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Adaptive Observer for Switching Linear Parameter-Varying (LPV) Systems ⋆

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Abstract: In this paper, the problem of joint state and parameter estimation in switching Linear Parameter-Varying (LPV) systems is considered. The proposed solution relies on an adaptive observer, which is designed finding a solution to a system of Trilinear Matrix Inequalities (TMIs). It is shown that, under certain assumptions, the TMIs can be reduced to Linear Matrix Inequalities (LMIs) that can be solved using available software. An example of a four wheeled omnidirectional mobile robot subject to unknown offsets in the motor voltages is used to illustrate the efficiency of the proposed approach.

Keywords: Adaptive observer, continuous-time systems, switching linear parameter-varying (LPV) systems, state and parameter estimation.

1. INTRODUCTION

Adaptive observers simultaneously observe the state and identify all the parameters of a system using only the input and the output signals (Lüders and Narendra, 1973). Many variations of this technique have been proposed in the literature, see e.g. (Li et al., 2011) and (Farrell and Polycarpou, 2006). Adaptive observers are of great interest, as they can guarantee stable estimation for systems, whose dynamics change during operation, e.g. due to faults (Zhang et al., 2008), or systems with poorly modeled dynamics and high modeling uncertainty (Reppa et al., 2013).

In the last decades, the Linear Parameter-Varying (LPV) paradigm has become a standard formalism in systems and control (Shamma, 2012). By embedding the system nonlinearities in some varying parameters that depend on endogenous signals (e.g. system states), gain-scheduling of nonlinear systems can be performed using extensions of linear techniques (in this case, the system is referred to as *quasi-LPV*). The LPV analysis and synthesis conditions can be formulated as Linear Matrix Inequalities (LMIs) using a single Lyapunov function over the entire parameter space (Becker and Packard, 1994). However, in some cases, due to the loss of feasibility of the LMIs or the inherent switching modes of the system, it may be needed to split the parameter region into subregions, and switch among them during the LPV system operation. Thus, the LPV system is transformed into a new class of system, referred to as *switching LPV system* (Lu and Wu, 2004).

In this paper, motivated by the design of adaptive observers for time-varying systems (Zhang, 2002), an adaptive observer is proposed within the context of switching LPV systems. Notice that the adaptive observer proposed by Zhang (2002) was used by Gáspár et al. (2006) for identifying LPV systems. However, in Gáspár et al. (2006), only the state observer part of the overall scheme is designed using LMI-based techniques. In contrast, in the present paper, the whole scheme is formulated in an LPV framework, such that the switching LPV adaptive observer can be designed finding a solution to a system of LMIs using efficient solvers available nowadays (Löfberg, 2004), (Sturm, 1999).

The paper is structured as follows: Section 2 presents the problem formulation. In Section 3, the proposed switching LPV adaptive observer is presented. The design conditions are given in Section 4. Section 5 presents the application example. Simulation results are shown in Section 6. Finally, the main conclusions are summarized in Section 7.

2. PROBLEM FORMULATION

Consider a continuous-time switching LPV system of the form:

$$\dot{x}(t) = A_\sigma(\vartheta(t))\,x(t) + B_\sigma(\vartheta(t))\,u(t) + \Psi_\sigma(\vartheta(t))\,\xi(t) \quad (1)$$
$$\dot{\xi}(t) = E_\sigma(\vartheta(t))\,\xi(t) \quad (2)$$
$$y(t) = C_\sigma(\vartheta(t))\,x(t) \quad (3)$$

where $x(t) \in \mathbb{R}^{n_x}$, $u(t) \in \mathbb{R}^{n_u}$, $y(t) \in \mathbb{R}^{n_y}$ are, respectively, the state, input and output of the system, $\xi(t) \in \mathbb{R}^{n_\xi}$ is an unknown parameter vector, $A_\sigma(\vartheta(t))$, $B_\sigma(\vartheta(t))$, $C_\sigma(\vartheta(t))$, $E_\sigma(\vartheta(t))$, $\Psi_\sigma(\vartheta(t))$ are known matrices of appropriate sizes whose structure and dependence on the vector of varying parameters $\vartheta(t) \in \Theta \subset \mathbb{R}^{n_\vartheta}$ depend on the value of the switching signal $\sigma \in \{1,\ldots,S\} \subset \mathbb{N}^+$, that is assumed to be known. It is also assumed that the parameter set Θ is partitioned into a finite number of subsets $\{\Theta_i\}_{i\in\{1,\ldots,S\}}$ by means of a family of switching surfaces. The value of the switching signal σ determines which parameter subset is active, and thus determines the dynamic behavior of the system.

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If (1) is rewritten as:

$$\begin{aligned}\dot{x}(t) &= \left[A_\sigma(\vartheta(t)) + L_\sigma(\vartheta(t))C_\sigma(\vartheta(t))\right]x(t) \\ &+ B_\sigma(\vartheta(t))u(t) - L_\sigma(\vartheta(t))y(t) + \Psi_\sigma(\vartheta(t))\xi(t)\end{aligned} \tag{4}$$

with some matrices $L_\sigma(\vartheta(t))$, it can be seen that two different *exogenous excitations* contribute to the generation of $x(t)$, namely the *known* exogenous excitation $B_\sigma(\vartheta(t))u(t) - L_\sigma(\vartheta(t))y(t)$ and the *unknown* exogenous excitation $\Psi_\sigma(\vartheta(t))\xi(t)$.

Thus, $x(t)$ can be split into $x(t) = x_u(t) + x_\xi(t)$, with:

$$\begin{aligned}\dot{x}_u(t) &= \left[A_\sigma(\vartheta(t)) + L_\sigma(\vartheta(t))C_\sigma(\vartheta(t))\right]x_u(t) \\ &+ B_\sigma(\vartheta(t))u(t) - L_\sigma(\vartheta(t))y(t)\end{aligned} \tag{5}$$

$$\begin{aligned}\dot{x}_\xi(t) &= \left[A_\sigma(\vartheta(t)) + L_\sigma(\vartheta(t))C_\sigma(\vartheta(t))\right]x_\xi(t) \\ &+ \Psi_\sigma(\vartheta(t))\xi(t)\end{aligned} \tag{6}$$

The problem considered in this paper is the joint estimation of $x_u(t)$, $x_\xi(t)$ and $\xi(t)$ from $u(t)$, $y(t)$ and $\vartheta(t)$.

Remark 1: It is worth noting that while $\vartheta(t)$ is assumed to be a known parameter vector that can be used to schedule the control system matrices, $\xi(t)$ is an unknown parameter vector that may represent an unexpected change of dynamics due, e.g, to a fault.

3. SWITCHING LPV ADAPTIVE OBSERVER

Let us estimate $x_u(t)$ using the observer:

$$\begin{aligned}\dot{\hat{x}}_u(t) &= \left[A_\sigma(\vartheta(t)) + L_\sigma(\vartheta(t))C_\sigma(\vartheta(t))\right]\hat{x}_u(t) \\ &+ B_\sigma(\vartheta(t))u(t) - L_\sigma(\vartheta(t))y(t)\end{aligned} \tag{7}$$

while $x_\xi(t)$ is estimated using:

$$\begin{aligned}\dot{\hat{x}}_\xi(t) &= \left[A_\sigma(\vartheta(t)) + L_\sigma(\vartheta(t))C_\sigma(\vartheta(t))\right]\hat{x}_\xi(t) \\ &+ \Psi_\sigma(\vartheta(t))\hat{\xi}(t) + \omega(t)\end{aligned} \tag{8}$$

where $\hat{\xi}(t)$ is an estimation of $\xi(t)$ computed using:

$$\begin{aligned}\dot{\hat{\xi}}(t) &= E_\sigma(\vartheta(t))\hat{\xi}(t) + \Gamma_\sigma(\vartheta(t))\big(y(t) \\ &- C_\sigma(\vartheta(t))\hat{x}_u(t) - C_\sigma(\vartheta(t))\hat{x}_\xi(t)\big)\end{aligned} \tag{9}$$

with $\Gamma_\sigma(\vartheta(t)) \in \mathbb{R}^{n_\xi \times n_y}$, while $\omega(t)$ is an extra term added in order to compensate the effect of the estimation error made by $\hat{\xi}(t)$, computed as follows:

$$\begin{aligned}\omega(t) &= \Omega_\sigma(\vartheta(t))\Gamma_\sigma(\vartheta(t))\big(y(t) \\ &- C_\sigma(\vartheta(t))\hat{x}_u(t) - C_\sigma(\vartheta(t))\hat{x}_\xi(t)\big)\end{aligned} \tag{10}$$

with $\Omega_\sigma(\vartheta(t)) \in \mathbb{R}^{n_x \times n_\xi}$.

If the following changes of variables are introduced:

$$\tilde{x}_u(t) \triangleq \hat{x}_u(t) - x_u(t) \tag{11}$$

$$\tilde{x}_\xi(t) \triangleq \hat{x}_\xi(t) - x_\xi(t) \tag{12}$$

$$\tilde{\xi}(t) \triangleq \hat{\xi}(t) - \xi(t) \tag{13}$$

then, the estimation error dynamics is described by:

$$\begin{bmatrix}\dot{\tilde{x}}_u(t) \\ \dot{\tilde{x}}_\xi(t) \\ \dot{\tilde{\xi}}(t)\end{bmatrix} = \Xi_\sigma(\vartheta(t))\begin{bmatrix}\tilde{x}_u(t) \\ \tilde{x}_\xi(t) \\ \tilde{\xi}(t)\end{bmatrix} \tag{14}$$

with:

$$\Xi_\sigma(\vartheta(t)) = \begin{bmatrix}\Xi_\sigma^{ALC}(\vartheta(t)) & 0 \\ \Xi_{\sigma,12}(\vartheta(t)) & \Xi_{\sigma,22}(\vartheta(t))\end{bmatrix} \tag{15}$$

$$\Xi_{\sigma,12}(\vartheta(t)) = \begin{bmatrix}\Xi_\sigma^{\Omega\Gamma C}(\vartheta(t)) \\ \Xi_\sigma^{\Gamma C}(\vartheta(t))\end{bmatrix} \tag{16}$$

$$\Xi_{\sigma,22}(\vartheta(t)) = \begin{bmatrix}\Xi_\sigma^{ALC}(\vartheta(t)) + \Xi_\sigma^{\Omega\Gamma C}(\vartheta(t)) & \Psi_\sigma(\vartheta(t)) \\ \Xi_\sigma^{\Gamma C}(\vartheta(t)) & E_\sigma(\vartheta(t))\end{bmatrix} \tag{17}$$

where:

$$\Xi_\sigma^{ALC}(\vartheta(t)) = A_\sigma(\vartheta(t)) + L_\sigma(\vartheta(t))C_\sigma(\vartheta(t)) \tag{18}$$

$$\Xi_\sigma^{\Omega\Gamma C}(\vartheta(t)) = -\Omega_\sigma(\vartheta(t))\Gamma_\sigma(\vartheta(t))C_\sigma(\vartheta(t)) \tag{19}$$

$$\Xi_\sigma^{\Gamma C}(\vartheta(t)) = -\Gamma_\sigma(\vartheta(t))C_\sigma(\vartheta(t)) \tag{20}$$

Hence, the overall design consists in finding the switching LPV state observer gain $L_\sigma(\vartheta(t))$, the switching LPV adaptive observer gain $\Omega_\sigma(\vartheta(t))$ and the switching LPV parameter estimator gain $\Gamma_\sigma(\vartheta(t))$ such that (14) is stable and satisfies some constraints on the pole location.

Since the matrix $\Xi(\vartheta(t))$ defined in (15) is in lower block-triangular form, the *separation principle* is valid, and the stability of the system (14) can be obtained from the stability of the subsystems in the diagonal. Hence, the overall design can be split into two parts:

1. finding the gain $L_\sigma(\vartheta(t))$ such that $\Xi_\sigma^{ALC}(\vartheta(t))$ is stable and satisfies some constraints on the pole location $\forall\vartheta \in \Theta$;
2. finding the gains $\Omega_\sigma(\vartheta(t))$ and $\Gamma_\sigma(\vartheta(t))$ such that $\Xi_{\sigma,22}(\vartheta(t))$ is stable and satisfies some constraints on the pole location $\forall\vartheta \in \Theta$.

4. STABILITY ANALYSIS AND DESIGN OF THE ADAPTIVE OBSERVER

In this work, the unknown gains $L_\sigma(\vartheta(t))$, $\Omega_\sigma(\vartheta(t))$ and $\Gamma_\sigma(\vartheta(t))$ are designed considering that the switchings can happen arbitrarily in time, such that a common parameter-dependent Lyapunov function is employed to derive sufficient Linear Matrix Inequalities (LMI) conditions for the switching LPV systems He et al. (2010). A family of LPV gains can be designed according to the LMI conditions, such that each of them is suitable for the corresponding parameter region.

More specifically, the system (14) is stable if there exists a symmetric positive definite matrix function $X(\vartheta(t))$, which is smooth over the parameter set Θ, such that the function:

$$V(\tilde{x}(t), \vartheta(t)) = \tilde{x}^\top(t)X(\vartheta(t))\tilde{x}(t) \tag{21}$$

where $\tilde{x}(t) = \left[\,\tilde{x}_u(t)\ \tilde{x}_\xi(t)\ \tilde{\xi}(t)\,\right]^\top$, has negative definite derivative over the entire parameter trajectory. This is equivalent to the satisfaction of the following condition $\forall\vartheta \in \Theta$ and $\forall\sigma \in \{1,\ldots,S\}$:

$$\Xi_\sigma(\vartheta)X(\vartheta) + X(\vartheta)\Xi_\sigma(\vartheta)^T + \dot{X}(\vartheta) < 0 \tag{22}$$

In addition to stability, another specification taken into consideration in this paper is the satisfaction of pole location constraints. This is motivated by Ghersin and Sanchez-Peña (2002), where a connection between the idea of *poles* for LPV systems, defined as the set of all the poles of the LTI systems obtained by freezing $\vartheta(t)$ to all its possible values $\vartheta \in \Theta$, and the dynamic behavior of the system itself was found. The techniques used for the design and analysis of LPV systems using pole placement constraints are based on Chilali and Gahinet (1996), where subsets $\mathscr{D}$ of the complex plane, referred to as *LMI regions*, are introduced as:

$$\mathscr{D} = \{z \in \mathbb{C} : f_{\mathscr{D}}(z) < 0\} \tag{23}$$

where $f_{\mathscr{D}}$ is the *characteristic function*, defined as:

$$f_{\mathscr{D}}(z) = \alpha + z\beta + \bar{z}\beta^\top = [\alpha_{kl} + \beta_{kl}z + \beta_{lk}\bar{z}]_{k,l\in[1,m]} \tag{24}$$

with $\alpha = \alpha^\top \in \mathbb{R}^{m\times m}$ and $\beta \in \mathbb{R}^{m\times m}$. Hence, the system (14) has its poles in $\mathscr{D}$ if there exists a symmetric positive definite matrix function $X(\vartheta(t))$ such that $\forall\vartheta \in \Theta$ and $\forall\sigma \in \{1,\ldots,S\}$:

$$\left[\alpha_{kl} X(\vartheta) + \beta_{kl} \Xi_\sigma(\vartheta) X(\vartheta) + \beta_{lk} X(\vartheta) \Xi_\sigma(\vartheta)^\top\right]_{k,l\in[1,m]} < 0 \tag{25}$$

In the following, for design simplicity, the case of a fixed matrix function X will be analyzed.

4.1 State observer design

The problem of the state observer design is to find $L_\sigma(\vartheta(t))$ such that $\Xi_\sigma^{ALC}(\vartheta(t))^\top$ is stable and has its poles in $\mathscr{D}_L$, defined as:

$$\mathscr{D}_L = \left\{z \in \mathbb{C} : f_{\mathscr{D}_L}(z) = \alpha_L + z\beta_L + \bar{z}\beta_L^\top < 0\right\} \tag{26}$$

The problem is solved if there exists a symmetric positive definite matrix X_L such that $\forall\vartheta \in \Theta$ and $\forall\sigma \in \{1,\ldots,S\}$ the following inequalities hold:

$$\Xi_\sigma^{ALC}(\vartheta)^\top X_L + X_L \Xi_\sigma^{ALC}(\vartheta) < 0 \tag{27}$$

$$\left[\alpha_{L,kl} + \beta_{L,kl} \Xi_\sigma^{ALC}(\vartheta)^\top X_L + \beta_{L,lk} X_L \Xi_\sigma^{ALC}(\vartheta)\right]_{k,l\in[1,m]} < 0 \tag{28}$$

These conditions cannot be used directly for the analysis/design purpose, since they correspond to an infinite number of constraints. In order to reduce them to a finite number, the assumption that $A_\sigma(\vartheta(t))$ and $C_\sigma(\vartheta(t))$ are polytopic is made:

$$A_\sigma(\vartheta(t)) = \sum_{i=1}^{N_{\sigma,A}} \alpha_{\sigma,i}(\vartheta(t)) A_{\sigma,i} \tag{29}$$

$$C_\sigma(\vartheta(t)) = \sum_{i=1}^{N_{\sigma,C}} \gamma_{\sigma,i}(\vartheta(t)) C_{\sigma,i} \tag{30}$$

with $\alpha_{\sigma,i} \geq 0$, $\gamma_{\sigma,i} \geq 0$, $\sum_{i=1}^{N_{\sigma,A}} \alpha_{\sigma,i}(\vartheta(t)) = 1$ and $\sum_{i=1}^{N_{\sigma,C}} \gamma_{\sigma,i}(\vartheta(t)) = 1$. Then, if $L_\sigma(\vartheta(t))$ is designed to be polytopic as well, i.e.:

$$L_\sigma(\vartheta(t)) = \sum_{i=1}^{N_{\sigma,A}} \alpha_{\sigma,i}(\vartheta(t)) L_{\sigma,i} \tag{31}$$

conditions (27)-(28) are reduced to:

$$\left(\Xi_{\sigma,ij}^{ALC}\right)^\top X_L + X_L \Xi_{\sigma,ij}^{ALC} < 0 \tag{32}$$

$$\left[\alpha_{L,kl} X_L + \beta_{L,kl}\left(\Xi_{\sigma,ij}^{ALC}\right)^\top X_L + \beta_{L,lk} X_L \Xi_{\sigma,ij}^{ALC}\right]_{k,l\in[1,m]} < 0 \tag{33}$$

with:

$$\Xi_{\sigma,ij}^{ALC} = A_{\sigma,i} + L_{\sigma,i} C_{\sigma,j} \tag{34}$$

for $i = 1,\ldots,N_{\sigma,A}$, $j = 1,\ldots,N_{\sigma,C}$, $\sigma = 1,\ldots,S$. Inequalities (32)-(33) are Bilinear Matrix Inequalities (BMIs) that can be brought to LMI form introducing the change of variables:

$$T_{\sigma,i} \triangleq L_{\sigma,i}^\top X_L \tag{35}$$

Hence, given that the matrices X_L and $T_{\sigma,i}$ have been found, the gains $L_{\sigma,i}$ can be obtained as:

$$L_{\sigma,i} = \left(T_{\sigma,i} X_L^{-1}\right)^\top = X_L^{-1} T_{\sigma,i}^\top \tag{36}$$

4.2 Adaptive observer design

The problem of the adaptive observer design is to find $\Gamma_\sigma(\vartheta(t))$ and $\Omega_\sigma(\vartheta(t))$ such that:

$$\Delta_\sigma(\vartheta(t)) = \begin{bmatrix} \Xi_\sigma^{ALC}(\vartheta(t))^\top + \Xi_\sigma^{\Omega\Gamma C}(\vartheta(t))^\top & \Xi_\sigma^{\Gamma C}(\vartheta(t))^\top \\ \Psi_\sigma(\vartheta(t))^\top & E_\sigma(\vartheta(t))^\top \end{bmatrix} \tag{37}$$

is stable and has its poles in $\mathscr{D}_\xi$, defined as:

$$\mathscr{D}_\xi = \left\{z \in \mathbb{C} : f_{\mathscr{D}_\xi}(z) = \alpha_\xi + z\beta_\xi + \bar{z}\beta_\xi^T < 0\right\} \tag{38}$$

The problem is solved if there exists a symmetric positive definite matrix X_ξ such that $\forall\vartheta \in \Theta$ and $\forall\sigma \in \{1,\ldots,S\}$ the following hold:

$$\Delta_\sigma(\vartheta) X_\xi + X_\xi \Delta_\sigma(\vartheta)^\top < 0 \tag{39}$$

$$\left[\alpha_{\xi,kl} X_\xi + \beta_{\xi,kl} \Delta_\sigma(\vartheta) X_\xi + \beta_{\xi,lk} X_\xi \Delta_\sigma(\vartheta)^\top\right]_{k,l\in[1,m]} < 0 \tag{40}$$

Similarly to the previous case, we assume that $A_\sigma(\vartheta(t))$, $C_\sigma(\vartheta(t))$, $E_\sigma(\vartheta(t))$ and $\Psi_\sigma(\vartheta(t))$ are polytopic, characterized by (29)-(30) and:

$$E_\sigma(\vartheta(t)) = \sum_{i=1}^{N_{\sigma,E}} \varepsilon_{\sigma,i}(\vartheta(t)) E_{\sigma,i} \tag{41}$$

$$\Psi_\sigma(\vartheta(t)) = \sum_{i=1}^{N_{\sigma,\Psi}} \psi_{\sigma,i}(\vartheta(t)) \Psi_{\sigma,i} \tag{42}$$

with $\varepsilon_{\sigma,i} \geq 0$, $\psi_{\sigma,i} \geq 0$, $\sum_{i=1}^{N_{\sigma,E}} \varepsilon_{\sigma,i}(\vartheta(t)) = 1$ and $\sum_{i=1}^{N_{\sigma,\Psi}} \psi_{\sigma,i}(\vartheta(t)) = 1$. Then, if $\Gamma_\sigma(\vartheta(t))$ is chosen as a constant matrix for a fixed value of σ, and $\Omega_\sigma(\vartheta(t))$ is designed polytopic, as follows:

$$\Gamma_\sigma(\vartheta(t)) = \Gamma_\sigma \tag{43}$$

$$\Omega_\sigma(\vartheta(t)) = \sum_{i=1}^{N_{\sigma,A}} \alpha_i(\vartheta(t)) \sum_{\mu=1}^{N_{\sigma,E}} \varepsilon_\mu(\vartheta(t)) \sum_{n=1}^{N_{\sigma,\Psi}} \psi_n(\vartheta(t)) \Omega_{\sigma,i\mu n} \tag{44}$$

conditions (39)-(40) are reduced to:

$$\Delta_{\sigma,ij\mu n} X_\xi + X_\xi \Delta_{\sigma,ij\mu n}^\top < 0 \tag{45}$$

$$\left[\alpha_{\xi,kl} X_\xi + \beta_{\xi,kl} \Delta_{\sigma,ij\mu n} X_\xi + \beta_{\xi,lk} X_\xi \Delta_{\sigma,ij\mu n}^\top\right]_{k,l\in[1,m]} < 0 \tag{46}$$

for $i = 1,\ldots,N_{\sigma,A}$, $j = 1,\ldots,N_{\sigma,C}$, $\mu = 1,\ldots,N_{\sigma,E}$, $n = 1,\ldots,N_{\sigma,\Psi}$ and $\sigma = 1,\ldots,S$, and where:

$$\Delta_{\sigma,ij\mu n} = \begin{bmatrix} \left(\Xi_{\sigma,ij}^{ALC}\right)^\top + \left(\Xi_{\sigma,ij\mu n}^{\Omega\Gamma C}\right)^\top & \left(\Xi_{\sigma,j}^{\Gamma C}\right)^\top \\ \Psi_{\sigma,n}^\top & E_{\sigma,\mu}^\top \end{bmatrix} \tag{47}$$

$$\Xi_{\sigma,ij\mu n}^{\Omega\Gamma C} = -\Omega_{\sigma,i\mu n} \Gamma_\sigma C_{\sigma,j} \tag{48}$$

$$\Xi_{\sigma,j}^{\Gamma C} = -\Gamma_\sigma C_{\sigma,j} \tag{49}$$

with $\Xi_{\sigma,ij}^{ALC}$ defined as in (34). Conditions (45)-(46) are Trilinear Matrix Inequalities (TMIs) that need to be put in LMI form in order to be solved efficiently using the available solvers. If the Lyapunov matrix X_ξ is chosen to be full, as follows:

$$X_\xi = \begin{bmatrix} X_{\xi,11} & X_{\xi,12} \\ X_{\xi,12}^\top & X_{\xi,22} \end{bmatrix} \tag{50}$$

then the following terms arise: $\Gamma_\sigma^\top \Omega_{\sigma,i\mu n}^\top X_{\xi,11}$, $\Gamma_\sigma^\top X_{\xi,12}$, $X_{\xi,11}$, $\Gamma_\sigma^\top \Omega_{\sigma,i\mu n}^\top X_{\xi,12}$, $\Gamma_\sigma^\top X_{\xi,22}$, $X_{\xi,12}$, $X_{\xi,22}$, $X_{\xi,12}^\top \Omega_{\sigma,i\mu n}\Gamma$ and $X_{\xi,22}^\top \Gamma_\sigma^\top$. In this case, it is not possible to find a change of variables that brings the problem to LMI form. However, this issue can be handled, at the expense of introducing additional conservativeness, by choosing the Lyapunov matrix block-diagonal, i.e. forcing $X_{\xi,12} = 0$ in (50). With this choice, only the following terms remain: $\Gamma_\sigma^\top \Omega_{\sigma,i\mu n}^\top X_{\xi,11}$, $\Gamma_\sigma^\top X_{\xi,22}$, $X_{\xi,11}$, $X_{\xi,22}$ and

$X_{\xi,22}^{\top}\Gamma_{\sigma}^{\top}$. Then, if the following change of variables is performed:

$$\Lambda_{\sigma,i\mu n}^{\Omega} = \Gamma_{\sigma}^{\top}\Omega_{\sigma,i\mu n}^{\top}X_{\xi,11} \tag{51}$$

$$\Lambda_{\sigma}^{\Gamma} = \Gamma_{\sigma}^{\top}X_{\xi,22} \tag{52}$$

conditions (45)-(49) become:

$$\begin{bmatrix} A_{\sigma,i}^{\top}X_{\xi,11} + C_{\sigma,j}^{\top}L_{\sigma,i}^{\top}X_{\xi,11} - C_{\sigma,j}^{\top}\Lambda_{\sigma,i\mu n}^{\Omega} & -C_{\sigma,j}^{\top}\Lambda_{\sigma}^{\Gamma} \\ \Psi_n^{\top}X_{\xi,11} & E_{\mu}^{\top}X_{\xi,22} \end{bmatrix} + \begin{bmatrix} X_{\xi,11}A_{\sigma,i} + X_{\xi,11}L_{\sigma,i}C_{\sigma,j} - (\Lambda_{\sigma,i\mu n}^{\Omega})^{\top}C_{\sigma,j} & X_{\xi,11}\Psi_n \\ -(\Lambda_{\sigma}^{\Gamma})^{\top}C_{\sigma,j} & X_{\xi,22}E_{\mu} \end{bmatrix} < 0 \tag{53}$$

$$\left\{ \alpha_{\xi,kl}\begin{bmatrix} X_{\xi,11} & 0 \\ 0 & X_{\xi,22} \end{bmatrix} + \beta_{\xi,kl}\begin{bmatrix} A_{\sigma,i}^{\top}X_{\xi,11} + C_{\sigma,j}^{\top}L_{\sigma,i}^{\top}X_{\xi,11} - C_{\sigma,j}^{\top}\Lambda_{\sigma,i\mu n}^{\Omega} & -C_{\sigma,j}^{\top}\Lambda_{\sigma}^{\Gamma} \\ \Psi_n^{\top}X_{\xi,11} & E_{\mu}^{\top}X_{\xi,22} \end{bmatrix} + \beta_{\xi,lk}\begin{bmatrix} X_{\xi,11}A_{\sigma,i} + X_{\xi,11}L_{\sigma,i}C_{\sigma,j} - (\Lambda_{\sigma,i\mu n}^{\Omega})^{\top}C_{\sigma,j} & X_{\xi,11}\Psi_n \\ -(\Lambda_{\sigma}^{\Gamma})^{\top}C_{\sigma,j} & X_{\xi,22}E_{\mu} \end{bmatrix} \right\}_{k,l\in[1,m]} < 0 \tag{54}$$

Based on the solution of (53)-(54), $\Omega_{\sigma,i\mu n}$ and Γ_{σ} are obtained as:

$$\Gamma_{\sigma} = X_{\xi,22}^{-1}\Lambda_{\sigma}^{\Gamma} \tag{55}$$

$$\Omega_{\sigma,i\mu n}^{\top} = \left(\Lambda_{\sigma}^{\Gamma}X_{\xi,22}^{-1}\right)^{\dagger}\Lambda_{\sigma,i\mu n}^{\Omega}X_{\xi,11}^{-1} \tag{56}$$

Notice that in the special case where $n_y = n_{\xi}$, the pseudo-inverse becomes the inverse and:

$$\Omega_{\sigma,i\mu n}^{\top} = X_{\xi,22}\left(\Lambda_{\sigma}^{\Gamma}\right)^{-1}\Lambda_{\sigma,i\mu n}^{\Omega}X_{\xi,11}^{-1} \tag{57}$$

Otherwise, $\Lambda_{\sigma}^{\Gamma}X_{\xi,22}^{-1}\Omega_{\sigma,i\mu n}^{\top}X_{\xi,11} = \Lambda_{\sigma,i\mu n}^{\Omega}$ is a system of $n_y \times n_x$ equations with $n_{\xi} \times n_x$ variables. Hence, if $n_y < n_{\xi}$, infinite solutions could exist and the pseudo-inverse would select the minimum-norm solution. On the other hand, if $n_y > n_{\xi}$, the solution could not exist, and the pseudo-inverse would select the least squares solution. In this last case, the stability and the satisfaction of performance of the solution should be verified a posteriori.

5. APPLICATION EXAMPLE

In this paper, a four wheeled omnidirectional robot (Oliveira et al., 2009) is used as an example to show the effectiveness of the proposed technique. The robot, subject to unknown offsets in the motor voltages, is described by:

$$\dot{x} = v_x \tag{58}$$

$$\begin{aligned} \dot{v}_x = {} & \left(A_{11}c_{\theta}^2 + A_{22}s_{\theta}^2\right)v_x + \left(A_{11}s_{\theta}c_{\theta} - A_{22}s_{\theta}c_{\theta} - v_{\theta}\right)v_y \\ & -B_{21}s_{\theta}u_0 + B_{12}c_{\theta}u_1 - B_{23}s_{\theta}u_2 + B_{14}c_{\theta}u_3 \\ & +K_{11}c_{\theta}sign\left(v_xc_{\theta} + v_ys_{\theta}\right) - K_{22}s_{\theta}sign\left(-v_xs_{\theta} + v_yc_{\theta}\right) \\ & -B_{21}s_{\theta}\Delta u_0 + B_{12}c_{\theta}\Delta u_1 - B_{23}s_{\theta}\Delta u_2 + B_{14}c_{\theta}\Delta u_3 \end{aligned} \tag{59}$$

$$\dot{y} = v_y \tag{60}$$

$$\begin{aligned} \dot{v}_y = {} & \left(A_{11}s_{\theta}c_{\theta} - A_{22}s_{\theta}c_{\theta} + v_{\theta}\right)v_x + \left(A_{11}s_{\theta}^2 + A_{22}c_{\theta}^2\right)v_y \\ & +B_{21}c_{\theta}u_0 + B_{12}s_{\theta}u_1 + B_{23}c_{\theta}u_2 + B_{14}s_{\theta}u_3 \\ & +K_{11}s_{\theta}sign\left(v_xc_{\theta} + v_ys_{\theta}\right) + K_{22}c_{\theta}sign\left(-v_xs_{\theta} + v_yc_{\theta}\right) \\ & +B_{21}c_{\theta}\Delta u_0 + B_{12}s_{\theta}\Delta u_1 + B_{23}c_{\theta}\Delta u_2 + B_{14}s_{\theta}\Delta u_3 \end{aligned} \tag{61}$$

$$\dot{\theta} = v_{\theta} \tag{62}$$

$$\begin{aligned} \dot{v}_{\theta} = {} & A_{33}v_{\theta} + B_{31}u_0 + B_{32}u_1 + B_{33}u_2 + B_{34}u_3 + K_{33}sign(v_{\theta}) \\ & +B_{31}\Delta u_0 + B_{32}\Delta u_1 + B_{33}\Delta u_2 + B_{34}\Delta u_3 \end{aligned} \tag{63}$$

where (x, y) is the robot position, θ is the angle with respect to the defined front of the robot ($s_{\theta} \triangleq \sin\theta$ and $c_{\theta} \triangleq \cos\theta$), v_x, v_y and v_{θ} are the corresponding linear/angular velocities, and u_0, u_1, u_2 and u_3 the motor voltage applied to the wheels 1, 2, 3 and 4, respectively, while Δu_0, Δu_1, Δu_2 and Δu_3 are unexpected offsets in the motor voltages, that are assumed to be unknown. The values used for the coefficients A_{ii}, B_{ij}, K_{ii} are: $A_{11} = -3.3605$, $A_{22} = -3.4368$, $A_{33} = -5.7363$, $B_{12} = -B_{14} = -B_{21} = B_{23} = -0.3950$, $B_{31} = B_{32} = B_{33} = B_{34} = 3.6079$, $K_{11} = -0.8008$, $K_{22} = -0.9486$, $K_{33} = -6.0746$.

By using the reference model proposed in Rotondo et al. (2014) and defining the tracking errors $e_1 \triangleq x_r - x$, $e_2 \triangleq v_x^r - v_x$, $e_3 \triangleq y_r - y$, $e_4 \triangleq v_y^r - v_y$, $e_5 \triangleq \theta_r - \theta$, $e_6 \triangleq v_{\theta}^r - v_{\theta}$, the new inputs $\delta u_i \triangleq u_i^r - u_i$, $i = 0, 1, 2, 3$ and the unknown parameters $\xi_2 \triangleq B_{21}\Delta u_0 + B_{23}\Delta u_2$, $\xi_4 \triangleq B_{12}\Delta u_1 + B_{14}\Delta u_3$, $\xi_6 \triangleq B_{31}\Delta u_0 + B_{32}\Delta u_1 + B_{33}\Delta u_2 + B_{34}\Delta u_3$, where (x_r, y_r) is the reference vehicle position, θ_r is its angle, v_x^r, v_y^r and v_{θ}^r are the corresponding linear/angular velocities, and u_0^r, u_1^r, u_2^r, u_3^r are the reference inputs (feedforward actions), we obtain an error model that can be brought into a quasi-LPV representation:

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \\ \dot{e}_4 \\ \dot{e}_5 \\ \dot{e}_6 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \vartheta_1 & 0 & \vartheta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \vartheta_3 & 0 & A_{11} + A_{22} - \vartheta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & A_{33} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ -B_{21}\vartheta_4 & B_{12}\vartheta_5 & -B_{23}\vartheta_4 & B_{14}\vartheta_5 \\ 0 & 0 & 0 & 0 \\ B_{21}\vartheta_5 & B_{12}\vartheta_4 & B_{23}\vartheta_5 & B_{14}\vartheta_4 \\ 0 & 0 & 0 & 0 \\ B_{31} & B_{32} & B_{33} & B_{34} \end{pmatrix} \begin{pmatrix} \delta u_0 \\ \delta u_1 \\ \delta u_2 \\ \delta u_3 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \vartheta_4 & 0 & -\vartheta_5 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -\vartheta_5 & 0 & -\vartheta_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_6 \end{pmatrix} \tag{64}$$

where the vector of varying parameters is:

$$\vartheta(t) = \begin{pmatrix} \vartheta_1(t) \\ \vartheta_2(t) \\ \vartheta_3(t) \\ \vartheta_4(t) \\ \vartheta_5(t) \end{pmatrix} = \begin{pmatrix} A_{11}\cos^2\theta + A_{22}\sin^2\theta \\ (A_{11} - A_{22})\sin\theta\cos\theta - v_{\theta} \\ (A_{11} - A_{22})\sin\theta\cos\theta + v_{\theta} \\ \sin\theta \\ \cos\theta \end{pmatrix} \tag{65}$$

and the parameters ξ_1, ξ_3 and ξ_5 have been introduced in order to obtain $n_y = n_{\xi}$ (it is assumed that the state is directly measured), as follows:

$$\dot{x} = v_x + \xi_1 \qquad \dot{y} = v_y + \xi_3 \qquad \dot{\theta} = v_{\theta} + \xi_5$$

If a polytopic approximation of the matrix Ψ is searched, as in (42), some problems would arise due to the singularity occurring for $\vartheta_4 = 0$ and $\vartheta_5 = 0$, values for which this matrix becomes:

$$\Psi_{\vartheta_4=0,\vartheta_5=0} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{66}$$

and a solution to (39)-(40) could not exist. In fact, even though $\vartheta_4 = 0$ and $\vartheta_5 = 0$ do not represent a real operating point of the system, it is inevitably included in any convex polytopic approximation (42) of the admissible values of $\vartheta_4(t) = \sin\theta(t)$ and $\vartheta_5(t) = \cos\theta(t)$. In other words, any convex polytope containing the unit circle will contain the origin too and, as a consequence, the singularity of the matrix Ψ will be included in the design conditions.

However, this issue can be successfully tackled within the switching LPV framework, since the overall system can be represented by an interaction between different LPV systems through discrete switching events, which can depend on states or time. Similarly, the overall adaptive observer is obtained from different LPV adaptive observers that are switched when discrete switching events occur. In particular, in this case, the problem arising due to the singularity (66) can be avoided by splitting the subset of the parameter space generated by ϑ_4 and ϑ_5 in more regions, such that in each region the resulting polytopic approximation does not include the origin. In particular, in this work, the quadrants have been considered as regions, with $\theta = k\pi/2$, $k \in \mathbb{Z}$ being the switching condition, such that:

$$\sigma = \begin{cases} 1 & if \quad \cos\theta \geq 0 \; AND \; \sin\theta \geq 0 \\ 2 & if \quad \cos\theta \geq 0 \; AND \; \sin\theta < 0 \\ 3 & if \quad \cos\theta < 0 \; AND \; \sin\theta < 0 \\ 4 & if \quad \cos\theta < 0 \; AND \; \sin\theta \geq 0 \end{cases}$$

Then, a triangular polytopic approximation has been used in each region.

6. SIMULATION RESULTS

The polytopic approximation (29) of the state matrix in the quasi-LPV model (64) has been obtained by evaluating the bounds of ϑ_1, ϑ_2 and ϑ_3 when the input voltages applied to the motors are within the saturation limits $u_i \in [-u_i^{max}, u_i^{max}]$, $i = 0, \ldots, 3$.

The error observer has been designed using (32) and (33), to assure stability and pole clustering in:

$$\mathscr{D}_L = \{z \in \mathbb{C} : \mathrm{Re}(z) < -100\} \tag{67}$$

The adaptive observer has been designed using (53) and (54), to assure stability and pole clustering in:

$$\mathscr{D}_\xi = \left\{z \in \mathbb{C} : \mathrm{Re}(z) < -0.1, \mathrm{Re}(z)^2 + \mathrm{Im}(z)^2 < 40000\right\} \tag{68}$$

under the assumption that the unknown parameters ξ_i, $i = 1, \ldots, 6$ are constant, that is, $E_\sigma(\vartheta(t)) = 0_{6\times 6}$.

The results shown hereafter refer to a simulation which lasts 100 s, where the four wheeled mobile robot is driven by the model reference switching quasi-LPV error-feedback controller proposed in Rotondo et al. (2014) from the initial state:

$$(x(0), v_x(0), y(0), v_y(0), \theta(0), v_\theta(0))^\top = 0_{6\times 1}$$

to the desired trajectory, generated by the reference model from the initial reference state:

$$(x_r(0), v_x^r(0), y_r(0), v_y^r(0), \theta_r(0), v_\theta^r(0))^\top = (2, 0, 0, \pi/5, 0, \pi/10)^\top$$

and using reference inputs calculated such that the following circular trajectory is obtained:

$$x_r(t) = 2\cos(\theta_r(t)) \;\; y_r(t) = 2\sin(\theta_r(t)) \;\; \theta_r(t) = 2\pi t/20$$

The initial estimation of the unknown parameters has been chosen as:

$$\hat{\xi}_i(0) = 0.1, \; \forall\, i \in \{1, \ldots, 6\} \tag{69}$$

in order to show the convergence to zero of the estimations when the unknown parameters are zero too.

At time 50s, an offset $\Delta u_0 = 3$ appears in the voltage of the motor 0, such that $\xi_2 = B_{21}\Delta u_0 + B_{23}\Delta u_2 \to 1.185$ and $\xi_6 = B_{31}\Delta u_0 + B_{32}\Delta u_1 + B_{33}\Delta u_2 + B_{34}\Delta u_3 \to 10.824$. Fig. 1 shows the error estimation given by the error observer both when the proposed adaptive observer is applied (blue line) and when it is not applied (red line). It can be seen that the presence of the unknown parameters due to Δu_0 generates undesired oscillations and offsets if no countermeasures are taken. The introduction of the adaptive observer drives back, after a short transient, the estimation error to zero.

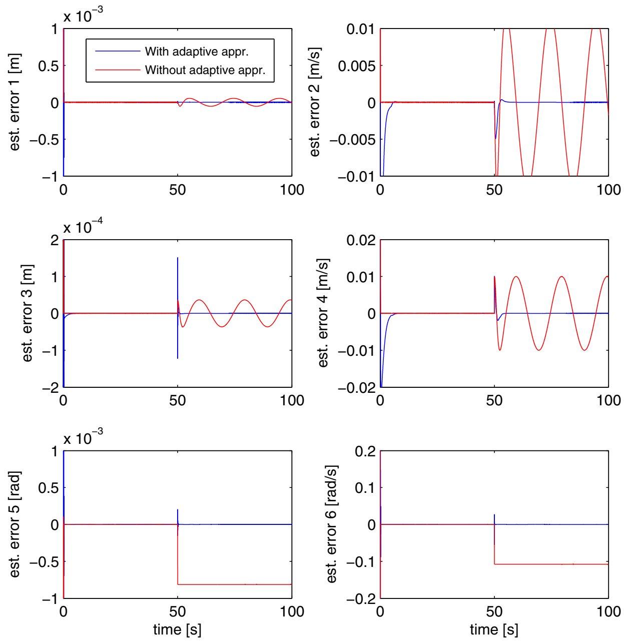

Fig. 1. Observer estimation error with and without the proposed adaptive observer.

The estimation of the unknown parameters is shown in Fig. 2 and Fig. 3, where it can be seen that the steady-state estimation matches the values of the unknown parameters, demonstrating the effectiveness of the proposed technique.

7. CONCLUSIONS

In this paper, an adaptive observer has been proposed within the context of switching LPV systems for solving the problem of joint state and parameter estimation. The state has been splitted in two parts, one excited by the known exogenous input, and the other excited by the unknown parameters. The overall estimation is provided by a switching LPV state observer, a switching LPV adaptive observer and a switching LPV parameter estimator. Thanks to the separation principle, it is possible to split the design into two parts: (1) designing the state observer, and (2) designing the adaptive observer and the parameter estimator. It has been shown that the design relies on the polytopic approximation of the different matrices and on solving a set of BMIs for the state observer and a set of TMIs for the adaptive observer/parameter estimator. While the BMIs can be transformed into LMIs through a standard change of variables, in the general case it is not possible to find a change of variables that transforms the TMIs into LMIs. However, an appropriate choice of the Lyapunov matrix allows to solve this issue, even though at the expense of introducing additional conservativeness.

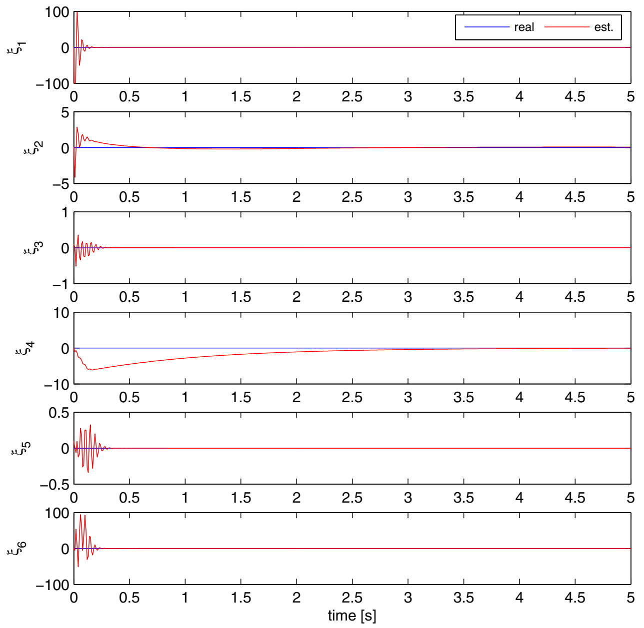

Fig. 2. Estimation of the unknown parameters at the beginning of the simulation.

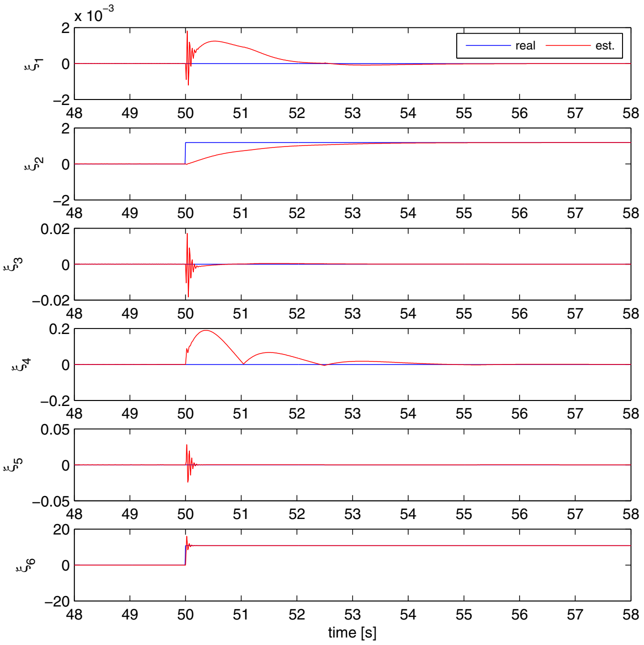

Fig. 3. Estimation of the unknown parameters around $t = 50\,s$.

The performance of the proposed approach has been assessed using a four wheeled omnidirectional mobile robot subject to unknown offsets in the motor voltages. It has been shown that the introduction of such offsets generates undesired oscillations and offsets in the robot response, if no countermeasures are taken. However, the application of the proposed adaptive observer drives back, after a short transient, the estimation error to zero.

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