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Stabilization of Time-Delay Markovian Jump Systems via Probability Rate Synthesis and State Feedback ⋆

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Abstract: This paper considers the stabilization problem for Markovian jump systems with time delays. Both the probability rate matrix and the state feedback control law are to be designed. A sufficient condition is established for such designs such that the resulting closed-loop Markovian jump system is stochastically stable. This condition is given in terms of a system of linear matrix inequalities with rank constraints, and can be solved using some existing algorithms. When the system has polytopic uncertainties, the robust stabilization problem is studied as well. Finally, a numerical example is given to show the validity of the proposed method.

Keywords: Markovian jump systems, time delay, state feedback, stabilization.

1. INTRODUCTION

Markovian jump systems are widely used to model those dynamic systems subject to abrupt changes in their structures and parameters. Examples of such systems include power systems, economic systems and production systems. During the past decades, many useful results have been obtained for Markovian jump systems, such as stability and stabilization (Feng et al. [2010], de Souza [2006]), model reduction (Zhang et al. [2003]), filtering (Shi et al. [1999a], Wu et al. [2008]), H_2 and H_∞ control (Dong and Yang [2008], Xu and Chen [2002]).

On the other hand, time delays are often inevitable in practice and exist in many practical systems such as communication systems and network systems. It is also well known that time delays are an important source of instability of control systems. Therefore there has been considerable interest in the study of time-delay systems in recent years; e.g., see Fridman and Shaked [2002] and Xu and Lam [2005]. For Markovian jump systems, the time delay issue is also widely investigated; e.g., see Boukas et al. [2002], Fei et al. [2009], Shi et al. [1999b], Sun et al. [2007] and Xu and Chen [2002].

In this paper, we consider the stabilization problem for a class of Markovian jump systems with time delays. Unlike previous techniques in the literature, our technique involves choosing appropriate probability rate matrices and state feedback control laws to guarantee the stability of closed-loop systems. This technique is motivated by the fact that sometimes engineers may have freedom in selecting the value of the probability rate matrix of a Markovian jump system (Feng et al. [2010], Shu et al. [2012]). For example, suppose we have a plant which is a linear time invariant system. If we design a Markovian type controller (with different control gains) for such a plant, an appropriate switching rule between these gains may be selected to help stabilize the system. In fact, the stabilizing technique developed here can be seen as an extension of the result in Feng et al. [2010] and Shu et al. [2012] to the class of Markovian jump systems with time delays.

Notation: $\mathbb{R}^m$, $\mathbb{R}^{m\times n}$ denote the sets of real $m\times 1$ vectors and real $m\times n$ matrices, respectively. The superscript "T" denotes the transpose for real matrices or real vectors. $\mathrm{diag}[M_1,\dots,M_N]$ denotes a block diagonal matrix with $M_1,\dots,M_N$ on its main diagonal. For real symmetric matrices X and Y, $X\geq Y$ (respectively, $X>Y$) means that $X-Y$ is positive semi-definite (respectively, positive definite). $\mathrm{E}(\cdot)$ stands for the mathematical expectation operator with respect to the complete probability space $(\Omega,\mathcal{F},P)$.

2. PROBLEM FORMULATION

Consider a time-delay Markovian jump system $\mathcal{S}$ of the following form:

$$\mathcal{S}:\begin{cases}\dot{x}(t)=A(\eta(t))x(t)+A_d(\eta(t))x(t-\tau)+B(\eta(t))u(t),\\ x(t)=\varrho(t),\quad \forall t\in[-\tau,0],\end{cases}\tag{1}$$

where $x(t)\in\mathbb{R}^n$ is the state, $u(t)\in\mathbb{R}^m$ is the control input. $\{\eta(t),\ t\geq 0\}$ describes the mode switching of the system $\mathcal{S}$ and is a continuous-time Markov process taking values in a finite set $\mathcal{M}\triangleq\{1,2,\dots,M\}$. The probability rate matrix of $\eta(t)$ is $\mathbf{Q}=[q_{\mu\nu}]\in\mathbb{R}^{M\times M}$, in which

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$$P(\eta(t+\triangle)=\nu \mid \eta(t)=\mu)=\begin{cases} q_{\mu\nu}\triangle+o(\triangle), & \nu\neq\mu, \\ 1+q_{\mu\mu}\triangle+o(\triangle), & \nu=\mu, \end{cases} \tag{2}$$

where $\triangle > 0$ and $o(\triangle)/\triangle \to 0$ as $\triangle$ approaches to zero. $q_{\mu\nu} \geq 0$, $\forall \nu \neq \mu$, and $q_{\mu\mu} = -\sum_{\nu=1,\nu\neq\mu}^{M} q_{\mu\nu} \leq 0$. The constant $\tau > 0$ represents the system's time delay. The initial condition of the system $\mathcal{S}$ is $(\eta_0, \varrho(t))$ where η_0 is the initial mode of $\eta(t)$ at time $t = 0$ and $\varrho(t)$ is a continuous vector-valued function defined on the interval $[-\tau, 0]$. We assume that $\int_{-\tau}^{0} \|\varrho(t)\|^2 \, dt < \infty$.

Definition 1. A square matrix $W = [w_{\mu\nu}]$ is called Metzler if it satisfies

$$w_{\mu\nu} \geq 0, \quad \text{if} \quad \mu \neq \nu.$$

Remark 2. The probability rate matrix $\mathbf{Q}$ is a Metzler matrix because its off-diagonal entries are nonnegative.

In the sequel, we will use the following stability definition for the system (1).

Definition 3. (Boukas et al. [2002]). The system (1) is said to be stochastically stable if there exists a constant $\lambda(\eta_0, \varrho(\cdot))$, such that

$$\begin{aligned} &\mathrm{E}\left(\int_0^\infty \|x(t)\|^2 \, dt \mid \eta_0,\ x(s)=\varrho(s),\ s\in[-\tau,0]\right) \\ &\leq \lambda(\eta_0, \varrho(\cdot)). \end{aligned} \tag{3}$$

For the Markovian jump system (1), our aim is to design a mode dependent state feedback controller

$$u(t) = K(\eta(t))x(t), \quad t \geq 0, \tag{4}$$

and a probability rate matrix $\mathbf{Q}$, such that the resulting closed-loop system is stochastically stable.

3. STABILITY ANALYSIS

In this section, we consider the stability problem of the free system in (1) (i.e., $u(t) \equiv 0$). The following result is taken from Boukas et al. [2002] and provides a sufficient condition for checking the stability of the free system in (1) when the probability rate matrix $\mathbf{Q}$ is given.

Lemma 4. (Boukas et al. [2002]). Given the time delay $\tau > 0$. The free system in (1) is stochastically stable if there exist positive definite matrices $P(\mu) > 0$, $\mu \in \mathcal{M}$, $R > 0$, $R_1 > 0$, such that

$$\begin{aligned} &[A(\mu)+A_d(\mu)]^T P(\mu) + P(\mu)[A(\mu)+A_d(\mu)] \\ &\quad + \sum_{\nu=1}^{M} q_{\mu\nu}P(\nu) + \tau\left[A^T(\mu)RA(\mu)+R_1\right] \\ &\quad\quad + 2\tau P(\mu)A_d(\mu)R^{-1}A_d^T(\mu)P(\mu) < 0, \end{aligned} \tag{5}$$

$$A_d^T(\mu)RA_d(\mu) \leq R_1. \tag{6}$$

The conditions in Lemma 4 can be easily recast as LMIs by using the Schur complement equivalence if the probability rate matrix $\mathbf{Q}$ is given. However, if the probability rate matrix $\mathbf{Q}$ is not known, the inequalities in (5), (6) cannot be transformed into LMIs directly and hence is generally very difficult to solve. In the following, we will deal with the stability analysis problem where the probability rate matrix $\mathbf{Q}$ is not known. We assume that the value of the probability rate matrix $\mathbf{Q}$ is allowed to be designed (or changed) to make the resulting system stable.

Theorem 5. Given the time delay $\tau > 0$. If there exist positive definite matrices $X(\mu) > 0$, $\bar{X}(\mu) > 0$, $\mu \in \mathcal{M}$, $Y > 0$, $\bar{Y} > 0$, $Z > 0$, $\bar{Z} > 0$, a scalar $\varepsilon > 0$, and a Metzler matrix $\bar{\mathbf{Q}} = [\bar{q}_{\mu\nu}] \in \mathbb{R}^{M\times M}$, such that the following rank constrained LMIs hold:

$$\begin{bmatrix} \mathcal{G}_{11}(\mu) & \mathcal{G}_{12}(\mu) & \mathcal{G}_{13}(\mu) & \mathcal{G}_{14}(\mu) & \mathcal{G}_{15}(\mu) & \mathcal{G}_{16}(\mu) \\ * & \mathcal{G}_{22}(\mu) & 0 & 0 & 0 & 0 \\ * & * & \mathcal{G}_{33}(\mu) & 0 & 0 & 0 \\ * & * & * & -Y & 0 & 0 \\ * & * & * & * & -\bar{Y} & 0 \\ * & * & * & * & * & -\bar{Z} \end{bmatrix} < 0, \tag{7}$$

$$A_d^T(\mu)YA_d(\mu) \leq Z, \tag{8}$$

$$\operatorname{rank}\left(\begin{bmatrix} X(\mu) & I \\ I & \bar{X}(\mu) \end{bmatrix}\right) \leq n, \ \operatorname{rank}\left(\begin{bmatrix} Y & I \\ I & \bar{Y} \end{bmatrix}\right) \leq n, \tag{9}$$

$$\operatorname{rank}\left(\begin{bmatrix} Z & I \\ I & \bar{Z} \end{bmatrix}\right) \leq n, \tag{10}$$

where

$$\begin{aligned} &\mathcal{G}_{11}(\mu) = -2X(\mu) - \sum_{\nu=1,\nu\neq\mu}^{M} X(\nu), \\ &\mathcal{G}_{12}(\mu) = I + \varepsilon\left[A(\mu)+A_d(\mu)\right]^T, \quad \mathcal{G}_{22}(\mu) = -\bar{X}(\mu), \\ &\mathcal{G}_{13}(\mu) = \left[(1+\bar{q}_{\mu1})I \ \ (1+\bar{q}_{\mu2})I \ \ \cdots \ \ (1+\bar{q}_{\mu M})I\right], \\ &\mathcal{G}_{33}(\mu) = -\operatorname{diag}\left[\bar{X}(1),\ \bar{X}(2),\ \cdots,\ \bar{X}(M)\right], \\ &\mathcal{G}_{14}(\mu) = \sqrt{2\tau}X(\mu)A_d(\mu), \quad \mathcal{G}_{15}(\mu) = \sqrt{\tau}\varepsilon A^T(\mu), \\ &\mathcal{G}_{16}(\mu) = \sqrt{\tau}\varepsilon I. \end{aligned}$$

then the free system in (1) is stochastically stable with the probability rate matrix given by $\mathbf{Q} = \frac{2}{\varepsilon}\bar{\mathbf{Q}}$.

Proof. The inequality (5) holds if and only if the following inequality holds for some sufficiently small scalar $\varepsilon > 0$.

$$\begin{aligned} &[A(\mu)+A_d(\mu)]^T P(\mu) + P(\mu)[A(\mu)+A_d(\mu)] \\ &+ \sum_{\nu=1}^{M} \frac{q_{\mu\nu}}{2}P(\nu) + \sum_{\nu=1}^{M} \frac{q_{\mu\nu}}{2}P(\nu) + \varepsilon\sum_{\nu=1}^{M}\left(\frac{q_{\mu\nu}}{2}P(\nu)\frac{q_{\mu\nu}}{2}\right) \\ &+ \varepsilon[A(\mu)+A_d(\mu)]^T P(\mu)[A(\mu)+A_d(\mu)] \\ &+ \tau\left[A^T(\mu)RA(\mu)+R_1\right] \\ &+ 2\tau P(\mu)A_d(\mu)R^{-1}A_d^T(\mu)P(\mu) < 0. \end{aligned} \tag{11}$$

By Schur complement equivalence, the inequality (11) can be further transformed into the following inequality:

$$\begin{bmatrix} \mathcal{H}_{11}(\mu) & \mathcal{H}_{12}(\mu) & \mathcal{H}_{13}(\mu) & \mathcal{H}_{14}(\mu) & \mathcal{H}_{15}(\mu) & \mathcal{H}_{16}(\mu) \\ * & \mathcal{H}_{22}(\mu) & 0 & 0 & 0 & 0 \\ * & * & \mathcal{H}_{33}(\mu) & 0 & 0 & 0 \\ * & * & * & -\varepsilon^{-2}R & 0 & 0 \\ * & * & * & * & -\varepsilon^2R^{-1} & 0 \\ * & * & * & * & * & -\varepsilon^2R_1^{-1} \end{bmatrix} < 0, \tag{12}$$

where

$$\mathcal{H}_{11}(\mu) = -2\varepsilon^{-1}P(\mu) - \varepsilon^{-1}\sum_{\nu=1,\nu\neq\mu}^{M} P(\nu),$$
$$\mathcal{H}_{12}(\mu) = I + \varepsilon\left[A(\mu) + A_d(\mu)\right]^T,$$
$$\mathcal{H}_{22}(\mu) = -\varepsilon P^{-1}(\mu),$$
$$\mathcal{H}_{13}(\mu) = \left[(1+\frac{\varepsilon}{2}q_{\mu 1})I \;\; (1+\frac{\varepsilon}{2}q_{\mu 2})I \;\; \cdots \;\; (1+\frac{\varepsilon}{2}q_{\mu M})I\right],$$
$$\mathcal{H}_{33}(\mu) = -\,\mathrm{diag}\left[\varepsilon P^{-1}(1),\ \varepsilon P^{-1}(2),\ \cdots,\ \varepsilon P^{-1}(M)\right],$$
$$\mathcal{H}_{14}(\mu) = \sqrt{2\tau}\varepsilon^{-1}P(\mu)A_d(\mu),$$
$$\mathcal{H}_{15}(\mu) = \sqrt{\tau}\varepsilon A^T(\mu),$$
$$\mathcal{H}_{16}(\mu) = \sqrt{\tau}\varepsilon I.$$

On the other hand, the inequality (6) is equivalent to the following inequality

$$A_d^T(\mu)\left(\varepsilon^{-2}R\right)A_d(\mu) \leq \varepsilon^{-2}R_1. \tag{13}$$

By defining $X(\mu) = \varepsilon^{-1}P(\mu)$, $\bar{X}(\mu) = \varepsilon P^{-1}(\mu)$, $\bar{q}_{\mu\nu} = \frac{\varepsilon}{2}q_{\mu\nu}$, $Y = \varepsilon^{-2}R$, $\bar{Y} = \varepsilon^{2}R^{-1}$, $Z = \varepsilon^{-2}R_1$ and $\bar{Z} = \varepsilon^{2}R_1^{-1}$ in the inequalities (12) and (13), we see that the inequalities (12) and (13) can be reformulated as the inequalities (7), (8) with matrix equality constraints $X(\mu)\bar{X}(\mu) = I$, $Y\bar{Y} = I$, $Z\bar{Z} = I$. Note that these matrix equality constraints are in fact equivalent to the rank constraints (9), (10). Therefore, if the inequalities (7), (8), (9), (10) hold, then the inequalities (5), (6) hold. By Lemma 4, the free system in (1) is stochastically stable if the probability rate matrix is chosen as $\mathbf{Q} = \frac{2}{\varepsilon}\bar{\mathbf{Q}}$. □

Theorem 5 provides us a method to find an appropriate probability rate matrix $\mathbf{Q}$ such that the free system in (1) is stochastically stable. Similar to Feng et al. [2010], the result in Theorem 5 can be extended to the free time-delay system in (1) with polytopic uncertainties. To illustrate this, assume that the free system in (1) has the following polytopic uncertainties:

$$\left[A(\mu)\ A_d(\mu)\right] = \sum_{\xi=1}^{\varpi}\alpha_\xi\left[A_\xi(\mu)\ A_{d\xi}(\mu)\right], \tag{14}$$

where $\mu \in \mathcal{M}$, $\alpha_\xi \geq 0$, $\sum_{\xi=1}^{\varpi}\alpha_\xi = 1$. Then we have the following corollary.

Corollary 6. Given the time delay $\tau > 0$. For the free time-delay system in (1) with polytopic uncertainties (14), if there exist positive definite matrices $X(\mu) > 0$, $\bar{X}(\mu) > 0$, $\mu \in \mathcal{M}$, $Y > 0$, $\bar{Y} > 0$, $Z > 0$, $\bar{Z} > 0$, a scalar $\varepsilon > 0$, and a Metzler matrix $\bar{\mathbf{Q}} = [\bar{q}_{\mu\nu}] \in \mathbb{R}^{M\times M}$, such that the following rank constrained LMIs hold:

$$\Xi_\xi(\mu) = \begin{bmatrix} \mathcal{G}_{11}(\xi,\mu) & \mathcal{G}_{12}(\xi,\mu) & \mathcal{G}_{13}(\xi,\mu) & \mathcal{G}_{14}(\xi,\mu) & \mathcal{G}_{15}(\xi,\mu) & \mathcal{G}_{16}(\xi,\mu) \\ * & \mathcal{G}_{22}(\xi,\mu) & 0 & 0 & 0 & 0 \\ * & * & \mathcal{G}_{33}(\xi,\mu) & 0 & 0 & 0 \\ * & * & * & -Y & 0 & 0 \\ * & * & * & * & -\bar{Y} & 0 \\ * & * & * & * & * & -\bar{Z} \end{bmatrix} < 0, \tag{15}$$

$$\Pi_{\xi,\varrho}(\mu) = A_{d\xi}^T(\mu)YA_{d\varrho}(\mu) \leq Z, \tag{16}$$

$$\mathrm{rank}\left(\begin{bmatrix} X(\mu) & I \\ I & \bar{X}(\mu)\end{bmatrix}\right) \leq n,\ \mathrm{rank}\left(\begin{bmatrix} Y & I \\ I & \bar{Y}\end{bmatrix}\right) \leq n, \tag{17}$$

$$\mathrm{rank}\left(\begin{bmatrix} Z & I \\ I & \bar{Z}\end{bmatrix}\right) \leq n, \tag{18}$$

where

$$\mathcal{G}_{11}(\xi,\mu) = -2X(\mu) - \sum_{\nu=1,\nu\neq\mu}^{M} X(\nu),$$
$$\mathcal{G}_{12}(\xi,\mu) = I + \varepsilon\left[A_\xi(\mu) + A_{d\xi}(\mu)\right]^T,\ \mathcal{G}_{22}(\xi,\mu) = -\bar{X}(\mu),$$
$$\mathcal{G}_{13}(\xi,\mu) = \left[(1+\bar{q}_{\mu 1})I \;\; (1+\bar{q}_{\mu 2})I \;\; \cdots \;\; (1+\bar{q}_{\mu M})I\right],$$
$$\mathcal{G}_{33}(\xi,\mu) = -\,\mathrm{diag}\left[\bar{X}(1),\ \bar{X}(2),\ \cdots,\ \bar{X}(M)\right],$$
$$\mathcal{G}_{14}(\xi,\mu) = \sqrt{2\tau}X(\mu)A_{d\xi}(\mu), \quad \mathcal{G}_{15}(\xi,\mu) = \sqrt{\tau}\varepsilon A_\xi^T(\mu),$$
$$\mathcal{G}_{16}(\xi,\mu) = \sqrt{\tau}\varepsilon I,$$

$\xi = 1,\cdots,\varpi$, $\varrho = 1,\cdots,\varpi$, then the free system in (1) is robustly stochastically stable with the probability rate matrix given by $\mathbf{Q} = \frac{2}{\varepsilon}\bar{\mathbf{Q}}$.

Proof. Multiplying both sides of the inequality (15) by α_ξ yields that

$$\alpha_\xi\Xi_\xi(\mu) < 0, \quad \xi = 1,\cdots,\varpi. \tag{19}$$

Then we have $\sum_{\xi=1}^{\varpi}\alpha_\xi\Xi_\xi(\mu) < 0$, shown at the bottom of this page, where

$$\sum_{\xi=1}^{\varpi}\alpha_\xi\Xi_\xi(\mu) = \begin{bmatrix} \sum_{\xi=1}^{\varpi}\alpha_\xi\mathcal{G}_{11}(\xi,\mu) & \sum_{\xi=1}^{\varpi}\alpha_\xi\mathcal{G}_{12}(\xi,\mu) & \sum_{\xi=1}^{\varpi}\alpha_\xi\mathcal{G}_{13}(\xi,\mu) & \sum_{\xi=1}^{\varpi}\alpha_\xi\mathcal{G}_{14}(\xi,\mu) & \sum_{\xi=1}^{\varpi}\alpha_\xi\mathcal{G}_{15}(\xi,\mu) & \sum_{\xi=1}^{\varpi}\alpha_\xi\mathcal{G}_{16}(\xi,\mu) \\ * & \sum_{\xi=1}^{\varpi}\alpha_\xi\mathcal{G}_{22}(\xi,\mu) & 0 & 0 & 0 & 0 \\ * & * & \sum_{\xi=1}^{\varpi}\alpha_\xi\mathcal{G}_{33}(\xi,\mu) & 0 & 0 & 0 \\ * & * & * & \sum_{\xi=1}^{\varpi}\alpha_\xi(-Y) & 0 & 0 \\ * & * & * & * & \sum_{\xi=1}^{\varpi}\alpha_\xi(-\bar{Y}) & 0 \\ * & * & * & * & * & \sum_{\xi=1}^{\varpi}\alpha_\xi(-\bar{Z}) \end{bmatrix}$$

$$\sum_{\xi=1}^{\varpi}\alpha_\xi\mathcal{G}_{11}(\xi,\mu) = -2X(\mu) - \sum_{\nu=1,\nu\neq\mu}^{M} X(\nu),$$
$$\sum_{\xi=1}^{\varpi}\alpha_\xi\mathcal{G}_{12}(\xi,\mu) = I + \varepsilon\left[A(\mu)+A_d(\mu)\right]^T,$$
$$\sum_{\xi=1}^{\varpi}\alpha_\xi\mathcal{G}_{22}(\xi,\mu) = -\bar{X}(\mu),$$
$$\sum_{\xi=1}^{\varpi}\alpha_\xi\mathcal{G}_{13}(\xi,\mu) = \left[(1+\bar{q}_{\mu 1})I \;\; (1+\bar{q}_{\mu 2})I \;\; \cdots \;\; (1+\bar{q}_{\mu M})I\right],$$
$$\sum_{\xi=1}^{\varpi}\alpha_\xi\mathcal{G}_{33}(\xi,\mu) = -\operatorname{diag}\left[\bar{X}(1),\ \bar{X}(2),\ \cdots,\ \bar{X}(M)\right],$$
$$\sum_{\xi=1}^{\varpi}\alpha_\xi\mathcal{G}_{14}(\xi,\mu) = \sqrt{2\tau}X(\mu)A_d(\mu),$$
$$\sum_{\xi=1}^{\varpi}\alpha_\xi\mathcal{G}_{15}(\xi,\mu) = \sqrt{\tau}\varepsilon A^T(\mu),$$
$$\sum_{\xi=1}^{\varpi}\alpha_\xi\mathcal{G}_{16}(\xi,\mu) = \sqrt{\tau}\varepsilon I, \quad \sum_{\xi=1}^{\varpi}\alpha_\xi(-Y) = -Y,$$
$$\sum_{\xi=1}^{\varpi}\alpha_\xi(-\bar{Y}) = -\bar{Y}, \quad \sum_{\xi=1}^{\varpi}\alpha_\xi(-\bar{Z}) = -\bar{Z}.$$

That is, the inequality (7) holds. On the other hand, the inequality (16) implies

$$\begin{aligned} A_d^T(\mu)YA_d(\mu) &= \sum_{\xi=1}^{\varpi}\sum_{\varrho=1}^{\varpi}\alpha_\xi\alpha_\varrho A_{d\xi}^T(\mu)YA_{d\varrho}(\mu) \\ &\leq \sum_{\xi=1}^{\varpi}\sum_{\varrho=1}^{\varpi}\alpha_\xi\alpha_\varrho Z = Z. \end{aligned} \tag{20}$$

That is, the inequality (8) holds. Therefore, based on Theorem 5, the free time-delay system in (1) with polytopic uncertainties (14) is robustly stochastically stable if the probability rate matrix is chosen as $\mathbf{Q} = \frac{2}{\varepsilon}\bar{\mathbf{Q}}$. □

4. STABILIZATION VIA PROBABILITY RATE SYNTHESIS AND STATE FEEDBACK

In this section, we are concerned with the problem of designing a stabilizing state feedback controller (4) and a corresponding probability rate matrix $\mathbf{Q}$ for the system (1). Firstly, applying the controller (4) to the system (1) yields the following closed-loop system:

$$\dot{x}(t) = \left(A(\eta(t)) + B(\eta(t))K(\eta(t))\right)x(t) + A_d(\eta(t))x(t-\tau), \tag{21}$$

$$x(t) = \varrho(t), \quad \forall t \in [-\tau, 0]. \tag{22}$$

Then we give the following Theorem 7 which provides a sufficient condition for the design of a state feedback stabilizing controller (4) and a corresponding probability rate matrix $\mathbf{Q}$.

Theorem 7. Given the time delay $\tau > 0$. If there exist positive definite matrices $X(\mu) > 0$, $\bar{X}(\mu) > 0$, $\mu \in \mathcal{M}$, $Y > 0$, $\bar{Y} > 0$, $Z > 0$, $\bar{Z} > 0$, a positive scalar $\varepsilon > 0$, a Metzler matrix $\bar{\mathbf{Q}} = [\bar{q}_{\mu\nu}] \in \mathbb{R}^{M\times M}$ and matrices $\tilde{K}(\mu) \in \mathbb{R}^{m\times n}$, $\mu \in \mathcal{M}$, such that the following rank constrained LMIs hold:

$$\begin{bmatrix} \mathcal{G}_{11}(\mu) & \mathcal{G}_{12}(\mu) & \mathcal{G}_{13}(\mu) & \mathcal{G}_{14}(\mu) & \mathcal{G}_{15}(\mu) & \mathcal{G}_{16}(\mu) & \mathcal{G}_{17}(\mu) \\ * & \mathcal{G}_{22}(\mu) & 0 & 0 & 0 & 0 & 0 \\ * & * & \mathcal{G}_{33}(\mu) & 0 & 0 & 0 & 0 \\ * & * & * & \mathcal{G}_{44}(\mu) & 0 & 0 & 0 \\ * & * & * & * & \mathcal{G}_{55}(\mu) & 0 & 0 \\ * & * & * & * & * & \mathcal{G}_{66}(\mu) & 0 \\ * & * & * & * & * & * & \mathcal{G}_{77}(\mu) \end{bmatrix} < 0, \tag{23}$$

$$A_d^T(\mu)YA_d(\mu) \leq Z, \tag{24}$$

$$\operatorname{rank}\left(\begin{bmatrix} X(\mu) & I \\ I & \bar{X}(\mu) \end{bmatrix}\right) \leq n, \tag{25}$$

$$\operatorname{rank}\left(\begin{bmatrix} Y & I \\ I & \bar{Y} \end{bmatrix}\right) \leq n, \ \operatorname{rank}\left(\begin{bmatrix} Z & I \\ I & \bar{Z} \end{bmatrix}\right) \leq n, \tag{26}$$

where

$$\begin{aligned}
&\mathcal{G}_{11}(\mu) = -3X(\mu) - \sum_{\nu=1,\nu\neq\mu}^{M} X(\nu),\\
&\mathcal{G}_{12}(\mu) = I + \varepsilon\left[A(\mu)+A_d(\mu)\right]^T,\\
&\mathcal{G}_{13}(\mu) = I + \left[B(\mu)\tilde{K}(\mu)\right]^T,\\
&\mathcal{G}_{22}(\mu) = \mathcal{G}_{33}(\mu) = -\bar{X}(\mu),\\
&\mathcal{G}_{14}(\mu) = \left[(1+\bar{q}_{\mu 1})I \;\; (1+\bar{q}_{\mu 2})I \;\; \cdots \;\; (1+\bar{q}_{\mu M})I\right],\\
&\mathcal{G}_{44}(\mu) = -\operatorname{diag}\left[\bar{X}(1),\ \bar{X}(2),\ \cdots,\ \bar{X}(M)\right],\\
&\mathcal{G}_{15}(\mu) = \sqrt{2\tau}X(\mu)A_d(\mu),\ \mathcal{G}_{55}(\mu) = -Y,\\
&\mathcal{G}_{16}(\mu) = \sqrt{\tau}\left[\varepsilon A(\mu) + B(\mu)\tilde{K}(\mu)\right]^T,\ \mathcal{G}_{66}(\mu) = -\bar{Y},\\
&\mathcal{G}_{17}(\mu) = \sqrt{\tau}\varepsilon I,\ \mathcal{G}_{77}(\mu) = -\bar{Z},
\end{aligned}$$

then the closed-loop system (21), (22) is stochastically stable if the controller is chosen to be $K(\mu) = \varepsilon^{-1}\tilde{K}(\mu)$, $\mu \in \mathcal{M}$ and the probability rate matrix is chosen to be $\mathbf{Q} = \frac{2}{\varepsilon}\bar{\mathbf{Q}}$.

Proof. By Lemma 4, if there exist positive matrices $P(\mu) > 0$, $\mu \in \mathcal{M}$, $R > 0$, $R_1 > 0$, such that the following inequalities hold:

$$\begin{aligned}
&\left[A(\mu) + B(\mu)K(\mu) + A_d(\mu)\right]^T P(\mu) \\
&+ P(\mu)\left[A(\mu) + B(\mu)K(\mu) + A_d(\mu)\right] + \sum_{\nu=1}^{M} q_{\mu\nu}P(\nu) \\
&+ \tau\left[(A(\mu)+B(\mu)K(\mu))^T R(A(\mu)+B(\mu)K(\mu)) + R_1\right] \\
&+ 2\tau P(\mu)A_d(\mu)R^{-1}A_d^T(\mu)P(\mu) < 0,
\end{aligned} \tag{27}$$

$$A_d^T(\mu)RA_d(\mu) \leq R_1, \tag{28}$$

then the closed-loop system (21), (22) is stochastically stable.

On the other hand, the inequality (27) holds if and only if the following inequality holds for some sufficiently small scalar $\varepsilon > 0$.

$$\begin{aligned}
&\left[A(\mu)+A_d(\mu)\right]^T P(\mu) + P(\mu)\left[A(\mu)+A_d(\mu)\right] \\
&+ \left[B(\mu)K(\mu)\right]^T P(\mu) + P(\mu)B(\mu)K(\mu) \\
&+ \sum_{\nu=1}^{M}\frac{q_{\mu\nu}}{2}P(\nu) + \sum_{\nu=1}^{M}\frac{q_{\mu\nu}}{2}P(\nu) + \varepsilon\sum_{\nu=1}^{M}\left(\frac{q_{\mu\nu}}{2}P(\nu)\frac{q_{\mu\nu}}{2}\right) +
\end{aligned}$$

$$\begin{aligned}&\varepsilon\left[A(\mu)+A_d(\mu)\right]^T P(\mu)\left[A(\mu)+A_d(\mu)\right]\\&+\varepsilon\left[B(\mu)K(\mu)\right]^T P(\mu)B(\mu)K(\mu)\\&+2\tau P(\mu)A_d(\mu)R^{-1}A_d^T(\mu)P(\mu)+\\&\tau\left[(A(\mu)+B(\mu)K(\mu))^T R(A(\mu)+B(\mu)K(\mu))+R_1\right]<0.\end{aligned} \tag{29}$$

By Schur complement equivalence, the inequality (29) is equivalent to the following inequality:

$$\begin{bmatrix}\mathcal{H}_{11}(\mu) & \mathcal{H}_{12}(\mu) & \mathcal{H}_{13}(\mu) & \mathcal{H}_{14}(\mu) & \mathcal{H}_{15}(\mu) & \mathcal{H}_{16}(\mu) & \mathcal{H}_{17}(\mu)\\ * & \mathcal{H}_{22}(\mu) & 0 & 0 & 0 & 0 & 0\\ * & * & \mathcal{H}_{33}(\mu) & 0 & 0 & 0 & 0\\ * & * & * & \mathcal{H}_{44}(\mu) & 0 & 0 & 0\\ * & * & * & * & \mathcal{H}_{55}(\mu) & 0 & 0\\ * & * & * & * & * & \mathcal{H}_{66}(\mu) & 0\\ * & * & * & * & * & * & \mathcal{H}_{77}(\mu)\end{bmatrix}<0, \tag{30}$$

where

$$\mathcal{H}_{11}(\mu)=-3\varepsilon^{-1}P(\mu)-\varepsilon^{-1}\sum_{\nu=1,\nu\neq\mu}^{M}P(\nu),$$
$$\mathcal{H}_{12}(\mu)=I+\varepsilon\left[A(\mu)+A_d(\mu)\right]^T,$$
$$\mathcal{H}_{13}(\mu)=I+\varepsilon\left[B(\mu)K(\mu)\right]^T,$$
$$\mathcal{H}_{22}(\mu)=\mathcal{H}_{33}(\mu)=-\varepsilon P^{-1}(\mu),$$
$$\mathcal{H}_{14}(\mu)=\left[(1+\frac{\varepsilon}{2}q_{\mu1})I\ \ (1+\frac{\varepsilon}{2}q_{\mu2})I\ \ \cdots\ \ (1+\frac{\varepsilon}{2}q_{\mu M})I\right],$$
$$\mathcal{H}_{44}(\mu)=-\operatorname{diag}\left[\varepsilon P^{-1}(1),\ \varepsilon P^{-1}(2),\ \cdots,\ \varepsilon P^{-1}(M)\right],$$
$$\mathcal{H}_{15}(\mu)=\sqrt{2\tau}\varepsilon^{-1}P(\mu)A_d(\mu),\ \mathcal{H}_{55}(\mu)=-\varepsilon^{-2}R,$$
$$\mathcal{H}_{16}(\mu)=\sqrt{\tau}\varepsilon\left[A(\mu)+B(\mu)K(\mu)\right]^T,\ \mathcal{H}_{66}(\mu)=-\varepsilon^2R^{-1},$$
$$\mathcal{H}_{17}(\mu)=\sqrt{\tau}\varepsilon I,\ \mathcal{H}_{77}(\mu)=-\varepsilon^2R_1^{-1}.$$

On the other hand, the inequality (28) holds if and only if the following inequality holds:

$$A_d^T(\mu)\left(\varepsilon^{-2}R\right)A_d(\mu)\le\varepsilon^{-2}R_1, \tag{31}$$

By defining $X(\mu)=\varepsilon^{-1}P(\mu)$, $\bar{X}(\mu)=\varepsilon P^{-1}(\mu)$, $\bar{q}_{\mu\nu}=\frac{\varepsilon}{2}q_{\mu\nu}$, $Y=\varepsilon^{-2}R$, $\bar{Y}=\varepsilon^2R^{-1}$, $Z=\varepsilon^{-2}R_1$, $\bar{Z}=\varepsilon^2R_1^{-1}$, $\tilde{K}(\mu)=\varepsilon K(\mu)$, $\mu\in\mathcal{M}$, in the inequalities (30), (31), we see that the inequalities (30), (31) can be reformulated as the inequalities (23), (24) with matrix equality constraints $X(\mu)\bar{X}(\mu)=I$, $Y\bar{Y}=I$, $Z\bar{Z}=I$. Note that these matrix equality constraints are in fact equivalent to the rank constraints (25), (26). Therefore, if the inequalities (23), (24), (25), (26) hold, then the inequalities (27), (28) hold. By Lemma 4, the resulting closed-loop system (21), (22) is stochastically stable if the controller is $K(\mu)=\varepsilon^{-1}\tilde{K}(\mu)$, $\mu\in\mathcal{M}$ and the probability rate matrix is $\mathbf{Q}=\frac{2}{\varepsilon}\bar{\mathbf{Q}}$. □

Similarly, Theorem 7 can be extended to the time-delay system (1) with polytopic uncertainties. To illustrate this, we assume that the system (1) has the following polytopic uncertainties:

$$\left[A(\mu)\ A_d(\mu)\ B(\mu)\right]=\sum_{\xi=1}^{\varpi}\alpha_\xi\left[A_\xi(\mu)\ A_{d\xi}(\mu)\ B_\xi(\mu)\right], \tag{32}$$

where $\mu\in\mathcal{M}$, $\alpha_\xi\ge0$, $\sum_{\xi=1}^{\varpi}\alpha_\xi=1$. Then we have the following corollary.

Corollary 8. Given the time delay $\tau>0$. For the time-delay system in (1) with polytopic uncertainties (32), if there exist positive definite matrices $X(\mu)>0$, $\bar{X}(\mu)>0$, $\mu\in\mathcal{M}$, $Y>0$, $\bar{Y}>0$, $Z>0$, $\bar{Z}>0$, a positive scalar $\varepsilon>0$, a Metzler matrix $\bar{\mathbf{Q}}=[\bar{q}_{\mu\nu}]\in\mathbb{R}^{M\times M}$ and matrices $\tilde{K}(\mu)\in\mathbb{R}^{m\times n}$, $\mu\in\mathcal{M}$, such that the following rank constrained LMIs hold:

$$\Xi_\xi(\mu)=\begin{bmatrix}\mathcal{G}_{11}(\xi,\mu) & \mathcal{G}_{12}(\xi,\mu) & \mathcal{G}_{13}(\xi,\mu) & \mathcal{G}_{14}(\xi,\mu) & \mathcal{G}_{15}(\xi,\mu) & \mathcal{G}_{16}(\xi,\mu) & \mathcal{G}_{17}(\xi,\mu)\\ * & \mathcal{G}_{22}(\xi,\mu) & 0 & 0 & 0 & 0 & 0\\ * & * & \mathcal{G}_{33}(\xi,\mu) & 0 & 0 & 0 & 0\\ * & * & * & \mathcal{G}_{44}(\xi,\mu) & 0 & 0 & 0\\ * & * & * & * & \mathcal{G}_{55}(\xi,\mu) & 0 & 0\\ * & * & * & * & * & \mathcal{G}_{66}(\xi,\mu) & 0\\ * & * & * & * & * & * & \mathcal{G}_{77}(\xi,\mu)\end{bmatrix}<0, \tag{33}$$

$$\Pi_{\xi,\varrho}(\mu)=A_{d\xi}^T(\mu)YA_{d\varrho}(\mu)\le Z, \tag{34}$$

$$\operatorname{rank}\left(\begin{bmatrix}X(\mu) & I\\ I & \bar{X}(\mu)\end{bmatrix}\right)\le n, \tag{35}$$

$$\operatorname{rank}\left(\begin{bmatrix}Y & I\\ I & \bar{Y}\end{bmatrix}\right)\le n,\ \operatorname{rank}\left(\begin{bmatrix}Z & I\\ I & \bar{Z}\end{bmatrix}\right)\le n, \tag{36}$$

where

$$\mathcal{G}_{11}(\xi,\mu)=-3X(\mu)-\sum_{\nu=1,\nu\neq\mu}^{M}X(\nu),$$
$$\mathcal{G}_{12}(\xi,\mu)=I+\varepsilon\left[A_\xi(\mu)+A_{d\xi}(\mu)\right]^T,$$
$$\mathcal{G}_{13}(\xi,\mu)=I+\left[B_\xi(\mu)\tilde{K}(\mu)\right]^T,$$
$$\mathcal{G}_{22}(\xi,\mu)=\mathcal{G}_{33}(\mu)=-\bar{X}(\mu),$$
$$\mathcal{G}_{14}(\xi,\mu)=\left[(1+\bar{q}_{\mu1})I\ \ (1+\bar{q}_{\mu2})I\ \ \cdots\ \ (1+\bar{q}_{\mu M})I\right],$$
$$\mathcal{G}_{44}(\xi,\mu)=-\operatorname{diag}\left[\bar{X}(1),\ \bar{X}(2),\ \cdots,\ \bar{X}(M)\right],$$
$$\mathcal{G}_{15}(\xi,\mu)=\sqrt{2\tau}X(\mu)A_{d\xi}(\mu),\ \mathcal{G}_{55}(\mu)=-Y,$$
$$\mathcal{G}_{16}(\xi,\mu)=\sqrt{\tau}\left[\varepsilon A_\xi(\mu)+B_\xi(\mu)\tilde{K}(\mu)\right]^T,\ \mathcal{G}_{66}(\mu)=-\bar{Y},$$
$$\mathcal{G}_{17}(\xi,\mu)=\sqrt{\tau}\varepsilon I,\ \mathcal{G}_{77}(\xi,\mu)=-\bar{Z},$$

$\xi=1,\cdots,\varpi$, $\varrho=1,\cdots,\varpi$, then the closed-loop system (21), (22) is stochastically stable if the controller is chosen to be $K(\mu)=\varepsilon^{-1}\tilde{K}(\mu)$, $\mu\in\mathcal{M}$ and the probability rate matrix is chosen to be $\mathbf{Q}=\frac{2}{\varepsilon}\bar{\mathbf{Q}}$.

Proof. The proof is very similar to the proof of Corollary 6, hence is omitted here. □

Remark 9. It is worth noting that, in both Theorem 5 and Theorem 7, an implicit equality constraint exists as follows:

$$\bar{q}_{\mu\mu}=-\sum_{\nu=1,\nu\neq\mu}^{M}\bar{q}_{\mu\nu},\quad \mu\in\mathcal{M}.$$

To eliminate this equality constraint, one may replace $\mathcal{G}_{13}(\mu)$ in (7) and $\mathcal{G}_{14}(\mu)$ in (23) by the following:

$$T(\mu)=[(1+\bar{q}_{\mu1})I\ \cdots\ (1+\bar{q}_{\mu(\mu-1)})I\ (1-\sum_{\nu=1,\nu\neq\mu}^{M}\bar{q}_{\mu\nu})I\ \ (1+\bar{q}_{\mu(\mu+1)})I\ \cdots\ (1+\bar{q}_{\mu M})I].$$

Remark 10. Although the rank constrained LMI problem (23), (24), (25), (26) is generally NP-hard, several numerical methods have been proposed for this problem, such as, alternating projections method (Grigoriadis and Skelton [1996]); sequential semidefinite programming (Fares et al. [2002]); augmented Lagrangian method (Fares et al. [2001]); cone complementarity linearization algorithm (Ghaoui et al. [1997]); Newton-like method (Orsi et al. [2006]). In the numerical example, we use the Newton-like method suggested in Orsi et al. [2006] to solve our problem. The corresponding Matlab toolbox LMIRank can be downloaded for free at the author's homepage (Orsi [2005]). Although this algorithm is not guaranteed to find a solution in all cases, it can be quite effective and often yields good results in applications; see Orsi et al. [2006] for details.

5. NUMERICAL EXAMPLE

In this section, we consider the stabilization problem of a Markovian jump system with constant time delay as follows:

$$A(1) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad A(2) = \begin{bmatrix} -2 & 2 \\ 0 & -1 \end{bmatrix}, B(1) = \begin{bmatrix} 0.2 \\ 1 \end{bmatrix},$$
$$A(3) = \begin{bmatrix} 0.1 & 1 \\ 0.3 & -0.5 \end{bmatrix}, A_d(1) = \begin{bmatrix} -1 & 0 \\ 0.5 & 1 \end{bmatrix}, B(2) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$A_d(2) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \quad A_d(3) = \begin{bmatrix} -1 & 0 \\ 0.5 & 1.5 \end{bmatrix}, B(3) = \begin{bmatrix} 0.1 \\ 0.8 \end{bmatrix}.$$

The time delay $\tau = 0.25$. By using Theorem 7, a feasible state feedback controller is found to be

$$K(1) = [-2.6951\ -11.5792],$$
$$K(2) = [-4.4455\ 7.5539],$$
$$K(3) = [-1.4005\ -10.6123],$$

and the corresponding probability rate matrix is found to be

$$\mathbf{Q} = \begin{bmatrix} -10.494245 & 4.730089 & 5.764156 \\ 0.017226 & -5.404557 & 5.387331 \\ 0.017226 & 0.017226 & -0.034452 \end{bmatrix}.$$

After obtaining the probability rate matrix $\mathbf{Q}$ and the feedback controllers $K(\mu)$, $\mu = 1, 2, 3$, we can deduce the stability of the resulting closed-loop system using the results in Theorem 7.

6. CONCLUSION

In this paper, we have studied the stabilization problem for a class of Markovian jump systems with constant time delays. A sufficient condition based on rank constrained linear matrix inequalities is proposed for the design of state feedback control laws and probability rate matrices. We also provide a numerical example to show the effectiveness of the proposed results.

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