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# Observer-based fuzzy control for nonlinear fractional-order systems via fuzzy T-S models: The $1 < \alpha < 2$ case ⋆

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**Abstract:** This paper presents observer-based fuzzy control for nonlinear fractional-order systems with the fractional order $\alpha$ satisfying $1 < \alpha < 2$ via fuzzy T-S models. Using the properties of the Kronecker product and LMI approach, the feedback and observer gain matrices are designed. By this method, the state of nonlinear system described as the fuzzy T-S model is convergent to the equilibrium and the observer error is convergent to zero. Finally, the simulation result of a numerical example is given to illustrate the effectiveness of this method.

## 1. INTRODUCTION

In recent years, fractional-order systems have attracted much attention (Machado et al. [2011]). Compared with integer-order systems, fractional-order systems can describe the dynamics of real-world systems better such as heat conduction, viscoelastic systems, electromagnetic wave, etc.(Hilfer [2000]). Some approximation methods of fractional-order operators have been proposed to implement the fractional-order controllers for fractional-order systems (Maione [2008], Oustaloup et al. [2000]). Many design methods for fractional-order PID controllers are studied since PID control is applied widely in the industry fields (Podlubny [1999]). Fractional-order PID controllers have been applied in many practical control systems (Zamani et al. [2009], Vinagre et al. [2007]). Other types of fractional-order controllers came into study recently such as the fractional-order optimal controller (Biswas & Sen [2011]), the fractional-order iterative learning controller (Li et al. [2011]) and the fractional-order sliding mode controller (Pisano et al. [2010],Yin et al. [2012]).

However, the aforementioned controllers are designed for linear fractional-order systems, or the fractional-order controllers are linear. Based on the LMI approach, the robust stability and stabilization for linear fractional-order systems were investigated by Lu & Chen [2010], Lu & Cao [2009] using state-space models. Lan & Zhou [2011] proposed the robust output control for linear fractional-order systems with the observers. If the fractional order $\alpha$ is between 1 and 2, the stability criterion of linear fractional-order systems is similar to the robust $\mathfrak{D}$-stability. The left half-plane contains the stable and unstable regions. But if the fractional order is between 0 and 1, the LMI conditions will be very complicated, presented by Lu & Chen [2010]. At least, we ensure that all eigenvalues of the system matrix are set in the left half-plane by the controller, the fractional-order system with $0 < \alpha < 1$ is stable, though this control method is conservative.

Although the requirements of Lyapunov functions for fractional-order systems have been proposed by Ahn & Chen [2008], Li & Chen [2010], an appropriate Lyapunov function is hard to be constructed for a nonlinear fractional-order system. Fuzzy T-S model was proposed by Takagi and Sugeno in 1985 (Takagi & Sugeno [1985]). Using the fuzzy sector nonlinearity concept, a nonlinear system can be described exactly as a set of local linear subsystems connected by fuzzy membership functions in the form of fuzzy T-S model (Ohtake et al. [2003]). The stability of fuzzy T-S systems and fuzzy control design have been investigated in many papers (Tseng et al. [2001], Liu & Zhang [2003], Fang [2006] and Rajesh & Kaimal [2007]). In a real-world system, not all the states of control systems are measurable, hence it is necessary to design an observer to estimate the states. Assuming that the premise variables are measurable, the output feedback controllers based on fuzzy observers via the fuzzy T-S model were proposed by Lin et al. [2005], Tong & Li [2002] and Chen et al. [2000]. Fuzzy control based on the fuzzy T-S models has been widely used in nonlinear integer-order systems. For nonlinear fractional-order systems, the fuzzy T-S method is still effective, but few papers on the control for nonlinear fractional-order systems have been reported.

Therefore, we introduce the fuzzy T-S model method into the nonlinear fractional-order systems in this paper. Assuming that the premise variables are measurable, the the observer-based fuzzy output control has been designed. By defining the augmented state vector consisting of the state and observer error, the feedback and observer gain matrices are obtained by solving a set of LMI conditions

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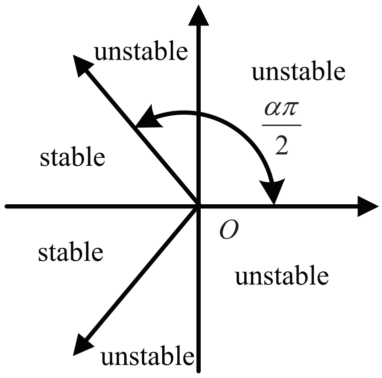


Fig. 1. Stable regions of fractional-order systems for $1 < \alpha < 2$

which are got by the properties of the Kronecker product. For $1 < \alpha < 2$, the stable region is in a sector area of the left half-plane. The stable region is smaller than integer-order systems which can be regarded as $\alpha = 1$ hence the LMIs are more complex than the integer-order systems.

## 2. PRELIMINARIES

There are many definitions of the fractional-order integral and differential operators such as the Grünwald-Letnikov, Riemann-Liouville and Caputo definitions. Because Caputo definition allows utilization of the initial values of classical integer-order derivatives with clear physical interpretation, the Caputo definition of fractional-order derivative for the function $f(t)$ is adopted in this paper as:

$$D^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)\mathrm{d}\tau}{(t-\tau)^{\alpha+1-m}}, \tag{1}$$

where $\alpha$ is a positive real number, $m$ is an integer satisfying $m - 1 \leq \alpha < m$ and $\Gamma(\cdot)$ is the Gamma function.

As defined above, a linear fractional-order system with the order $\alpha$ satisfying $1 < \alpha < 2$ is modeled in the form of the state space equation as:

$$D^{\alpha} x(t) = \tilde{A} x(t), \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the pseudo-state vector, $\alpha \in (1, 2)$ is the fractional order and $\tilde{A} \in \mathbb{R}^{n\times n}$ is the system matrix.

For the fractional-order system (2) for $1 < \alpha < 2$, we have the following Lemma to verify the stability.

*Lemma 1:* (Tavazoei & Haeri [2009]) For $0 < \alpha < 2$, let $\lambda_i$, $i = 1, 2, \ldots, n$ be the $i$th eigenvalue of $\tilde{A}$. The fractional-order system (2) is stable if and only if $|\arg(\lambda_i)| > (\alpha\pi)/2$ holds. The stable region of fractional-order systems for $1 < \alpha < 2$ is shown in Fig. 1.

It is seen in Fig. 1, some regions of the left half-plane are not stable, and it is not convenient to calculate all the eigenvalues to verity the stability by Lemma 1, hence the following lemma gives a stable criterion for $1 < \alpha < 2$.

*Lemma 2:* (Lu & Cao [2009]) For $1 < \alpha < 2$, the fractional-order system (2) is asymptotically stable if and only if there exists a positive definite matrix $\tilde{P} \in \mathbb{R}^{n\times n}$ such that

$$Sym\{\Theta \otimes (\tilde{A}\tilde{P})\} < 0, \tag{3}$$

where $Sym\{X\} = X^{\mathrm{T}} + X$, "$\otimes$" is the Kronecker product,

$$\Theta = \begin{bmatrix} \sin(\frac{\pi\alpha}{2}) & -\cos(\frac{\pi\alpha}{2}) \\ \cos(\frac{\pi\alpha}{2}) & \sin(\frac{\pi\alpha}{2}) \end{bmatrix}.$$

Some properties of the Kronecker product will be used later to design the fuzzy T-S controller as follows:

$$(A \otimes B)^{\mathrm{T}} = A^{\mathrm{T}} \otimes B^{\mathrm{T}},$$

$$(A + B) \otimes C = A \otimes C + B \otimes C,$$

where $A$, $B$ and $C$ are matrices with appropriate dimensions.

## 3. MAIN RESULTS

By the sector nonlinearity concept, nonlinear systems can be represented as T-S fuzzy systems described by a series of "IF-Then" rules. Each subsystem in T-S models is a linear system, hence, the linear control approaches can be applied for nonlinear systems by T-S models. We apply the T-S technique to control nonlinear fractional-order systems in this section.

Consider the following affine nonlinear fractional-order system as

$$\begin{aligned} D^{\alpha} x(t) &= f(x(t)) + g(x(t))u(t), \\ y(t) &= d(x(t)), \end{aligned} \tag{4}$$

where $\alpha$ is the fractional order satisfying $1 < \alpha < 2$, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^s$ is the input vector, $y(t) \in \mathbb{R}^t$ is the output vector, $f(\cdot) \in \mathbb{R}^n$, $g(\cdot) \in \mathbb{R}^{n\times s}$ and $d(\cdot) \in \mathbb{R}^t$ are nonlinear functions.

Using the sector nonlinearity concept, we get the following T-S fuzzy system of (4) for the $i$th rule as:

Rule $i$: If $z_1(t)$ is $M_{i1}$ and $\ldots$ and $z_p(t)$ is $M_{ip}$

$$\text{Then} \begin{cases} D^{\alpha} x = A_i x(t) + B_i u(t) \\ y(t) = C_i x(t) \end{cases}, i = 1, 2, \ldots, r, \tag{5}$$

where $r$ is the number of the fuzzy rules, $z_j(t)$ is the premise variable assumed to be measurable, which is related with the control input, some of states or output, $M_{ij}$ denotes the fuzzy set, $j = 1, 2, \ldots, p$, $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times s}$ and $C_i \in \mathbb{R}^{t\times n}$ are the system matrix, control matrix and output matrix of the $i$th subsystem.

Using the fuzzy inference with a singleton fuzzifier, product inference and a center-average defuzzifer, the global model is represented as:

$$\begin{aligned} D^{\alpha} x(t) &= \frac{\sum_{i=1}^{r} w_i(z(t))(A_i x(t) + B_i u(t))}{\sum_{i=1}^{r} w_i(z(t))}, \\ y(t) &= \frac{\sum_{i=1}^{r} w_i(z(t)) C_i x(t)}{\sum_{i=1}^{r} w_i(z(t))}, \end{aligned} \tag{6}$$

where $z(t) = [z_1(t), z_2(t), \ldots, z_p(t)]^{\mathrm{T}}$, $w_i(z(t)) = \prod_{j=1}^{p} M_{ij}(z_j(t))$, $M_{ij}(z_j(t))$ is the membership function of $z_j(t)$ in $M_{ij}$ and $w_i(z(t)) > 0$, $\sum_{i=1}^{r} w_i(z(t)) > 0$. Denote

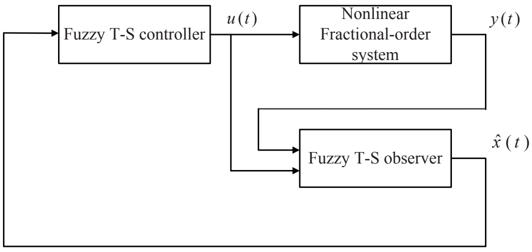


Fig. 2. Structure of the fuzzy T-S control system with the fuzzy T-S observer

$h_i(z) = w_i(z(t))/\sum_{i=1}^{r} w_i(z(t))$, then (6) can be rewritten as:

$$\begin{aligned} D^{\alpha}x(t) &= \sum_{i=1}^{r} h_i(z(t))(A_i x(t) + B_i u(t)), \\ y(t) &= \sum_{i=1}^{r} h_i(z(t))C_i x(t), \end{aligned} \tag{7}$$

and $h_i(z(t)) \geq 0$, $\sum_{i=1}^{r} h_i(z(t)) = 1$.

By supposing the premise variables are measurable, the observer of the fractional-order system (7) is designed as:

$$D^{\alpha}\hat{x}(t) = \sum_{i=1}^{r} h_i(z(t))[A_i\hat{x}(t) + B_i u(t) + G_i(y(t) - \hat{y}(t))], \tag{8}$$

where $\hat{x}(t)$ and $\hat{y}(t)$ are the estimations of $x(t)$ and $y(t)$ respectively. $\hat{y}(t) = \sum_{j=1}^{r} h_j(z(t))C_j\hat{x}(t)$, $G_i \in \mathbb{R}^{n\times t}$ is the observer gain matrix.

Therefore, the fuzzy control adopting $\hat{x}(t)$ is given by

$$u(t) = \sum_{j=1}^{r} h_j(z(t))K_j\hat{x}(t), \tag{9}$$

where $K_j \in \mathbb{R}^{s\times n}$ is the feedback gain matrix. The structure of the fuzzy T-S control system with the fuzzy T-S observer for the nonlinear system for $1 < \alpha < 2$ is shown in Fig. 2.

Denote $e(t) = x(t) - \hat{x}(t)$, we have

$$\begin{aligned} D^{\alpha}x(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} & h_i(z(t))h_j(z(t))[(A_i + B_iK_j)x(t) \\ & - B_iK_je(t)], \end{aligned} \tag{10}$$

and

$$D^{\alpha}e(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))h_j(z(t))[(A_i - G_iC_j)e(t)], \tag{11}$$

Let $\tilde{x}(t) = [x^{\mathrm{T}}(t), e^{\mathrm{T}}(t)]^{\mathrm{T}}$, then the fractional-order T-S fuzzy model for $\tilde{x}(t)$ is given by

$$D^{\alpha}\tilde{x}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))h_j(z(t)) \begin{bmatrix} A_i + B_iK_j & -B_iK_j \\ 0 & A_i - G_iC_j \end{bmatrix} \tilde{x}(t), \tag{12}$$

According to Lemma 2, we obtain the following theorem to design the feedback and observer gain matrices.

*Theorem 1.* For $1 < \alpha < 2$, the fractional-order T-S fuzzy system (5) with the fractional order $\alpha$ under the control $u = \sum_{j=1}^{r} h_j(z(t))K_j\hat{x}(t)$ and the observer (8) is asymptotically stable, if there exist two positive definite matrices $X \in \mathbb{R}^{n\times n}$ and $Y \in \mathbb{R}^{n\times n}$ and the matrices $S_j$ and $N_i$ satisfying

$$\Xi_{ij} = \begin{bmatrix} \sin(\frac{\pi\alpha}{2})\Omega_{1ij} & \sin(\frac{\pi\alpha}{2})\Omega_{3ij} & -\cos(\frac{\pi\alpha}{2})\Omega_{2ij} & -\cos(\frac{\pi\alpha}{2})\Omega_{3ij} \\ \sin(\frac{\pi\alpha}{2})\Omega_{4ij} & \sin(\frac{\pi\alpha}{2})\Omega_{5ij} & \cos(\frac{\pi\alpha}{2})\Omega_{4ij} & -\cos(\frac{\pi\alpha}{2})\Omega_{6ij} \\ \cos(\frac{\pi\alpha}{2})\Omega_{2ij} & \cos(\frac{\pi\alpha}{2})\Omega_{3ij} & \sin(\frac{\pi\alpha}{2})\Omega_{1ij} & \sin(\frac{\pi\alpha}{2})\Omega_{3ij} \\ -\cos(\frac{\pi\alpha}{2})\Omega_{4ij} & \cos(\frac{\pi\alpha}{2})\Omega_{6ij} & \sin(\frac{\pi\alpha}{2})\Omega_{4ij} & \sin(\frac{\pi\alpha}{2})\Omega_{5ij} \end{bmatrix} < 0, \tag{13}$$
$$i = 1, 2, \ldots, r, \quad j = 1, 2, \ldots, r,$$

where

$$\Omega_{1ij} = A_iX + B_iS_j + XA_i^{\mathrm{T}} + S_j^{\mathrm{T}}B_i^{\mathrm{T}},$$

$$\Omega_{2ij} = A_iX + B_iS_j - XA_i^{\mathrm{T}} - S_j^{\mathrm{T}}B_i^{\mathrm{T}},$$

$$\Omega_{3ij} = A_i,$$

$$\Omega_{4ij} = A_i^{\mathrm{T}},$$

$$\Omega_{5ij} = YA_i - N_iC_j + A_i^{\mathrm{T}}Y - C_j^{\mathrm{T}}N_i^{\mathrm{T}},$$

$$\Omega_{6ij} = YA_i - N_iC_j - A_i^{\mathrm{T}}Y + C_j^{\mathrm{T}}N_i^{\mathrm{T}}.$$

Moreover, the feedback and observer gain matrices are designed as $K_j = S_jX^{-1}$ and $G_i = Y^{-1}N_i$, respectively.

*Proof.* Let $P_1 \in \mathbb{R}^{n\times n}$ and $P_2 \in \mathbb{R}^{n\times n}$ be two positive definite matrices.

Define the positive definite matrix $P = \begin{bmatrix} P_1 & P_2 \\ 0 & P_2 \end{bmatrix} \in \mathbb{R}^{2n\times 2n}$, $\Phi_{ij} = \begin{bmatrix} A_i + B_iK_j & -B_iK_j \\ 0 & A_i - G_iC_j \end{bmatrix}$. Replace $\tilde{A}$ and $\tilde{P}$ defined in Lemma 2 with $\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))h_j(z(t))\Phi_{ij}$ and $P$ respectively. If the following LMI (14) holds, then $x(t)$ and $e(t)$ converge to the equilibrium.

$$Q = Sym\{\Theta \otimes (\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))h_j(z(t))\Phi_{ij}P)\} < 0. \tag{14}$$

Using the properties of the Kronecker product, we have

$$
\begin{aligned}
Q =& Sym\{\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))h_j(z(t))[\Theta \otimes (\Phi_{ij}P)]\} \\
=& \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))h_j(z(t))Sym\{[\Theta \otimes (\Phi_{ij}P)]\} \\
=& \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))h_j(z(t))\times \\
& Sym\{[\Theta \otimes \begin{bmatrix} A_iP_1 + B_iK_jP_1 & A_iP_2 \\ 0 & A_iP_2 - G_iC_jP_2 \end{bmatrix}]\}.
\end{aligned} \tag{15}
$$

Let $S_j = K_jP_1$, then we have

$$
\begin{aligned}
Q =& \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))h_j(z(t))Sym\{\Theta \otimes (\Phi_{ij}P)\} \\
=& \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))h_j(z(t)) \begin{bmatrix} \sin(\frac{\pi\alpha}{2})\Lambda_{1ij} & \sin(\frac{\pi\alpha}{2})\Lambda_{3ij} & -\cos(\frac{\pi\alpha}{2})\Lambda_{2ij} & -\cos(\frac{\pi\alpha}{2})\Lambda_{3ij} \\ \sin(\frac{\pi\alpha}{2})\Lambda_{4ij} & \sin(\frac{\pi\alpha}{2})\Lambda_{5ij} & \cos(\frac{\pi\alpha}{2})\Lambda_{4ij} & -\cos(\frac{\pi\alpha}{2})\Lambda_{6ij} \\ \cos(\frac{\pi\alpha}{2})\Lambda_{2ij} & \cos(\frac{\pi\alpha}{2})\Lambda_{3ij} & \sin(\frac{\pi\alpha}{2})\Lambda_{1ij} & \sin(\frac{\pi\alpha}{2})\Lambda_{3ij} \\ -\cos(\frac{\pi\alpha}{2})\Lambda_{4ij} & \cos(\frac{\pi\alpha}{2})\Lambda_{6ij} & \sin(\frac{\pi\alpha}{2})\Lambda_{4ij} & \sin(\frac{\pi\alpha}{2})\Lambda_{5ij} \end{bmatrix}.
\end{aligned} \tag{16}
$$

where $\Lambda_{1ij} = A_iP_1 + B_iS_j + P_1A_i^{\mathrm{T}} + S_j^{\mathrm{T}}B_i^{\mathrm{T}}$, $\Lambda_{2ij} = A_iP_1 + B_iS_j - P_1A_i^{\mathrm{T}} - S_j^{\mathrm{T}}B_i^{\mathrm{T}}$, $\Lambda_{3ij} = A_iP_2$, $\Lambda_{4ij} = P_2A_i^{\mathrm{T}}$, $\Lambda_{5ij} = A_iP_2 - G_iC_jP_2 + P_2A_i^{\mathrm{T}} - P_2C_j^{\mathrm{T}}G_i^{\mathrm{T}}$, $\Lambda_{6ij} = A_iP_2 - G_iC_jP_2 - P_2A_i^{\mathrm{T}} + P_2C_j^{\mathrm{T}}G_i^{\mathrm{T}}$.

Define $U = \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & P_2^{-1} & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & P_2^{-1} \end{bmatrix}$ and let $X = P_1$, $Y = P_2^{-1}$ and $N_i = YG_i$, where $I_n$ is an identity matrix with $n$-dimension. Multiply the left and right sides of $Q$ by $U$, yields

$$
\begin{aligned}
\Upsilon =& UQU \\
=& \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))h_j(z(t)) \begin{bmatrix} \sin(\frac{\pi\alpha}{2})\Omega_{1ij} & \sin(\frac{\pi\alpha}{2})\Omega_{3ij} & -\cos(\frac{\pi\alpha}{2})\Omega_{2ij} & -\cos(\frac{\pi\alpha}{2})\Omega_{3ij} \\ \sin(\frac{\pi\alpha}{2})\Omega_{4ij} & \sin(\frac{\pi\alpha}{2})\Omega_{5ij} & \cos(\frac{\pi\alpha}{2})\Omega_{4ij} & -\cos(\frac{\pi\alpha}{2})\Omega_{6ij} \\ \cos(\frac{\pi\alpha}{2})\Omega_{2ij} & \cos(\frac{\pi\alpha}{2})\Omega_{3ij} & \sin(\frac{\pi\alpha}{2})\Omega_{1ij} & \sin(\frac{\pi\alpha}{2})\Omega_{3ij} \\ -\cos(\frac{\pi\alpha}{2})\Omega_{4ij} & \cos(\frac{\pi\alpha}{2})\Omega_{6ij} & \sin(\frac{\pi\alpha}{2})\Omega_{4ij} & \sin(\frac{\pi\alpha}{2})\Omega_{5ij} \end{bmatrix},
\end{aligned} \tag{17}
$$

where $\Omega_{1ij} = A_iX + B_iS_j + XA_i^{\mathrm{T}} + S_j^{\mathrm{T}}B_i^{\mathrm{T}}$, $\Omega_{2ij} = A_iX + B_iS_j - XA_i^{\mathrm{T}} - S_j^{\mathrm{T}}B_i^{\mathrm{T}}$, $\Omega_{3ij} = A_i$, $\Omega_{4ij} = A_i^{\mathrm{T}}$, $\Omega_{5ij} = YA_i - N_iC_j + A_i^{\mathrm{T}}Y - C_j^{\mathrm{T}}N_i^{\mathrm{T}}$, $\Omega_{6ij} = YA_i - N_iC_j - A_i^{\mathrm{T}}Y + C_j^{\mathrm{T}}N_i^{\mathrm{T}}$.

Note that $U > 0$, hence the condition $\Upsilon < 0$ is equivalent to $Q < 0$.

Define

$$
\Xi_{ij} = \begin{bmatrix} \sin(\frac{\pi\alpha}{2})\Omega_{1ij} & \sin(\frac{\pi\alpha}{2})\Omega_{3ij} & -\cos(\frac{\pi\alpha}{2})\Omega_{2ij} & -\cos(\frac{\pi\alpha}{2})\Omega_{3ij} \\ \sin(\frac{\pi\alpha}{2})\Omega_{4ij} & \sin(\frac{\pi\alpha}{2})\Omega_{5ij} & \cos(\frac{\pi\alpha}{2})\Omega_{4ij} & -\cos(\frac{\pi\alpha}{2})\Omega_{6ij} \\ \cos(\frac{\pi\alpha}{2})\Omega_{2ij} & \cos(\frac{\pi\alpha}{2})\Omega_{3ij} & \sin(\frac{\pi\alpha}{2})\Omega_{1ij} & \sin(\frac{\pi\alpha}{2})\Omega_{3ij} \\ -\cos(\frac{\pi\alpha}{2})\Omega_{4ij} & \cos(\frac{\pi\alpha}{2})\Omega_{6ij} & \sin(\frac{\pi\alpha}{2})\Omega_{4ij} & \sin(\frac{\pi\alpha}{2})\Omega_{5ij} \end{bmatrix}.
$$

From (17), if the LMI condition $\Xi_{ij} < 0$ holds for $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, r$, then $\Upsilon < 0$ holds, it means that $Q < 0$. According to Lemma 2, the fractional-order system on $\tilde{x}(t)$ is asymptotically stable, namely $x(t) \to 0$ and $e(t) \to 0$ as $t \to +\infty$.

## 4. ILLUSTRATIVE EXAMPLE

In this section, a fuzzy controller with the fuzzy observer is designed to stabilize the following nonlinear fractional-order system as

$$\begin{aligned} D^{1.2}x_1(t) =& -2x_1(t) + x_2^2(t) \\ D^{1.2}x_2(t) =& x_1(t) - 3x_2(t) + x_2^2(t) + (1 + x_2(t))u(t) \\ y(t) =& x_2(t) \end{aligned} \tag{18}$$

Assuming that $x_2(t) \in [a,\ b]$ is measurable and $x_1(t)$ is unknown, then let $z(t) = x_2(t)$, hence the fuzzy T-S model for (18) is exactly represented as follows.

Rule 1: If $x_2(t)$ is $a$

$$\text{Then} \begin{cases} D^{1.2}x = A_1x(t) + B_1u(t) \\ y(t) = C_1x(t) \end{cases},$$

Rule 2: If $x_2(t)$ is $b$

$$\text{Then} \begin{cases} D^{1.2}x = A_2x(t) + B_2u(t) \\ y(t) = C_2x(t) \end{cases},$$

where $x(t) = [x_1(t), x_2(t)]^{\mathrm{T}}$, $y(t) = x_2(t)$. The premise membership functions and the consequent matrices in each subsystem are given as:

$$M_1(x_2(t)) = \frac{b - x_2(t)}{b - a},\ M_2(x_2(t)) = \frac{x_2(t) - a}{b - a},$$

$$A_1 = \begin{bmatrix} -2 & a \\ 1 & -3 + a \end{bmatrix},\ A_2 = \begin{bmatrix} -2 & b \\ 1 & -3 + b \end{bmatrix},\ B_1 = \begin{bmatrix} 0 \\ 1 + a \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0 \\ 1 + b \end{bmatrix},\ C_1 = [0\ 1],\ C_2 = [0\ 1].$$

The initial values of $x(t)$ are set as $x(0) = [2, -3]^{\mathrm{T}}$. The maximum and minimum values of $x_2(t)$ are set as $a = -5$ and $b = 5$. We solve the LMIs in Theorem 1 by the LMI Toolbox in MATLAB, the feedback and observer gain matrices are calculated as:

$$K_1 = [-0.3748, -1.0044],\ K_2 = [-0.3433, -1.0186],$$

$$G_1 = \begin{bmatrix} -5.4000 \\ -7.1812 \end{bmatrix},\ G_2 = \begin{bmatrix} 4.6003 \\ 2.8197 \end{bmatrix}.$$

Supposing $x_1(t)$ is unmeasurable and $x_2(t)$ is measurable which exists in the membership functions, the controller and observer for the nonlinear fractional-order system (18) are designed as:

the controller:

$$u(t) = \sum_{j=1}^{2} h_jK_j\hat{x}(t), \tag{19}$$

the observer:

$$D^{1.2}\hat{x}(t) = \sum_{i=1}^{2} h_i[A_i\hat{x}(t) + B_iu(t) + G_i(y(t) - \hat{y}(t))], \tag{20}$$

where $\hat{y}(t) = \sum_{j=1}^{2} h_jC_j\hat{x}(t)$, $\hat{x}(t) = [\hat{x}_1(t), \hat{x}_2(t)]^{\mathrm{T}}$ and

$$h_1 = \frac{b - x_2(t)}{b - a},\ h_2 = \frac{x_2(t) - a}{b - a}.$$

Assuming the initial values of the fuzzy observer are set as 0, the responses of $x(t)$ and $e(t)$ are shown in Fig. 3 and Fig. 4. Fig. 5 illustrates the control input.

The simulation results show that the state $x(t)$ converges to the equilibrium. The nonlinear fractional-order system with the order $\alpha = 1.2$ is stable under the fuzzy observer

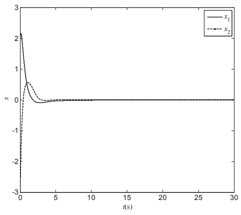

Fig. 3. Responses of the state $x(t)$ for $\alpha = 1.2$

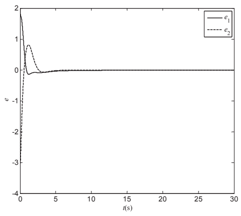

Fig. 4. Responses of the observer error $e(t)$ for $\alpha = 1.2$

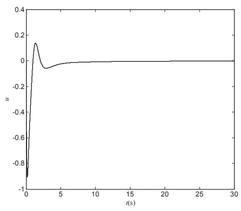

Fig. 5. Control input for $\alpha = 1.2$

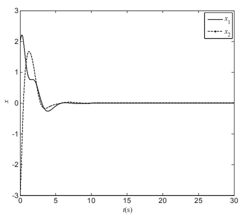

Fig. 6. Responses of the state $x(t)$ for $\alpha = 1.5$

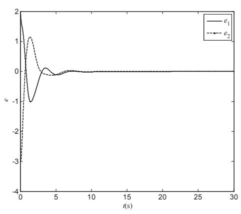

Fig. 7. Responses of the observer error $e(t)$ for $\alpha = 1.5$

designed by Theorem 1 and the observer error $e(t)$ converges to zero.

We increase the fractional order as $\alpha = 1.5$ and other parameters and the initial values are not changed. The feedback and observer gain matrices are calculated as:

$$K_1 = [-0.15,\ -0.9149],\ K_2 = [0.0211,\ -0.9248],$$

$$G_1 = \begin{bmatrix} -5.8504 \\ -6.7416 \end{bmatrix},\ G_2 = \begin{bmatrix} 4.1772 \\ 3.2576 \end{bmatrix}.$$

The controller and observer are also adopted (19) and (20), then the responses of the state and observer are shown in Fig. 6 and Fig. 7. Fig. 8 illustrates the control input.

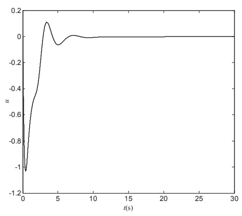

Fig. 8. Control input for $\alpha = 1.5$

As shown in Fig. 6 and Fig. 7, the fractional-order system with $\alpha = 1.5$ is also stable by Theorem 1 and the observer error converges to zero. But if we set $\alpha = 1.6$, the feasible solution of the common matrices $X$ and $Y$ can not be got by the LMI conditions in Theorem 1, because the stable region becomes smaller.

## 5. CONCLUSION

In this paper, an output control for nonlinear fractional-order systems based on the fuzzy observer via fuzzy T-S models has been proposed. The fuzzy control for integer-order control based on the fuzzy T-S model is widely studied, but few researches for fractional-order systems have been studied so far. By supposing that the premise variables are measurable, we designed fuzzy observers and controllers for nonlinear fractional-order systems for $1 < \alpha < 2$ by the LMI approach. Using the properties of the Kronecker product and Lemma 2, the LMI conditions are presented to calculate the feedback and observer gain matrices. The stable region is decided by the fractional order as illustrated in Fig 1. With the increase of the number of the fuzzy rules and the fractional-order, the common positive definite matrices $X$ and $Y$ are hard to obtain, hence the future work is to relax the LMI conditions for fractional-order systems.

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