, if, given any $\varepsilon > 0$, there exists some $\delta = \delta(\varepsilon) > 0$, such that $\|x(t_0)\| < \delta$ implies that $\|x(t)\| < \varepsilon$, $t \geqslant t_0$.

Definition 5. The system (1) is said to be globally asymptotically stable, if, for any $\delta > 0$, $\|x(t_0)\| < \delta$ implies that $\lim_{t\to\infty} \|x(t)\| = 0$, $t \geqslant t_0$.

Definition 6. The system (1) is said to be globally exponentially stable, if, for any $\delta > 0$, and some scalars $\zeta > 0$, $\eta > 0$, $\|x(t_0)\| < \delta$ implies that $\|x(t)\| \leqslant \zeta e^{-\eta(t-t_0)}$, $t \geqslant t_0$.

Remark 7. Note that in the above definitions, the vector norm $\|x\|$ can be any type of the norm of x. However, to make the subsequent proof easy to understand, we use the following norm:

$$\|x\| = \sum_{i=1}^{n} |x_i|,$$

where x_i is the ith element of $x \in \mathbb{R}^n$.

We end this section by introducing a lemma which plays a critical role in the subsequent proof.

Lemma 8. (Luenberger 1979) Let $A \in \mathbf{M}$, then the following holds true.
(1) $A \in \mathbf{M} \Leftrightarrow e^{At} \succeq 0$ for $t \geqslant 0$;
(2) if a vector $v \succeq 0$, then $e^{At}v \succeq 0$ for $t \geqslant 0$.

3. MAIN RESULTS

In this section, a necessary and sufficient condition of the positivity with respect to the system (1) is first given, and then, a sufficient criterion of the stability for the system (1) is proposed.

Proposition 9. The system (1) is an impulsive positive system if and only if $A \in \mathbf{M}$ and $(I + B) \succeq 0$.

Proof. In the system (1), any initial condition $x_0 \succeq 0$, and for any $t \in [t_{k-1}, t_k)$ ($k \in N$), the solution to (1) is given by

$$x(t) = e^{A(t-t_{k-1})}x(t_{k-1}), \tag{2}$$

when $t = t_k$, we have

$$x(t_k) = (I + B)x(t_k^-),\ k \in N. \tag{3}$$

(*Sufficiency*) By (2) and (3), we can obtain that for $t \in [t_0, t_1)$, $x(t) = e^{A(t-t_0)}x(t_0) = e^{A(t-t_0)}x_0$. Since $A \in \mathbf{M}$, from Lemma 8, we have $e^{A(t-t_0)} \succeq 0$ for $t \in [t_0, t_1)$. Thus, one has $x(t) \succeq 0$ for $t \in [t_0, t_1)$.

When $t = t_1$, we have $x(t_1) = (I + B)x(t_1^-)$. Since $(I + B) \succeq 0$ and $x(t_1^-) \succeq 0$, one has $x(t_1) \succeq 0$.

So for $t \in [t_1, t_2)$, $x(t) = e^{A(t-t_1)}x(t_1)$. Since $A \in \mathbf{M}$ and $x(t_1) \succeq 0$, we have $x(t) \succeq 0$ for $t \in [t_1, t_2)$.

When $t = t_2$, we have $x(t_2) = (I + B)x(t_2^-)$. Since $(I + B) \succeq 0$ and $x(t_2^-) \succeq 0$, one has $x(t_2) \succeq 0$.

By repeating the same procedure, we can easily obtain the general result as follows:

For all $t \in [t_{k-1}, t_k)$ $(k \in N)$, $x(t) \succeq 0$, i.e., $x(t) \succeq 0$ for all $t \geqslant t_0$.

By Definition 3, the system (1) is an impulsive positive system.

(*Necessity*) Since the system (1) is an impulsive positive system, for $t \in [t_0, t_1)$, we know that

$$x(t) = e^{A(t-t_0)}x_0 \succeq 0. \tag{4}$$

Suppose $e^{A(t-t_0)}$ dissatisfies $e^{A(t-t_0)} \succeq 0$, since the initial condition x_0 is any vector and $x_0 \succeq 0$, there exists a x_0 such that $x(t)$ dissatisfies $x(t) \succeq 0$. This is contradictory with (4).

Consequently, there must be $e^{A(t-t_0)} \succeq 0$. From Lemma 8, we have $A \in \mathbf{M}$.

when $t = t_1$, we have

$$x(t_1) = (I + B)x(t_1^-) = (I + B)e^{A(t_1-t_0)}x_0 \succeq 0. \tag{5}$$

Suppose $I + B$ dissatisfies $(I + B) \succeq 0$, since the initial condition $x_0 \succeq 0$ is any vector and $e^{A(t_1-t_0)} \succeq 0$, there exists a x_0 such that $x(t_1)$ dissatisfies $x(t_1) \succeq 0$. This is contradictory with (5).

Therefore, there must be $(I + B) \succeq 0$.

This completes the whole proof.

In what follows, a sufficient criterion of the stability for the system (1) is presented.

Theorem 10. Consider the impulsive positive system (1) with $A \in \mathbf{M}$ and $(I + B) \succeq 0$, let α and β $(\beta > 0)$ be given constants, if there exists a vector $v \succ 0$ such that the following inequalities hold:

$$(A^T - \alpha I)v \preceq 0, \tag{6}$$

$$[(I + B)^T - \beta I]v \preceq 0, \tag{7}$$

then,

(i) if $\alpha < 0$ and there exists a constant γ $(0 \leqslant \gamma < -\alpha)$, such that

$$\ln\beta - \gamma(t_k - t_{k-1}) \leqslant 0,\ k \in N, \tag{8}$$

then the system (1) is globally exponentially stable;

(ii) if $\alpha = 0$, one has if $0 < \beta < 1$, the system (1) is globally asymptotically stable; if $\beta = 1$ the system (1) is stable;

(iii) if $\alpha > 0$ and there exists a constant γ $(\gamma \geqslant 1)$, such that

$$\ln\beta\gamma + \alpha(t_k - t_{k-1}) \leqslant 0,\ k \in N, \tag{9}$$

then when $\gamma = 1$, the system (1) is stable; when $\gamma > 1$, the system (1) is globally asymptotically stable.

Proof. From Definition 3, we have $x^T(t) \succeq 0$, and choose a linear copositive Lyapunov function in the form of

$$V(t) = x^T(t)v,$$

where $v \succ 0$.

When $t \in [t_{k-1}, t_k)$, $k \in N$, the time derivative of $V(t)$ with respect to (1) is

$$\dot{V}(t) = \dot{x}^T(t)v = x^T(t)A^Tv.$$

By (6), we can obtain $x^T(t)A^Tv \leqslant \alpha x^T(t)v$.

This implies

$$\dot{V}(t) \leqslant \alpha x^T(t)v = \alpha V(t),$$

so that

$$V(t) \leqslant V(t_{k-1})e^{\alpha(t-t_{k-1})}. \tag{10}$$

On the other hand, it follows from (1) that when $t = t_k$, $k \in N$, we have

$$V(t_k) = x^T(t_k)v = x^T(t_k^-)(I + B)^Tv.$$

From (7), we can obtain

$$V(t_k) \leqslant \beta x^T(t_k^-)v = \beta V(t_k^-). \tag{11}$$

From (10) and (11) , we know that for $t \in [t_0, t_1)$, $V(t) \leqslant e^{\alpha(t-t_0)}V(t_0)$, which leads to $V(t_1) \leqslant e^{\alpha(t_1-t_0)}V(t_0)$. When $t = t_1$, we have $V(t_1) \leqslant \beta V(t_1^-) \leqslant \beta e^{\alpha(t_1-t_0)}V(t_0)$. So for $t \in [t_1, t_2)$, $V(t) \leqslant e^{\alpha(t-t_1)}V(t_1) \leqslant \beta e^{\alpha(t-t_0)}V(t_0)$.

In general, for $t \in [t_{k-1}, t_k)$ $(k \in N)$,

$$V(t) \leqslant \beta^{k-1}e^{\alpha(t-t_0)}V(t_0),$$

i.e.,

$$x^T(t)v \leqslant \beta^{k-1}e^{\alpha(t-t_0)}x^T(t_0)v. \tag{12}$$

Then, taking the vector 1-norm on both sides of (12), defining $a = \min\{v\}$, $b = \max\{v\}$ and denoting $c = \|x_0\| \frac{b}{a} > 0$, we can obtain that

$$\|x(t)\| \leqslant c\beta^{k-1}e^{\alpha(t-t_0)}. \tag{13}$$

We now consider three cases as follows:

Case 1: when $\alpha < 0$ and there exists a constant γ $(0 \leqslant \gamma < -\alpha)$, such that $\ln\beta - \gamma(t_k - t_{k-1}) \leqslant 0$, $k \in N$, it then follows from (13) that for $t \in [t_{k-1}, t_k)$,

$$\begin{aligned}
\|x(t)\| &\leqslant c\beta^{k-1}e^{-\gamma(t-t_0)}e^{(\alpha+\gamma)(t-t_0)} \\
&\leqslant c\beta^{k-1}e^{-\gamma(t_{k-1}-t_0)}e^{(\alpha+\gamma)(t-t_0)} \\
&= c\beta e^{-\gamma(t_1-t_0)}\beta e^{-\gamma(t_2-t_1)}\dots \\
&\quad \beta e^{-\gamma(t_{k-1}-t_{k-2})}e^{(\alpha+\gamma)(t-t_0)} \\
&\leqslant ce^{(\alpha+\gamma)(t-t_0)}.
\end{aligned}$$

Thus, the system (1) is globally exponentially stable by Definition 6, which implies that the conclusion (i) of Theorem 10 holds.

Case 2: when $\alpha = 0$, inequality (13) becomes $\|x(t)\| \leqslant c\beta^{k-1}$ for $t \in [t_{k-1}, t_k)$.

Thus, from Definition 4 and Definition 5, we can easily show that when $0 < \beta < 1$, the system (1) is globally asymptotically stable; when $\beta = 1$ the system (1) is stable, which implies that the conclusion (ii) of Theorem 10 holds.

Case 3: when $\alpha > 0$ and there exists a constant γ $(\gamma \geqslant 1)$, such that $\ln\beta\gamma + \alpha(t_k - t_{k-1}) \leqslant 0$, $k \in N$, it then follows from (13) that for $t \in [t_{k-1}, t_k)$,

$$\begin{aligned}\|x(t)\| &\leqslant c\beta^{k-1}e^{\alpha(t-t_0)}\\ &\leqslant c\beta^{k-1}e^{\alpha(t_k-t_0)}\\ &= \frac{c}{\gamma^{k-1}}\beta\gamma e^{\alpha(t_1-t_0)}\beta\gamma e^{\alpha(t_2-t_1)}\cdots\\ &\quad \beta\gamma e^{\alpha(t_{k-1}-t_{k-2})}e^{\alpha(t_k-t_{k-1})}\\ &\leqslant \frac{c}{\gamma^{k-1}}e^{\alpha(t_k-t_{k-1})}.\end{aligned}$$

Since $t_k - t_{k-1}$ is finite, from Definition 4 and Definition 5, we know that if $\gamma = 1$, the system (1) is stable; when $\gamma > 1$, the system (1) is globally asymptotically stable, which implies that the conclusion (iii) of Theorem 10 holds.

This has completed the proof of Theorem 10.

Remark 11. For Theorem 10, suppose that the impulsive interval is a constant, i.e., $t_k - t_{k-1} = \tau$ for $k \in N$, then two conclusions are provided as follows:

(1) since the inequality (8) becomes $\ln\beta - \gamma\tau \leqslant 0$, we can obtain the lower bound for τ is

$$\tau_{lower} = \frac{\ln\beta}{\gamma}.$$

Therefore, in order to ensure the stability of the system (1), there must be $\tau \geqslant \tau_{lower}$ to guarantee that the impulses do not occur too frequently.

(2) since the inequality (9) becomes $\ln\beta\gamma + \alpha\tau \leqslant 0$, we can obtain the upper bound for τ is

$$\tau_{upper} = \frac{-\ln\beta\gamma}{\alpha}.$$

Therefore, in order to ensure the stability of the system (1), there must be $\tau \leqslant \tau_{upper}$ to guarantee that the frequency of impulses should not be too low.

Remark 12. From Theorem 10, it is known that the stability of impulsive positive systems not only is affected by the system matrix A and the impulsive gain B, but also is influenced by the impulsive intervals $t_k - t_{k-1}$ for $k \in N$.

4. NUMERICAL EXAMPLES

In this section, several numerical examples will be presented to demonstrate the applicability and validity of our theoretical results.

Example 1: Consider the system (1) described by

$$A = \begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix},\ B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

it is obvious that $A \in \mathbf{M}$ and $(I + B) \succeq 0$.

Letting $\alpha = -2$, $\beta = 3$. Solving inequalities (6) and (7) gives $v = [\,55.8513\ 105.4970\,]^T$.

Since $\alpha < 0$, in order to guarantee the system is globally exponentially stable, there must exist a constant γ ($0 \leqslant \gamma < 2$) such that $\ln\beta - \gamma(t_k - t_{k-1}) \leqslant 0$, $k \in N$.

Select γ=1 and suppose that the impulsive interval is a constant, i.e., $t_k - t_{k-1} = \tau_1$, so we have

$$\tau_1 \geqslant \frac{\ln\beta}{\gamma} = 1.0986.$$

Thus, when $\tau_1 \geqslant 1.0986$, the system is globally exponentially stable. The simulation result is shown in Figure 1.

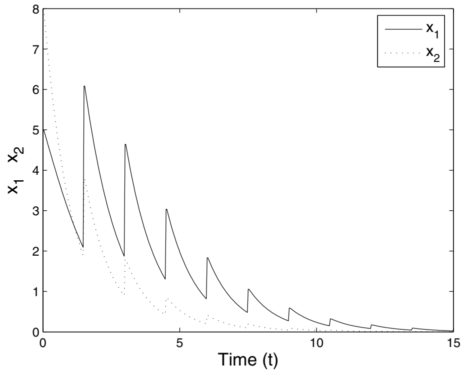

Fig. 1. Evolution of the states $x(t)$ in the first example when $t_k = t_{k-1} + 1.5$.

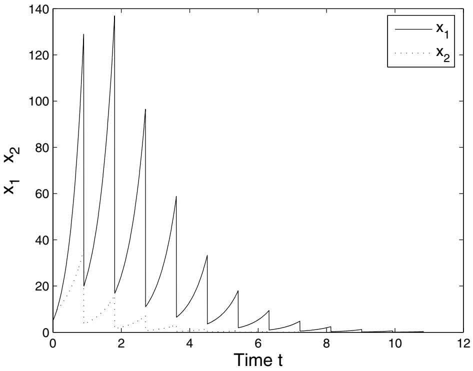

Fig. 2. Evolution of the states $x(t)$ in the second example when $t_k = t_{k-1} + 0.9$.

From Fig. 1, when τ_1=1.5 $\geqslant$ 1.0986, the system is globally exponentially stable.

Example 2: Consider the system (1) described by

$$A = \begin{bmatrix} 0.5 & 1 \\ 0 & 0.5 \end{bmatrix},\ B = \begin{bmatrix} -0.9 & 0.2 \\ 0 & -0.9 \end{bmatrix},$$

it is obvious that $A \in \mathbf{M}$ and $(I + B) \succeq 0$.

Letting $\alpha = 1$, $\beta = 0.2$. Solving inequalities (6) and (7) gives $v = [\,0.5675\ 1.6574\,]^T$.

Since $\alpha > 0$, in order to guarantee the system is globally asymptotically stable, there must exist a constant γ ($\gamma > 1$), such that $\ln\beta\gamma + \alpha(t_k - t_{k-1}) \leqslant 0$, $k \in N$.

Select γ=1.5 and suppose that the impulsive interval is a constant, i.e., $t_k - t_{k-1} = \tau_2$, so we have

$$\tau_2 \leqslant \frac{-\ln\beta\gamma}{\alpha} = 1.2040.$$

Thus, when $\tau_2 \leqslant 1.2040$, the system is globally asymptotically stable. The simulation result is shown in Figure 2.

From Fig. 2, when τ_2=0.9 $\leqslant$ 1.2040, the system is globally asymptotically stable.

5. CONCLUSION

In this paper, an impulsive positive system model has been put forward for the first time. A sufficient criterion for the stability of impulsive positive systems has been derived on the basis of the linear copositive Lyapunove function method. Two numerical examples have been presented to illustrate the theoretical results. In our future work, the stability problem of impulsive positive systems with time delays will be considered.

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