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Self-organising disturbance attenuation in unidirectionally coupled synchronised systems

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Abstract: This paper proposes a self-organising networked controller for disturbance attenuation in multi-agent systems. A disturbance affecting a single agent has some effect on all neighbouring agents through the communication network. To avoid large disturbance effects, the proposed self-organising controller switches off the communication whenever the effect of the disturbance on the corresponding agent exceeds a given bound. As a consequence, the structure of the networked controller is adjusted to the current disturbance. It is proved that the proposed controller bounds the effect of arbitrary disturbances on all undisturbed agents. The results are illustrated by their application to a robot formation.

Keywords: Self-organising controller, synchronization, autonomous agents, networked control system, vehicle platoon.

1. INTRODUCTION

This paper deals with the disturbance attenuation in synchronised multi-agent systems. A networked controller should make the output $y_i(t)$ of all agents P_i, $(i = 0, 1, ..., N)$ follow the synchronous trajectory $y_s(t) = y_0(t)$, which is prescribed by the leading agent P_0 (Fig. 1). The main aim is to show that the disturbance attenuation of the overall system can be improved if the local controllers C_i adjust their mutual communication to the current disturbances. If a disturbance $d_i(t)$ is too large, the controller C_i of the affected agent temporarily interrupts sending its output $y_i(t)$ towards its neighbours (dashed arrow in the figure). Accordingly the set $\mathcal{Y}_j$ of information that the neighbouring controllers C_j receive changes. As the controllers C_i decide upon the interruption of the communication independently of each other based only on the current information that is locally available at the corresponding agent, the networked controller has the property of self-organisation.

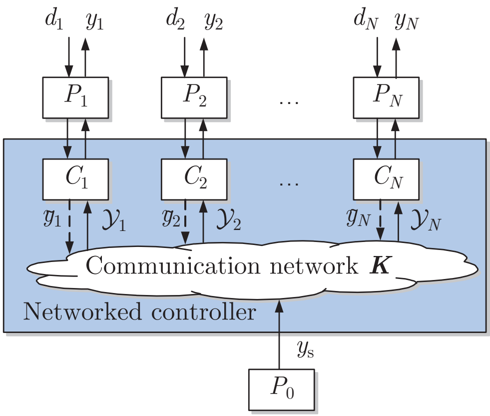

Fig. 1: Synchronisation of autonomous agents by a networked controller

To precisely formulate the performance requirements, consider the situation, where all agents start in the same initial state

$$\boldsymbol{x}_{10} = \boldsymbol{x}_{20} = ... = \boldsymbol{x}_{N0} = \boldsymbol{x}_{s0} \tag{1}$$

and, hence, the undisturbed agents follow the synchronous trajectory:

$$y_1(t) = y_2(t) = ... = y_N(t) = y_s(t). \tag{2}$$

A self-organising networked controller shall be found that bounds the effect of any disturbance $d_i(t)$ on all undisturbed agents such that the synchronisation error of the undisturbed agents does not exceed a bound $\bar{s}$:

$$|y_i(t) - y_s(t)| \leq \bar{s}, \quad i \in \bar{\mathcal{D}},\ t \geq 0, \tag{3}$$

where $\bar{\mathcal{D}} = \{i \,|\, d_i(t) = 0, t \geq 0\}$ is the set of indices of the undisturbed agents.

Main idea: Cutting the communication from disturbed towards undisturbed agents. As the disturbances $d_i(t)$ of the agents P_i, $(i \notin \bar{\mathcal{D}})$ may be arbitrarily large, the control aim (3) can only be satisfied if the undisturbed agents P_i, $(i \in \bar{\mathcal{D}})$ are decoupled from the disturbances $d_i(t)$, $(i \notin \bar{\mathcal{D}})$. This decoupling necessitates to cut the communication links from the controllers of the disturbed agents towards the controllers of the undisturbed agents. The paper proposes self-organising controllers C_i, $(i = 1, 2, ..., N)$ that interrupt their communication towards the controllers of the neigbouring agents in time intervals in which the effect of the disturbance $d_i(t)$ on the agent output $y_i(t)$ exceeds a bound $\bar{r}$.

The main problem to be solved when elaborating such a control scheme results from the fact that the synchronisation error $y_i(t) - y_{si}(t)$ at agent P_i can be brought about either by a disturbance $d_i(t)$ acting on that agent P_i or by a change of the information $\mathcal{Y}_i$ that the controller C_i receives and uses to determine the local reference trajectory $y_{si}(t)$ (for details cf. eqn. (13)). Only in the first situation the controller C_i should interrupt its communication towards other controllers. The first result of this paper proves that every local observer O_i (described by eqn. (16)), which is included in the local controller C_i, reconstructs the effect of the disturbance $d_i(t)$ on the agent P_i (Theorem 1). Hence, the observer output $r_i(t)$ can be used to decide upon the communication.

The second result is a method for evaluating the maximum effect of any set of disturbances on the undisturbed agents (Theorem 2), which can be used to choose the switching threshold $\bar{r}$ so as to satisfy the control aim (3).

Literature survey. In the control literature on synchronisation, the focus has been laid on the design of distributed controllers for sets of identical subsystems [2, 4, 7, 8, 9]. For the synchronisation with time-varying communication structures it has been shown that in order to synchronise autonomous agents in leader-follower structures, the communication graph has to possess a spanning tree with the leader as root node. The connectivity of the communication graph has to be retained during sufficiently long time intervals or, for stochastic coupling, in the sense of average couplings [3, 6, 10].

All these results concern communication structures, the time dependence of which is given and does not occur as a reaction on the current disturbances. The novelty of the method developed in this paper lies in the fact that the time variation of the communication network is invoked deliberately by the agents in order to satisfy the control aim (3).

Notation. Scalars are denoted by italics $(k_{ij}, y_i(t))$, vectors by lower case boldface letters $(\boldsymbol{y}(t), \boldsymbol{x}(t))$ and matrices by upper case boldface letters $(\boldsymbol{A}, \boldsymbol{K})$. $\boldsymbol{k}^{\mathrm{T}}$ is the transposed vector, $\boldsymbol{I}_r$ an r-dimensional unity matrix and $\otimes$ the symbol of the Kronecker product. In structured matrices, sometimes the vanishing blocks are suppressed for the clarity of notation. The relations $>$, $\leq$, $>$, $\geq$ and $|.|$ apply elementwise for vectors and matrices.

Sets are denoted by calligraphic letters like $\mathcal{P}$. $|\mathcal{P}|$ denotes the cardinality (number of elements) of the set $\mathcal{P}$.

2. MODELS

2.1 Agent model

The agents are described by the state-space model

$$P_i : \begin{cases} \dot{\boldsymbol{x}}_i(t) = \boldsymbol{A}\boldsymbol{x}_i(t) + \boldsymbol{b}u_i(t) + \boldsymbol{e}d_i(t), \ \boldsymbol{x}_i(0) = \boldsymbol{x}_{i0} \\ y_i(t) = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}_i(t) \end{cases} \tag{4}$$

$(i = 1, 2, ..., N)$ with

- $u_i(t)$ – scalar input of the i-th agent,
- $d_i(t)$ – scalar disturbance input to the i-th agent,
- $\boldsymbol{x}_i(t)$ – n-dimensional state vector,
- $y_i(t)$ – scalar output of the i-th agent.

They should be synchronised at the trajectory $y_{\mathrm{s}}(t)$ generated by the leader

$$P_0 : \begin{cases} \dot{\boldsymbol{x}}_{\mathrm{s}}(t) = \boldsymbol{A}\boldsymbol{x}_{\mathrm{s}}(t), \quad \boldsymbol{x}_{\mathrm{s}}(0) = \boldsymbol{x}_{\mathrm{s}0} \\ y_{\mathrm{s}}(t) = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}_{\mathrm{s}}(t). \end{cases} \tag{5}$$

The following assumptions are used:

- The communication is instantaneous and lossless.
- The agents may only communicate their output $y_i(t)$ to each other.
- The agents P_i are completely controllable by the input $u_i(t)$ and completely observable through the output $y_i(t)$.

2.2 Communication structure

The communication structure of the controller is represented by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = \{0, 1, 2, ..., N\}$ denoting the set of vertices (agents) and $\mathcal{E}$ the set of directed edges (communication links used). The vertex $0 \in \mathcal{V}$ represents the reference system P_0. A directed edge $(i \to j) \in \mathcal{E}$ shows that the output $y_i(t)$ of agent P_i is communicated to the controller C_j.

The sets of predecessors of agent P_i are defined as

$$\mathcal{P}_i = \{j \,|\, (j \to i) \in \mathcal{E}\}, \quad i = 0, 1, ..., N. \tag{6}$$

Clearly, $\mathcal{P}_0 = \emptyset$ holds. The graph $\mathcal{G}$ is called *connected*, if there exist paths in the communication graph from the leader P_0 to all followers P_i, $(i = 1, 2, ..., N)$.

This paper restricts the communication to be *unidirectional*, which means that any agent P_i can send its information only towards agents P_j with a larger index j. In the illustrations, the couplings are restricted towards the first m followers (cf. top of Fig. 3). Hence, the set of predecessors of the agent P_i is

$$\bar{\mathcal{P}}_i = \{j \,|\, \max(0, i-m) \leq j < i\}, \quad i = 1, 2, ..., N. \tag{7}$$

Time-varying communication graph. The communication structure of the networked controller proposed in this paper changes over time, which will be explicitly stated as $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t))$ and $\mathcal{P}_i(t) = \{j \,|\, (j \to i) \in \mathcal{E}(t)\}$. The changes are restricted to the interruption of some communication links. If the agent P_i interrupts sending its output information $y_i(t)$ towards its neighbours, all edges starting in the vertex i are deleted from the graph $\mathcal{G}$.

A bar is used to indicate the *basic communication structure*, which is represented by the graph $\bar{\mathcal{G}} = (\mathcal{V}, \bar{\mathcal{E}})$ with all communications links used in the undisturbed system. For $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t))$ the relation $\mathcal{E}(t) \subseteq \bar{\mathcal{E}}, \ (t \geq 0)$ holds. $\bar{\mathcal{P}}_i$ denotes the "basic" set of predecessors of the node i and for all "reduced" sets the relation $\mathcal{P}_i(t) \subseteq \bar{\mathcal{P}}_i, \ (t \geq 0, \ i = 1, 2, ..., N)$ is valid. The deletion of the vertex j from the set $\bar{\mathcal{P}}_i$ is symbolised by $\mathcal{P}_i(t) = \bar{\mathcal{P}}_i \backslash \{j\}$. For $j \notin \bar{\mathcal{P}}_i$ the operator "\" does not have any effect.

Representation of the coupling structure by a labeled graph. For the determination of the reference signal of the agents, a labeled graph $\bar{\mathcal{G}} = (\mathcal{V}, \bar{\mathcal{E}}, \boldsymbol{K})$ is used, where each edge $(j \to i) \in \bar{\mathcal{E}}$ has a real label $k_{ij} > 0$. For $(j \to i) \notin \bar{\mathcal{E}}$, the label is defined to be $k_{ij} = 0$ (cf. eqn. (13)), in particular $k_{ii} = 0, \ (i = 0, 1, ..., N)$. Due to the unidirectional couplings,

$$k_{ij} = 0, \quad i < j \tag{8}$$

holds. All labels k_{ij} form the $(N+1, N+1)$-matrix $\boldsymbol{K} = (k_{ij})$.

For the time-varying labeled graph $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t), \boldsymbol{K})$ the labels k_{ij} remain the same for all edges $(j \to i) \in \mathcal{E}(t)$ as in $\bar{\mathcal{E}}$. The in-degree is generalised to be

$$|\mathcal{P}_i(t)| = \sum_{j \in \mathcal{P}_i(t)} k_{ij}. \tag{9}$$

Assumption 2.1. The basic communication graph $\bar{\mathcal{G}}$ is assumed to have the property

$$\bar{\mathcal{P}}_i \neq \emptyset, \quad i = 1, 2, ..., N. \tag{10}$$

Hence, the basic communication graph is connected.

2.3 Networked controller

The networked controller consists of the communication network and the local controllers C_i, $(i = 1, 2, ..., N)$, each of which includes a feedback F_i, a state observer O_i, and a decision component D_i (Fig. 2).

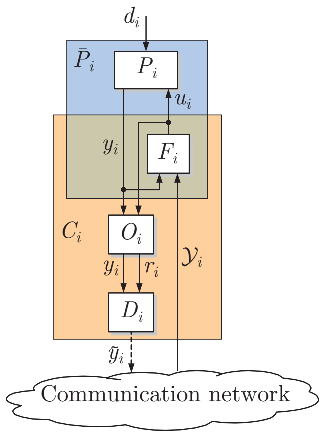

Fig. 2: Analysis of a single agent

The *feedback component* determines the local control error

$$e_i(t) = y_{si}(t) - y_i(t) \tag{11}$$

and feeds it back to the agent input:

$$F_i : \quad u_i(t) = -k(y_i(t) - y_{si}(t)), \quad i = 1, 2, ..., N. \tag{12}$$

The local reference trajectory $y_{si}(t)$ is determined by using the information $\mathcal{Y}_i(t) = \{y_j(t) \,|\, j \in \mathcal{P}_i(t)\}$ received according to

$$y_{si}(t) = \frac{1}{|\mathcal{P}_i(t)|} \sum_{j \in \mathcal{P}_i(t)} k_{ij} y_j(t) = \sum_{j \in \mathcal{P}_i(t)} \tilde{k}_{ij}(t) y_j(t) \tag{13}$$

with $y_0(t) = y_s(t)$ and $\tilde{k}_{ij}(t) = \frac{k_{ij}}{|\mathcal{P}_i(t)|}$ for $\mathcal{P}_i(t) \neq \emptyset$. Consequently,

$$y_{si}(t) = 0, \quad \text{if } \mathcal{P}_i(t) = \emptyset. \tag{14}$$

Due to eqn. (9), the modified weightings $\tilde{k}_{ij}$ satisfy the relation

$$\sum_{j \in \mathcal{P}_i(t)} \tilde{k}_{ij}(t) = 1. \tag{15}$$

The elements $\tilde{k}_{ij}$, $(i, j = 0, 1, ..., N)$ constitute the matrix $\tilde{\boldsymbol{K}}(t) = (\tilde{k}_{ij}(t))$.

The *state observer*

$$O_i : \begin{cases} \dot{\hat{\boldsymbol{x}}}_i(t) = \boldsymbol{A}\hat{\boldsymbol{x}}_i(t) + \boldsymbol{b}u_i(t) + \boldsymbol{l}(y_i(t) - \hat{y}_i(t)) \\ \hat{\boldsymbol{x}}_i(0) = \hat{\boldsymbol{x}}_{i0} \\ \hat{y}_i(t) = \boldsymbol{c}^{\mathrm{T}}\hat{\boldsymbol{x}}_i(t) \\ r_i(t) = y_i(t) - \boldsymbol{c}^{\mathrm{T}}\hat{\boldsymbol{x}}_i(t) \end{cases} \tag{16}$$

generates the output signal $r_i(t)$ to be used by the decision component.

The information sent by the *decision component* D_i towards the controller of the neighbouring agents is denoted by $\tilde{y}_i(t)$. As long as the agent is not sufficiently disturbed, the equality $\tilde{y}_i(t) = y_i(t)$ holds. If the observer output exceeds a given bound $\bar{r}$ $(|r_i(t)| > \bar{r})$, the decision component D_i interrupts the communication and sends the symbolic value ε to indicate that no output information is available:

$$\tilde{y}_i(t) = \begin{cases} y_i(t) & \text{if } |r_i(t)| \leq \bar{r} \\ \varepsilon & \text{otherwise.} \end{cases} \tag{17}$$

If $\mathcal{I}(t)$ denotes the set of the indices of all agents that have interrupted their communication at time t

$$\mathcal{I}(t) = \{i \,|\; |r_i(t)| > \bar{r}\}, \tag{18}$$

then the set of agents that currently send information towards the agent P_i is given by

$$\mathcal{P}_i(t) = \bar{\mathcal{P}}_i \setminus \mathcal{I}(t), \quad i = 1, 2, ..., N. \tag{19}$$

As eqn. (13) is applied to the information $\tilde{y}_i(t)$ received over the communication network, it has to be written as

$$y_{si}(t) = \frac{1}{|\mathcal{P}_i(t)|} \sum_{j \in \mathcal{P}_i(t)} k_{ij}\tilde{y}_j(t) = \sum_{j \in \mathcal{P}_i(t)} \tilde{k}_{ij}(t)\, \tilde{y}_j(t). \tag{20}$$

As $\tilde{y}_j(t) = y_j(t)$ holds for all $j \in \mathcal{P}_i(t)$, the result is the same. A distinction has to be made between both versions of this equation in Section 4 in the analysis of the disturbance behaviour of the overall system after communication links have been switched off.

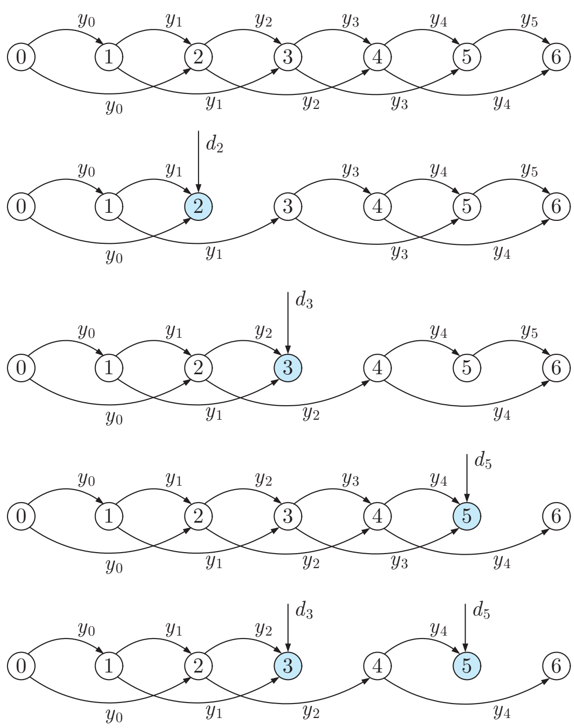

Fig. 3: Five different structures of the networked controller that appear due to different disturbance situations $(m = 2)$

The effect of the self-organising adaptation of the information structure of the networked controller is illustrated in Fig. 3. Without the switching (19), the disturbances have influence, through the communication network, on all following agents. As the communication network does not only contribute to the synchronisation of the overall system, but also to the penetration of a disturbance through the overall system, the communication is interrupted if the effect of the disturbance exceeds the bound $\bar{r}$.

2.4 Controlled agents

The controlled agent (4), (12) is described by

$$\bar{P}_i : \begin{cases} \dot{\boldsymbol{x}}_i(t) = (\boldsymbol{A} - \boldsymbol{b}k\boldsymbol{c}^{\mathrm{T}})\boldsymbol{x}_i(t) + \boldsymbol{b}k y_{\mathrm{s}i}(t) + \boldsymbol{e}d_i(t) \\ \boldsymbol{x}_i(0) = \boldsymbol{x}_{i0} \\ y_i(t) = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}_i(t). \end{cases} \tag{21}$$

The feedback gain k should satisfy the following assumption:

Assumption 2.2. The feedback gain k is chosen so as to make the matrix $\boldsymbol{A} - \boldsymbol{b}k\boldsymbol{c}^{\mathrm{T}}$ asymptotically stable.

Note that such a stabilising feedback gain k exists if and only if the agents are synchronisable [4].

Standard analysis methods for the synchronisation of identical agents show that under the Assumptions 2.1 and 2.2, the undisturbed overall system is asymptotically synchronised for all initial states $\boldsymbol{x}_{i0}$:

$$\lim_{t\to\infty} |y_i(t) - y_j(t)| = 0, \quad \forall i, j = 1, 2, ..., N. \tag{22}$$

3. RECONSTRUCTION OF THE DISTURBANCE EFFECT

This section investigates how the effect of the disturbance $d_i(t)$ on the behaviour of the controlled agent $\bar{P}_i$ can be reconstructed by means of the observer O_i. As the agent (21) has the two inputs $d_i(t)$ and $y_{\mathrm{s}i}(t)$, its output can be represented as sum

$$y_i(t) = \underbrace{\boldsymbol{c}^{\mathrm{T}}\,\mathrm{e}^{(\boldsymbol{A} - \boldsymbol{b}k\boldsymbol{c}^{\mathrm{T}})t}\,\boldsymbol{x}_{i0}}_{y_{0i}(t)} + \underbrace{g_{\mathrm{yd}} * d_i}_{y_{\mathrm{d}i}(t)} + \underbrace{g_{\mathrm{yy}} * y_{\mathrm{s}i}}_{y_{\mathrm{y}i}(t)} \tag{23}$$

of the free motion $y_{0i}(t)$, the disturbance behaviour $y_{\mathrm{d}i}(t)$ and the reference behaviour $y_{\mathrm{y}i}(t)$ with the impulse responses

$$g_{\mathrm{yd}}(t) = \boldsymbol{c}^{\mathrm{T}}\,\mathrm{e}^{(\boldsymbol{A} - \boldsymbol{b}k\boldsymbol{c}^{\mathrm{T}})t}\,\boldsymbol{e} \tag{24}$$

$$g_{\mathrm{yy}}(t) = \boldsymbol{c}^{\mathrm{T}}\,\mathrm{e}^{(\boldsymbol{A} - \boldsymbol{b}k\boldsymbol{c}^{\mathrm{T}})t}\,\boldsymbol{b}k \tag{25}$$

and the asterisk denoting the convolution operation.

Theorem 1. Consider the observer O_i represented by eqn. (16) for

$$\boldsymbol{l} = \boldsymbol{b}k \tag{26}$$

with k satisfying Assumption 2.2. The observer output $r_i(t)$ asymptotically describes the disturbance behaviour $y_{\mathrm{d}i}(t)$ of the controlled agent $\bar{P}_i$:

$$\lim_{t\to\infty} |r_i(t) - y_{\mathrm{d}i}(t)| = 0. \tag{27}$$

Proof. Equations (12), (16) and (21) yield

$$\begin{pmatrix} \dot{\boldsymbol{x}}_i(t) \\ \dot{\hat{\boldsymbol{x}}}_i(t) \end{pmatrix} = \begin{pmatrix} \boldsymbol{A} - \boldsymbol{b}k\boldsymbol{c}^{\mathrm{T}} & \boldsymbol{O} \\ \boldsymbol{l}\boldsymbol{c}^{\mathrm{T}} - \boldsymbol{b}k\boldsymbol{c}^{\mathrm{T}} & \boldsymbol{A} - \boldsymbol{l}\boldsymbol{c}^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_i(t) \\ \hat{\boldsymbol{x}}_i(t) \end{pmatrix} + \begin{pmatrix} \boldsymbol{b}k \\ \boldsymbol{b}k \end{pmatrix} y_{\mathrm{s}i}(t) + \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{0} \end{pmatrix} d_i(t)$$

$$\begin{pmatrix} \boldsymbol{x}_i(0) \\ \hat{\boldsymbol{x}}_i(0) \end{pmatrix} = \begin{pmatrix} \boldsymbol{x}_{i0} \\ \hat{\boldsymbol{x}}_{i0} \end{pmatrix}$$

$$r_i(t) = (\boldsymbol{c}^{\mathrm{T}} \quad -\boldsymbol{c}^{\mathrm{T}}) \begin{pmatrix} \boldsymbol{x}_i(t) \\ \hat{\boldsymbol{x}}_i(t) \end{pmatrix}.$$

This model is transformed by

$$\begin{pmatrix} \boldsymbol{x}_i(t) \\ \boldsymbol{e}_i(t) \end{pmatrix} = \begin{pmatrix} \boldsymbol{I} & \\ \boldsymbol{I} & -\boldsymbol{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_i(t) \\ \hat{\boldsymbol{x}}_i(t) \end{pmatrix}$$

and reduced to its controllable and observable part

$$\Delta O_i : \begin{cases} \dot{\boldsymbol{e}}_i(t) = (\boldsymbol{A} - \boldsymbol{l}\boldsymbol{c}^{\mathrm{T}})\boldsymbol{e}_i(t) + \boldsymbol{e}d_i(t) \\ \boldsymbol{e}_i(0) = \boldsymbol{x}_{i0} - \hat{\boldsymbol{x}}_{i0} \\ r_i(t) = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{e}_i(t). \end{cases} \tag{28}$$

For the observer feedback (26) this model leads to

$$r_i(t) = \underbrace{\boldsymbol{c}^{\mathrm{T}}\,\mathrm{e}^{(\boldsymbol{A} - \boldsymbol{b}k\boldsymbol{c}^{\mathrm{T}})t}(\boldsymbol{x}_{i0} - \hat{\boldsymbol{x}}_{i0})}_{r_{0i}(t)} + g_{\mathrm{yd}} * d_i$$

with $g_{\mathrm{yd}}(t)$ defined by eqn. (24). Due to Assumption 2.2, the first addend vanishes asymptotically. As the second addend coincides with the second term in eqn. (23), the claim (27) is obtained. □

Interpretation. Theorem 1 shows that the observer can be used to reconstruct the effect of the unknown disturbance $d_i(t)$ on the behaviour of the controlled agent $\bar{P}_i$. After the free motion $r_{0i}(t)$ of the observer, which is initiated by the unknown state difference $\boldsymbol{x}_{i0} - \hat{\boldsymbol{x}}_{i0}$, has asymptotically vanished, the observer O_i delivers the "pure" disturbance effect $y_{\mathrm{d}i}(t)$. This fact is true independently of the reference signal $y_{\mathrm{s}i}(t)$ that the agent should follow. Consequently, the observer is a means to decide whether a non-vanishing control error $y_{\mathrm{s}i}(t) - y_i(t)$ is brought about by a changing reference signal $y_{\mathrm{s}i}(t)$ or by the disturbance $d_i(t)$ affecting the agent $\bar{P}_i$. In both cases a non-vanishing control error occurs. However, in the first case the observer generates an asymptotically vanishing output ($r_i(t) \to 0$), whereas in the second case the observer output shows the effect of the disturbance on the agent output ($r_i(t) \to y_{\mathrm{d}i}(t)$).

Therefore, the following assumption is used:

Assumption 3.1. The observer feedback is chosen according to eqn. (26).

4. DISTURBANCE BEHAVIOUR OF THE OVERALL SYSTEM

This section analyses the disturbance behaviour of the overall system and shows that this behaviour is improved if disturbed agents interrupt the communication to other agents according to eqn. (17). The following situation is considered in the analysis:

- The agents are initially synchronised due to the common initial state (1).

- The agents are classified as disturbed ($i \notin \bar{\mathcal{D}}$) and undisturbed ($i \in \bar{\mathcal{D}}$) and this fact is represented by the scalars

$$\alpha_i = \begin{cases} 1 & \text{if } i \notin \bar{\mathcal{D}} \\ 0 & \text{otherwise,} \end{cases} \qquad i = 0, 1, ..., N.$$

The following assumption is made:

Assumption 4.1. The overall system remains connected, which implies

$$\mathcal{P}_i(t) \neq \emptyset, \quad i = 1, 2, ..., N, \ t \geq 0.$$

The investigations of this section assume that the transient behaviour $r_{i0}(t)$ of all observers O_i has vanished: $r_i(t) = y_{\mathrm{d}i}(t)$. They deal with the deviations

$$\begin{aligned} \Delta y_i(t) &= y_i(t) - y_{\mathrm{s}}(t) \\ \Delta \tilde{y}_i(t) &= \tilde{y}_i(t) - y_{\mathrm{s}}(t) \\ \Delta y_{\mathrm{s}i}(t) &= y_{\mathrm{s}i}(t) - y_{\mathrm{s}}(t), \quad i = 0, 1, ..., N \end{aligned}$$

of the three local signals $y_i(t)$, $\tilde{y}_i(t)$, $y_{si}(t)$ from the synchronous trajectory $y_{\mathrm{s}}(t)$. Equation (17) leads to

$$\Delta \tilde{y}_i(t) = \begin{cases} \Delta y_i(t) & \text{if } |r_i(t)| \leq \bar{r} \\ 0 & \text{otherwise,} \end{cases} \tag{29}$$

where $\Delta \tilde{y}_i(t)$ is defined to vanish if the output $y_i(t)$ is not sent to the neighbouring agents.

Theorem 2. Consider a synchronised overall system (4), (5), (12), (16), (17), (20) subject to an arbitrary disturbance $\boldsymbol{d}(t)$. Under the Assumption 3.1 the effect of the disturbances on the synchronised system is bounded from above by

$$|\Delta \tilde{\boldsymbol{y}}(t)| \leq \overline{\Delta \boldsymbol{y}} = \left(\boldsymbol{I} - M_{\mathrm{yy}} \boldsymbol{K}_{\mathrm{d}}\right)^{-1} \alpha \bar{r} \tag{30}$$

with

$$M_{\mathrm{yy}} = \int_0^\infty |g_{\mathrm{yy}}(\tau)| \, \mathrm{d}\tau$$

$$\boldsymbol{K}_{\mathrm{d}} = (k_{\mathrm{d}ij}) \quad \text{with} \quad k_{\mathrm{d}ij} = \frac{k_{ij}}{|\bar{\mathcal{P}}_i| - \sum_{j \in \bar{\mathcal{P}}_i} \alpha_j}$$

$$\alpha = (\alpha_0 \ \ \alpha_1 \ldots \alpha_N)^{\mathrm{T}}.$$

The inverse matrix exists, because $\boldsymbol{K}_{\mathrm{d}}$ is a lower-triangular matrix with vanishing diagonal elements (cf. eqn. (15)). Hence, the matrix $\boldsymbol{I} - M_{\mathrm{yy}} \boldsymbol{K}_{\mathrm{d}}$ to be inverted has the eigenvalue 1 with multiplicity $N + 1$. Furthermore, this matrix is an M-matrix [1] and, hence, its inverse is a nonnegative matrix.

Interpretation. Theorem 2 shows that the effect of an arbitrary disturbance $\boldsymbol{d}(t)$ on a synchronised system can be bounded by a self-organising controller. The communication is broken down by the control units of the disturbed agents for the time interval in which the effect of the disturbance exceeds a threshold $\bar{r}$. Note that for this decision the controller uses only locally available information.

Equation (30) is relevant only for the components $\Delta \tilde{y}_i(t)$, $i \in \bar{\mathcal{D}}$ of the undisturbed agents, because for the other agents, eqn. (29) defines this signal to vanish. Hence, the theorem provides an upper bound for the deviation of the output of the undisturbed agents from the synchronous trajectory.

With the result of the theorem, a switching threshold $\bar{r}$ can be determined such that the requirement (3) on the disturbance effect is satisfied. Whether or not this requirement can be met, depends upon the disturbance situation. If, for example, only the k-th agent is disturbed and the effect of the disturbance $d_k(t)$ on the neighbouring $(k+1)$-st agent should be bounded, eqn. (30) yields

$$\overline{\Delta y_{k+1}} = \tilde{\boldsymbol{e}}^{\mathrm{T}} \left(\boldsymbol{I} - M_{\mathrm{yy}} \boldsymbol{K}_{\mathrm{d}}\right)^{-1} \tilde{\alpha} \bar{r}$$

with $\tilde{\boldsymbol{e}}^{\mathrm{T}}$ denoting a vector with the $(k+1)$-st element equal to 1 and all other elements vanishing and with $\tilde{\alpha}$ being a vector with the elements $\alpha_k = 1$ and $\alpha_i = 0$, $(i \neq k)$. Hence, requirement (3) on the $(k+1)$-st agent is satisfied if the switching threshold is chosen to be

$$\bar{r} < \frac{\bar{s}}{\tilde{\boldsymbol{e}}^{\mathrm{T}} \left(\boldsymbol{I} - M_{\mathrm{yy}} \boldsymbol{K}_{\mathrm{d}}\right)^{-1} \tilde{\alpha}}.$$

5. EXAMPLE: ROBOT POSITIONING PROBLEM

Consider the positioning problem for N robots illustrated by Fig. 4. Robot P_0 generates the reference position

$$y_{\mathrm{s}}(t) = s_0 + \bar{v} t \tag{31}$$

that all other robots should assume. The networked controller has the communication structure shown on top of Fig. 3.

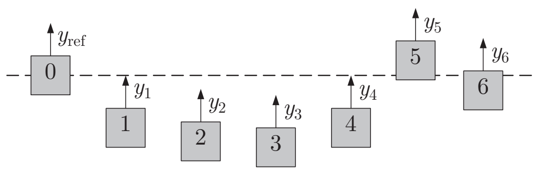

Fig. 4: Positioning problem for robots

For initially synchronised robots, Fig. 5 shows the disturbance $d_3(t)$ applied to the Robot P_3 together with the output $r_3(t)$ of the observer O_3. Around $t = 0$ the non-zero observer output results from the deviation of the initial state $\hat{\boldsymbol{x}}_{30}$ of the observer from the initial state $\boldsymbol{x}_{30}$ of Robot P_3. When after time $t = 100$ s the disturbance brings about deviations of the robot position $y_3(t)$ from the synchronous trajectory $y_{\mathrm{s}3}(t) = y_{\mathrm{s}}(t)$ as indicated by the signal $r_3(t)$, the communication of the position from Robot P_3 towards the neighbouring robots is interrupted. The time intervals, in which this interruption occurs ($\tilde{y}_3(t) = \varepsilon$), are marked by black bars in the lower part of the figure. The tolerance used in the switching rule (18), (19) was set to $\bar{r} = 1\,\mathrm{m}$.

It is interesting to see that the interruption of communication does occur only in short time intervals compared to the long time interval in which the disturbance $d_3(t)$ has a large magnitude. The reason for the intermediate recovery of communication is the ability of the local feedback F_3 to attenuate a constant disturbance $d_3(t)$.

Figure 6 shows the behaviour of the overall system with three disturbances affecting the Robots 2, 3 and 5. The black bars in the second subplot show the time intervals in which the communication is interrupted. The performance of the overall

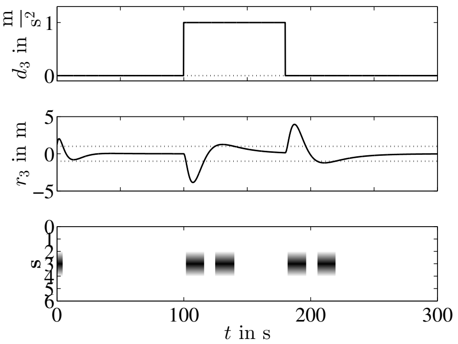

Fig. 5: Disturbance $d_3(t)$ and observer output $r_3(t)$

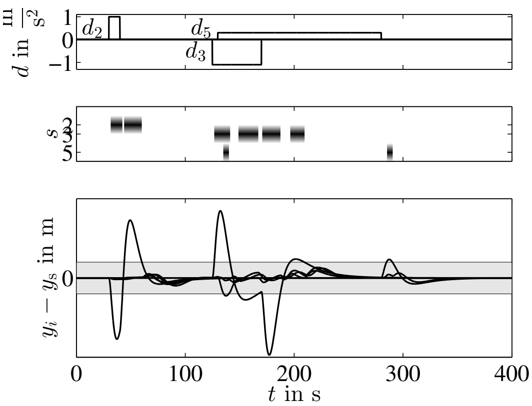

Fig. 6: Behaviour of the overall system with self-organising controller

system with switching controller shown in the lower part of the figure has to be compared with the performance of the robot formation controlled by the non-switching controller shown in Fig. 7. The grey band marks the switching threshold $\bar{r}$.

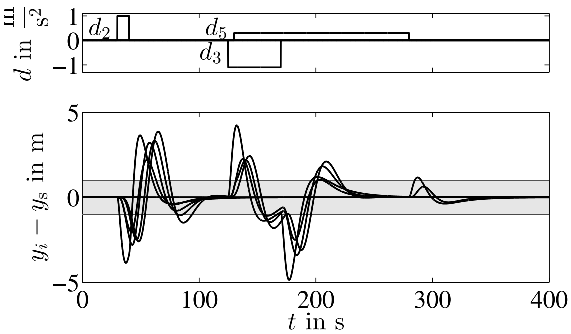

Fig. 7: Behaviour of the overall system with non-switching controller subject to the same disturbance as in Fig. 6

As the disturbances $d_2(t)$, $d_3(t)$ and $d_5(t)$ act in different time intervals, the robots react to these disturbances in different, possibly overlapping time intervals with interrupting their communication. Hence, five different controller structures occur (Fig. 3), all of which satisfy Assumption 4.1. Hence, the undisturbed robots are synchronised, i. e. satisfy the requirement (22).

Figure 3 illustrates that self-organisation leads to a control structure that is adjusted to the current disturbances affecting three robots. Note that the robots act completely independently of each other and by using local information only.

6. CONCLUSIONS AND OUTLOOK

The paper proposes a self-organising networked controller where each local unit C_i has three typical components:

- an observer to evaluate the current state of the agent,
- a decision logic that decides when to change the communication and
- a local feedback that solves the control task for the agent.

These three components have been taylored to the disturbance attenuation task considered here but can be extended to other control tasks.

The results have been derived in this paper for identical single-input single-output agents, but they can be generalised without difficult problems for multiple-input multiple-output systems and for agents with individual dynamics.

The restriction to cycle-free coupling graphs simplify the analysis in the sense that cutting communication links cannot endanger the stability of the overall system. The extension to communication graphs with cycles poses the difficult problem that before switching off communication links the local controllers C_i have to test whether this action will jeopardise the stability and, hence, the synchrony of the overall system. The local controllers have to be enabled to carry out this test by using local information only. The second problem is to prove that switching off the communication leads indeed to an improved disturbance behaviour of the remaining agents.

The analysis of this paper is valid only if the Assumption 4.1 holds. This assumption ensures that the overall system remains synchronisable with the reduced communication structure. Whether or not the overall system satisfies this requirement depends upon the current disturbance. In the basic communication graph defined in eqn. (7), disturbances acting on m neigbouring agents and reaching a simultaneous interruption of the communication make the overall system fall apart into independent subsystems. Then the overall system cannot be synchronised any longer, which demonstrates the necessity of Assumption 4.1.

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APPENDIX: PROOF OF THEOREM 2

Equations (15) and (20) yield

$$\Delta y_{\mathrm{s}i}(t) = \sum_{j \in \mathcal{P}_i(t)} \tilde{k}_{ij}(t) \Delta \tilde{y}_j(t). \tag{32}$$

$$\begin{aligned}\max_t |\Delta y_{\mathrm{s}i}(t)| &\leq \sum_{j \in \mathcal{P}_i(t)} \max_t \tilde{k}_{ij}(t) \cdot \max_t |\Delta \tilde{y}_j(t)| \\ &\leq \sum_{j \in \bar{\mathcal{P}}_i} \frac{k_{ij}}{\min_t |\mathcal{P}_i(t)|} \cdot \max_t |\Delta \tilde{y}_j(t)|. \\ &\leq \sum_{j \in \bar{\mathcal{P}}_i} \frac{k_{ij}}{|\bar{\mathcal{P}}_i| - \sum_{j \in \bar{\mathcal{P}}_i} \alpha_j} \max_t |\Delta \tilde{y}_j(t)|. \end{aligned} \tag{33}$$

Next, assuming that the reference signal $y_{\mathrm{s}i}(t)$ of the agent deviates from the synchronous signal $y_{\mathrm{s}}(t)$ by $\Delta y_{\mathrm{s}i}(t)$, a bound on the signal $\Delta \tilde{y}_i(t)$ should be found. For the leading agent P_0, the relation $\Delta \tilde{y}_0(t) = 0$, $(t \geq 0)$ holds. For all other agents, the model (21) of the initially synchronised controlled agent yields

$$\bar{P}_i: \begin{cases} \dot{\boldsymbol{x}}_i(t) = (\boldsymbol{A} - \boldsymbol{b}k\boldsymbol{c}^{\mathrm{T}})\boldsymbol{x}_i(t) + \boldsymbol{b}k(y_{\mathrm{s}}(t) + \Delta y_{\mathrm{s}i}(t)) \\ \qquad\quad + \boldsymbol{e}d_i(t), \quad \boldsymbol{x}_i(0) = \boldsymbol{x}_{\mathrm{s}0} \\ y_i(t) = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}_i(t). \end{cases}$$

After introducing the difference state $\Delta \boldsymbol{x}_i(t) = \boldsymbol{x}_i(t) - \boldsymbol{x}_{\mathrm{s}}(t)$, the following representation of $\Delta y_i(t)$ is obtained:

$$\Delta\bar{P}_i: \begin{cases} \Delta\dot{\boldsymbol{x}}_i(t) = (\boldsymbol{A} - \boldsymbol{b}k\boldsymbol{c}^{\mathrm{T}})\Delta\boldsymbol{x}_i(t) + \boldsymbol{b}k\Delta y_{\mathrm{s}i}(t) \\ \qquad\quad + \boldsymbol{e}d_i(t), \quad \Delta\boldsymbol{x}_i(0) = \boldsymbol{0} \\ \Delta y_i(t) = \boldsymbol{c}^{\mathrm{T}}\Delta\boldsymbol{x}_i(t). \end{cases}$$

Hence,

$$\Delta y_i(t) = g_{\mathrm{yy}} * \Delta y_{\mathrm{s}i} + y_{\mathrm{d}i}(t)$$

holds with the impulse response $g_{\mathrm{yy}}(t)$ defined in eqn. (25). A bound for $|\Delta y_i(t)|$ can be obtained as follows:

$$\begin{aligned}|\Delta y_i(t)| &= \left| \int_0^t g_{\mathrm{yy}}(t-\tau)\, \Delta y_{\mathrm{s}i}(\tau)\, \mathrm{d}\tau + y_{\mathrm{d}i}(t) \right| \\ &\leq \int_0^\infty |g_{\mathrm{yy}}(\tau)|\, \mathrm{d}\tau \cdot \max_t |\Delta y_{\mathrm{s}i}(t)| + |y_{\mathrm{d}i}(t)|.\end{aligned}$$

The information $\tilde{y}_i(t)$ sent to the neighbouring agents deviates from the synchronous trajectory by $\Delta \tilde{y}_i(t)$ with

$$\max_t |\Delta \tilde{y}_i(t)| \leq \begin{cases} M_{\mathrm{yy}} \cdot \max_t |\Delta y_{\mathrm{s}i}(t)| + \alpha_i \bar{r} & \text{if } |r_i(t)| \leq \bar{r} \\ 0 & \text{otherwise.} \end{cases} \tag{34}$$

The factor α_i has been introduced in order to indicate that for all undisturbed agents $\bar{P}_i$, $(i \in \bar{\mathcal{D}})$ with $\alpha_i = 0$ the relation $y_{\mathrm{d}i}(t) = 0$ holds and, hence, the upper bound does not have a term related to the disturbance $d_i(t)$.

Equations (33) and (35) yield

$$\begin{aligned}\max_t |\Delta \tilde{y}_i(t)| &\leq M_{\mathrm{yy}} \cdot \max_t |\Delta y_{\mathrm{s}i}(t)| + \alpha_i \bar{r} \\ &\leq M_{\mathrm{yy}} \sum_{j \in \bar{\mathcal{P}}_i} \frac{k_{ij}}{\min_t |\mathcal{P}_i(t)|} \cdot \max_t |\Delta \tilde{y}_j(t)| + \alpha_i \bar{r}\end{aligned}$$

and in vector notation

$$\begin{pmatrix} \max_t |\Delta \tilde{y}_0(t)| \\ \max_t |\Delta \tilde{y}_1(t)| \\ \vdots \\ \max_t |\Delta \tilde{y}_N(t)| \end{pmatrix} \leq M_{\mathrm{yy}} \boldsymbol{K}_{\mathrm{d}} \begin{pmatrix} \max_t |\Delta \tilde{y}_0(t)| \\ \max_t |\Delta \tilde{y}_1(t)| \\ \vdots \\ \max_t |\Delta \tilde{y}_N(t)| \end{pmatrix} + \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix} \bar{r}$$

with

$$\min_t |\mathcal{P}_i(t)| = \bar{\mathcal{P}}_i - \sum_{j \in \bar{\mathcal{P}}_i} \alpha_j.$$

The inequality can be reformulated as

$$\begin{pmatrix} \max_t |\Delta \tilde{y}_0(t)| \\ \max_t |\Delta \tilde{y}_1(t)| \\ \vdots \\ \max_t |\Delta \tilde{y}_N(t)| \end{pmatrix} \leq (\boldsymbol{I} - M_{\mathrm{yy}} \boldsymbol{K}_{\mathrm{d}})^{-1} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix} \bar{r}$$

where the inverse matrix exists and is nonnegative. Hence, the final result (30) follows. □